Previous exams: Fall 1994 and spring 1993

The suggested answers below are written with references to a previous curriculum.

Exam spring 1993

Question 1

In answering this question it is not necessary to derive model results, but you should explain the assumptions underlying the results you use.

(a) Discuss how a corporation should take risk into account when it evaluates whether to start a risky real investment project. You may assume that all the revenue from the project comes in a future period. There is no flexibility attached to the project: If the corporation decides to invest, all quantities (production and input factors) are fixed for the lifetime of the project.

(b) In relation to (a), discuss in particular the statement, "When the corporation adjusts the project value for risk, it should use the beta value of its own shares."

(d) In relation to (a), discuss in particular the statement, "If uncertain variable costs will be incurred in the same period as the project revenue, it will be an advantage to the owners if these are positively correlated with the market portfolio."

Exam fall 1994

(Questions (a)–(d))

Consider a number of shares which all have uncertain returns. You may assume that the return will appear in the form of increased prices of the shares, not as dividend payouts. Consider the set of those portfolios which may be composed from the shares. You may, without proof, assume that the set of the possible returns of these portfolios over one period may be described as a hyperbola in a diagram with the return's standard deviation on one axis and the return's expected value on the other. The area within the hyperbola is included.

(a) Show that when there is also a security with a risk free return, there may — under specific assumptions — for each share be derived a linear relationship between the expected return and a risk measure called beta.

(b) Show that one may then derive an equation giving the equilibrium value of the share at the beginning of the period expressed as a present value. The present value is obtained from dividing a certainty equivalent by the expression "one plus the risk free interest rate." The certainty equivalent is the expected value of the share at the end of the period minus a term which depends on a measure of risk.

(c) Assume now that the number of shares is very large, and that an exogenous event makes the value one period ahead of one of the shares more uncertain. More precisely the future value is multiplied by a stochastic variable which has an expected value of unity, and which is stochastically independent of everything else in the economy. Show how this affects the value of the share today, the expected return on the share, and the variance of this return.

(d) Explain why (but not how) the answer to (c) would be different if the share in focus had been a substantial part of the economy.

Answer to Question 1 of exam, spring 1993.

(The notation below is more sloppy than usual, dropping the tildes over stochastic variables.)

(a)

The basic model to use for this purpose is the capital asset pricing model (CAPM). One might think of some alternative theories:

- The time-state-preference model would be better than the CAPM if a complete set of markets for state-contingent claims existed. But in practice, the CAPM is more useful.
- Since there is no flexibility, the option valuation theory is not relevant.
- The weighted average cost of capital expressions are only relevant in the version which is an extension of the CAPM, bringing in the additional value of debt when there is a debt subsidy from the tax system. But this additional value as it is presented in Copeland and Weston has no particular relation to uncertainty.
- Limited (or "poor") diversification does not belong in 1 (a),(b), or (d), but in 1 (c). However, the CAPM is a special case of limited diverisfication, so there is little harm in starting out with that model.

Let "now" be time zero, and the end of the first period be time one. The CAPM relies on the following assumptions:

1. Agents' utilities depend only on the mean and the variance of their wealths at time one. When they maximize expected utility, this may rely on an assumption of normally distributed returns or on a quadratic utility function.
2. An agent's wealth at time one is the value of a portfolio of shares and riskless bonds composed from the agent's exogenous wealth at time zero.
3. The model considers competitive equilibrium in a stock market with an exogenously given interest rate. There is a fixed number of assets, which are perfectly divisible and tradable. Trade takes place only once, at time zero.
4. Everyone believes in the same, exogenously given probability distribution for the stock values at time one. All information is simultaneously available to everyone.
5. Short sales are allowed, there are no taxes and no transaction costs.

This means that the P_{j1}'s have exogenous probability distributions. The P_{j0}'s are exogenous to the agents, but endogenous in the model, as in any model of a competitive market equilibrium. The same goes for $R_j \equiv P_{j1}/P_{j0}$, and if the vector of P_{j1}'s has a multivariate normal distribution, then so has the vector of R_j's.

From the assumptions one derives the equilibrium relationship,

$$E(R_j) = R_0 + \beta_j[E(R_m) - R_0]$$

where $R_j = 1+$ the rate of return on asset j, the riskless asset has number zero, R_m is the return on the market portfolio, and $\beta_j = \operatorname{cov}(R_j, R_m)/\operatorname{var}(R_m)$. In equilibrium the expected return is equal to the riskless rate plus a risk premium, which depends on the return's covariance with the market portfolio, multiplied by $\lambda \equiv [E(R_m) - R_0]/\operatorname{var}(R_m)$. This reflects that all investors will compose their portfolio of risky assets in the same way, with return R_m, and that the covariance expresses how much asset j contributes to the risk of this portfolio.

When asset j is shares in a corporation, its return is equal to the value of equity at $t = 1$ divided by the value of equity $t = 0$. For simplicity we shall now consider an all-equity, no-debt, corporation. The value at $t = 1$ is simply the sum of the values of the corporation's projects. The valuation at $t = 0$ is given as

$$P_{j0} = \frac{1}{R_0}[E(P_{j1}) - \lambda \operatorname{cov}(P_{j1}, R_m)],$$

where the P_{jt}'s may be interpreted as the total values of the whole corporation. In relation to the CAPM, the expression in square brackets may be called the certainty equivalent of P_{j1}, but it is a different certainty equivalent from the one which is used in expected utility theory.

This valuation formula is additive, in the sense that the value of a sum of P_{j1}'s is equal to the sum of their values. (This could be shown at this point.) The value of adding a project can thus be found by valuing the project separately, as long as the project is small enough to leave the probability distribution of R_m approximately unchanged. There is no reason to consider the whole value of the firm, and different firms will find the same value of the project.

If we let P_{p1} be the project's cash flow at $t = 1$, then it remains to estimate $E(P_{p1})$ and $\operatorname{cov}(P_{p1}, R_m)$. Although this task is not simple, there may exist data to help. For instance, the covariance may be estimated from data for similar prices historically.

If the two estimates can be found, it remains to compare the valuation at $t = 0$ of the project's cash flow at $t = 1$ to the necessary investment at $t = 0$. If $R_p \equiv P_{p1}/I$, then it does not in general fall on the SML. But if I is replaced by the equilibrium valuation of the project, it will fall on the SML.

(b)

The correct beta, or covariance, to use is that which is determined by the project's characteristics. This may or may not be equal to that determined by the corporation's pre-existing activities. If the project will produce the same output as produced before, but by another technology, the project beta is equal to the corporation's beta. In practice this may often be helpful. But if there is a different output, the beta will be different, depending on the covariance of the new output price with the market portfolio.

The reference here is chapter 7F in Copeland and Weston. They argue against an existing practice, namely to use the same required expected rate of return for all projects within a firm.

(d)

If we let $P_{p1} \equiv Y - B$, where Y is revenue and B is cost, then the project beta depends on $\operatorname{cov}(Y - B, R_m) = \operatorname{cov}(Y, R_m) - \operatorname{cov}(B, R_m)$. It is clear that when the latter covariance is high, this reduces the project beta, so the statement is true. The intuitive explanation is that the obligation to pay B is an easier burden if the absolute value of B is high when R_m is high, than vice versa, since a high R_m means that the ability to pay is higher.

Answer to Questions (a)–(d) of exam, fall 1994

(a) (Weight 20/100.) Already the introduction assumes that short sales are allowed. While the introduction just concerns an opportunity set, part (a) asks for an equlibrium relationship, which (in the standard CAPM) is based on some additional assumptions:

- The shares to be considered, comprise all available investment opportunities in the economy, except for the riskless opportunity.
- Each agent is just interested in the mean and the variance of his/her wealth at the end of the period, and all have the same beliefs about means, variances, and covariances.
- There are no transaction costs.

The derivation of the results will also neglect taxation. Let R_j be defined as 1+ the rate of return on share j, and let R_0 be 1+ the riskless interest rate. The equation to derive is

$$E(R_j) = R_0 + \beta_j[E(R_m) - R_0], \text{ der } \beta_j \equiv \frac{\operatorname{cov}(R_j, R_m)}{\operatorname{var}(R_m)}. \tag{1}$$

(This formula looks the same if R_j had been defined as the *rate* of return.) The derivation in the book (Copeland & Weston, 3rd ed., p. 181 and 195–197) relies on combinations of R_0 and risky portfolios giving straight lines in the diagram. This is justified as follows:

Let $R_p = aR_j + (1-a)R_0$, where R_j is the return on a share or a portfolio of risky securities. Then

$$E(R_p) = aE(R_j) + (1-a)R_0,$$

and

$$\mathrm{var}(R_p) = a^2 \,\mathrm{var}(R_j) = a^2\sigma_j^2.$$

When $a > 0$, this means

$$\frac{d\sigma_p}{da} = \sigma_j$$

(where $\sigma_p \equiv \sqrt{\mathrm{var}(R_p)}$), while $a < 0$ implies

$$\frac{d\sigma_p}{da} = -\sigma_j,$$

When we vary a, we obtain a curve in the (σ, μ) diagram with a slope which in absolute value is equal to

$$\frac{dE(R_p)}{d\sigma_p} = \frac{dE(R_p)/da}{d\sigma_p/da} = \frac{E(R_p) - R_0}{\sigma_j}.$$

Since this is a constant, independent of a, the curve must be a straight line, or more precisely, one ray as long as $a > 0$, another ray when $a < 0$. (A diagram would be welcome at this point.)

Furthermore, those combinations which are efficient, must be on that ray which is tangent to the hyperbola. This is that ray through $(0, R_0)$ which is highest in the diagram. Any points to the north west (the preferred direction) of this are unattainable.

Call the tangency point (σ_m, μ_m), where $\mu_m \equiv E(R_m)$. The ray has the formula

$$E(R_p) = R_0 + \frac{\mu_m - R_0}{\sigma_m}\sigma_p. \tag{2}$$

All agents will demand the same combination of risky assets, namely that which gives rise to the tangency point. The weights in their portfolios of risky assets are the same for all agents, and this will thus also be the weights in the total portfolio of risky assets in the economy, the market portfolio.

Consider a combination k of any share (or portfolio) j which is available in the market, and one portfolio φ on the efficient frontier,

$$R_k = aR_j + (1-a)R_\varphi.$$

It can be shown that combinations of two risky portfolios or shares in general will give rise to a (new, "little") hyperbola when the fraction a varies. When one of the portfolios is on the frontier (the "big" hyperbola), the "little" hyperbola cannot cross the big one, by definition of the efficient frontier. There is a tangency between the two in $(\sigma_\varphi, \mu_\varphi)$.

The expected rate of return and the standard deviation of the combinations have the formulae

$$E(R_k) = aE(R_j) + (1-a)E(R_\varphi)$$

and

$$\sigma_k = \sqrt{a^2\sigma_j^2 + 2a(1-a)\sigma_{j\varphi} + (1-a)^2\sigma_\varphi^2},$$

where $\sigma_{j\varphi} \equiv \mathrm{cov}(R_j, R_\varphi)$.

We can now find the slope of the hyperbola in the point where it is tangent to the efficient frontier,

$$\left.\frac{dE(R_k)/da}{d\sigma_k/da}\right|_{a=0} = \frac{E(R_j) - E(R_\varphi)}{(\sigma_{j\varphi} - \sigma_\varphi^2)/\sigma_\varphi}.$$

This must also be valid when $R_\varphi \equiv R_m$, where we have another expression for the slope,

$$\frac{E(R_m) - R_0}{\sigma_m}.$$

From this, (1) follows.

(b) (Weight 10/100.) This is straight forward: Let S_{jt} be the value of share j at time t, where $t = 0, 1$ are the "beginning of the period" and the "end of the period," respectively. Then $R_j \equiv S_{j1}/S_{j0}$, and we may multiply both sides of (1) with S_{j0}/R_0. (Observe that $S_{j0}\,\text{cov}(R_j, R_m) = \text{cov}(S_{j1}, R_m)$.) Then we subtract the covariance term on both sides, and are left with

$$S_{j0} = \frac{1}{R_0}\left\{E(S_{j1}) - \frac{\text{cov}(S_{j1}, R_m)}{\text{var}(R_m)}[E(R_m) - R_0]\right\}, \tag{3}$$

which is the equation we are asked for.

(c) (Weight 20/100.) Instead of S_{j1} we now get ZS_{j1}, where $E(Z) = 1$, and Z is stochastically independent of everything else, in particular of the vector (S_{j1}, R_m). We shall consider both $E(R_m)$ and $\text{cov}(S_{j1}, R_m)$ as unchanged by the event. This is a good approximation when share j is a small fraction of R_m. We find expected end-of-period value $E(ZS_{j1}) = E(Z)E(S_{j1}) = E(S_{j1})$, while the covariance between the end-of-period value and R_m is $\text{cov}(ZS_{j1}, R_m) = E(ZS_{j1}R_m) - E(ZS_{j1})E(R_m) = E(Z)E(S_{j1}R_m) - E(Z)E(S_{j1})E(R_m) = \text{cov}(S_{j1}, R_m)$. Equation (3) shows that the value at $t = 0$ is unchanged.

The expected return on the share is $E(ZR_j) = E(R_j)$ as before.

Assume $\text{var}(Z) > 0$. The variance of the return will become $\text{var}(ZR_j) = E[(ZR_j)^2] - [E(ZR_j)]^2 = E[Z^2R_j^2] - [E(Z)E(R_j)]^2 = E(Z^2)E(R_j^2) - [E(Z)]^2[E(R_j)]^2$ which is strictly greater than $[E(Z)]^2E(R_j^2) - [E(Z)]^2[E(R_j)]^2 = \text{var}(R_j)$ since $\text{var}(Z) > 0 \Leftrightarrow E(Z^2) > [E(Z)]^2$. In other words, the variance of the return increases.

(d) (Weight 10/100.) Part (c) showed that share j moves to the right in a (σ, μ) diagram when its future value becomes more uncertain. If share j is a substantial part of the economy, and thus has a large weight in R_m, then the hyperbola mentioned in the introduction must change as a consequence of the event mentioned in part (c). R_m will change, and the discussion based on (3) will be more complicated.