1

Let X be a strong Markov process whose generator $A = A_X$ can be written as

$$A\, f(x) = \sum_i \lambda_i \left[f(x+z_i) - f(x) - f'(x)\cdot z_i \right],$$

$\{\lambda_i, z_i\}$ given with $\lambda_i \geq 0$.

(a) Show that if $f \in C^1$ is concave, then $Af \leq 0$. You can assume X one-dimensional.

(<u>Hint:</u> $f(x+z_i) - f(x) - f'(x)\cdot z_i$

$$= \int_x^{x+z_i} \underbrace{\left(f'(y) - f'(x)\right)}_{\substack{\text{sign if } z_i \geq 0? \\ \text{sign if } z_i < 0?}} \cdot dy$$

)

(b) Let g be C^1, and consider the problem

$$\sup E\left[g(X(\tau)) \right]$$

where "sup" is taken over

(case I) all stopping times

(case II) all bounded stopping times.

(i) Give an interpretation of the difference between case I and case II. In which case should the criterion be modified - and how - in order to be rigorous?

(ii) In each case I and II: when can we conclude that the smallest concave majorant of g is <u>superoptimal</u>?

1 (b) cont'd:

(iii) What would happen if X were replaced by a Brownian motion? Would this increase (weakly, i.e. "$\geq$") the value function or decrease it, or could either happen?

→ Give an example where the value function would remain the same.

(c) Consider the problem

$$\sup_\tau E\left[e^{-\delta\tau} g(X(\tau)) \right].$$

(i) Test the function

$$F(x) = a - K\left(e^{q(x-c)} + e^{-q(x-c)} \right)$$

for superoptimality.

(ii) Use your reasoning from (b) (iii) to try to guess what will happen if X is replaced by a Brownian motion in this problem.

2)

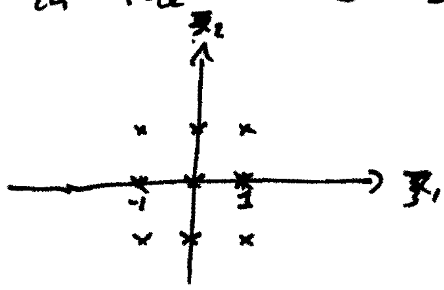
Let a 2-dimensional system be given by

$$X_{t+1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} X_t + w_t$$

where we first assume we observe

$$z_t = X_t + v_t .$$

We assume that w_t and v_t both take values in the 3×3 set



i.e. each coordinate is -1, 0 or 1.

Each w_t and each v_t is drawn independently of everything else, with the following probabilities

ξ_2 \ ξ_1	-1	0	1
-1	p	q	p
0	q	r	q
1	p	q	p

So that

r = 1 - 4p -4q
(corrected)

a) Write down the eq's for the Kalman filter

Here, I have not had time to do any calculations myself. Sorry, we might be in for a surprise.

(b) Assuming $X_0 \cancel{(0)} = \begin{pmatrix}0\\0\end{pmatrix}$, solve for $\cancel{X_1}$, $\hat{X}_1^- \cancel{(1)}$, $\hat{X}_2^-$ and $\hat{X}_3^-$, and $\hat{X}_1$ and $\hat{X}_2$

(c) Is it possible to give a general form?

(d) Assume we only observe the <u>first coordinate of Z</u>.

Repeat (a) - (c) for this case.