University of Oslo / Department of Economics / NCF

ECON5160: The exam solved

Problem 1

a) The transition matrix for Y_n will have $\mathbf{I}_2$ in the upper–left corner. Hence both 1 and 2 will be absorbing states for Y, and so Y cannot have a limiting distribution. Therefore, neither can X.

b) (i) By communication and nonzeroness of $\mathbf{V}$, states $x > k + 2$ are transient. $\mathbf{S}$ is symmetric, hence doubly stochastic, and so

$$\boldsymbol{\pi} = (\overbrace{\frac{1}{k}, \ldots, \frac{1}{k}}^{k \text{ components}}, 0, 0, \ldots).$$

(ii) Here $\mathbf{W}$ is symmetric too. The first k components must be equal, and the last two must be equal: $\boldsymbol{\pi} = (a/k, ..., a/k, b/2, b/2)$, so that $a + b = 1$. Hence, for all $a \in [0, 1]$,

$$\boldsymbol{\pi} = (\overbrace{\frac{a}{k}, \ldots, \frac{a}{k}}^{k \text{ components}}, \frac{1-a}{2}, \frac{1-a}{2},)$$

will be a stationary distribution.
(***Note:*** The problem set might have been a bit unclear as to whether there could be more than one.)

(iii) By non-uniqueness, case (ii) cannot be limiting. Case (i) need not be limiting (example: $\mathbf{S} = \mathbf{J}$) but can be (example: $\mathbf{S} = \left[\begin{smallmatrix} .5 & .5 \\ .5 & .5 \end{smallmatrix}\right]$).

c) (i) States 1 and 2 do not communicate with states > 2, so then $X_\nu < 2$ and no transition. $\beta(1) = \beta(2) = \delta(1) = \delta(2) = 0$.
States $3, \ldots, k + 2$ communicate with each other, but not with other states. The long-run time average is uniform over these states, so the only zero intensities here, are $\delta(3) = \beta(k + 2) = 0$.
States $r > k + 2$ are transient, so $\beta(r) = 0$. We will have $\delta(r) = 0$ if and only if $\mathbf{V} = \mathbf{0}$, since then we are absorbed in class $\{1, 2\}$.

(ii) The long-run time average of the class $\{3, \ldots, k + 2\}$ is uniform over k elements, and for the minimal and maximal elements, $k-1$ of these are larger (resp. smaller). So $\Pr[X_\nu > 3|X(0) = 3] = (k-1)/k = \Pr[X_\nu < k+2|X(0) = k+2]$. The common value is then $\beta(3) = \delta(k + 2) = \lambda \cdot (1 - 1/k)$.

Problem 2

a) The probability u_k starting from k satisfies $u_k = pu_{k+1} + (1-p)u_{k-1}$ and with $u_0 = 0$, the solutions are $(\frac{1-p}{p})^k$ and 1. The former solves the problem whenever < 1, i.e. $p < 1/2$. (***Note:*** This is the «gambler's ruin» problem in TK ch. III p. 142 ff. The limit transition is treated lightly in the course and is not expected – indeed, a reference to TK ch. III formula (5.14) p. 144, which states the result, may be expected.)
For the expectation, we have $\mathsf{E}[\tau] = \mathsf{E}[T_1] = 1 + p\mathsf{E}[T_2] + (1-p) \cdot 0 = 1 + 2p\mathsf{E}[\tau]$ whose only solutions are $\mathsf{E}[\tau] = (1-2p)^{-1}$ and $\mathsf{E}[\tau] = \infty$, the former being the solution if and only if positive, i.e. $p < 1/2$.

b) We have $\mathsf{E}[W(t)] = \lambda_I t - \lambda_C t = (\lambda_I - \lambda_C)t$, and by independence of I and C, $\mathsf{var}\, W(t) = \mathsf{var}\, I(t) + \mathsf{var}\, C(t) = (\lambda_I + \lambda_C)t$. As for the distribution, we have

$$\begin{aligned}\sum_{c=0}^{\infty} \mathsf{Pr}[I = w + c]\mathsf{Pr}[C = c] &= \sum_{c=\max\{0,-w\}}^{\infty} \frac{(\lambda_I t)^{w+c} e^{-\lambda_I t}}{(w+c)!} \frac{(\lambda_C t)^c e^{-\lambda_C t}}{c!} \\ &= \sum_{c=\max\{0,-w\}}^{\infty} \frac{\lambda_I^{w+c} \lambda_C^c}{c!(w+c)!} t^{w+2c} e^{-(\lambda_I + \lambda_C)t}.\end{aligned}$$

c) By the Markov property, $\mathsf{E}[V_w] = w\mathsf{E}[V_1]$. The wealth at the nth jump time is distributed as $1 - X_n$ in part a), with $p = \lambda_C/(\lambda_I + \lambda_C)$ being the probability for decrease in X, and we are searching n to be the first jump for which $X_n = 0$ – i.e. τ in part a). Now

$$V_1 = (\text{number of jumps}) \times (\text{mean time between jumps})$$

and by independence between those two factors,

$$\begin{aligned}\mathsf{E}[V_1] &= (1-2p)^{-1} \cdot \frac{1}{\lambda_I + \lambda_C} \qquad \text{(with } p \text{ as above)} \\ &= \frac{1}{(1 - \frac{2}{\lambda_C + \lambda_I})(\lambda_I + \lambda_C)} \\ &= \frac{1}{\lambda_I - \lambda_C}\end{aligned}$$

d) $Y(t) = W(t) - (\lambda_I - \lambda_C)t$ is a martingale, and the optional sampling theorem gives $0 = \mathsf{E}[Y(V_1)] = W(V_1) - (\lambda_I - \lambda_C)\mathsf{E}[V_1]$ if $\mathsf{E}[V_1] < \infty$, in which case $\mathsf{E}[V_1] = (\lambda_I - \lambda_C)^{-1}$.

Note: The above solution has – on a few occations – a slight unrigorousness which should be accepted from the students too. E.g. in a), we can infer from $\mathsf{E}[\tau] = 2p\mathsf{E}[\tau]$ that $\mathsf{E}[\tau] = (1-2p)^{-1}$ if $\mathsf{E}[\tau] \in (0, \infty)$, but we have not really made any argument that $\mathsf{E}[\tau] < \infty$ if $(1-2p)^{-1} \in (0, \infty)$.

Problem 3

a) (i) $\mathsf{A}_X f(x) = -\frac{k}{2} x f'(x) + \frac{1}{2}(\sigma(k))^2 x f''(x)$.

(ii) $\mathrm{d}Y(t) = \mathsf{A}_X f(X(t))\,\mathrm{d}t +$ «$\mathrm{d}B$-term», so we solve $\mathsf{A}_X f(x) = 0$, i.e. $\frac{k}{(\sigma(k))^2} = f''(x)/f'(x)$, with solution $C_1 \exp(\frac{k}{(\sigma(k))^2} x) + C_2$.
Integrability of $Y(t) = C_1 \exp(\frac{k}{(\sigma(k))^2} X(t)) + C_2$: with $k < 0$, $|Y| \le C_1 + C_2$, since $X \ge 0$.

b) (i) The HJB equation is

$$-\delta v(x) + \sup_k\{-\frac{k}{2} x v'(x) + \frac{1}{2}(\sigma(k))^2 x v''(x) + x\sqrt{k}\}$$

(***Note:*** both «v» and «V» – corresponding to sufficiency and necessity, respectively – should be accepted.)

(ii) (Using necessity with $v = V$:) The left hand side is concave in k, with stationary point when $V'(x) = k^{-1/2}$, so that the optimal $k^* = \max\{k_0, (V')^{-2}\}$ – which means

$$k^* = \begin{cases} (V'(x))^{-2} & \text{for } k_0 = 0 \\ \max\{3, (V'(x))^{-2}\} & \text{for } k_0 = 3 \end{cases}$$

c) (i) Testing $v(x) = \gamma x$, we get $0 = -\delta\gamma x + \sup_k\{-\frac{k}{2} x\gamma + x\sqrt{k}\}$ with $k^* = \max\{k_0, \gamma^{-2}\}$. Inserting, and canceling x, yields $0 = -\delta\gamma - \frac{\gamma}{2}\max\{k_0, \gamma^{-2}\} + \sqrt{\max\{k_0, \gamma^{-2}\}}\}$. The right hand side is continuous in γ, and $\gamma \to 0^+$ yields ∞ while $\gamma \to \infty$ yields $\sqrt{k_0} - \infty$. The intermediate value theorem grants a zero for some $\gamma > 0$.

(ii) This is sufficient to solve (C), and (C) has only one solution.

Problem 4

We shift the probabilities by considering the process $\mathrm{d}Y(t) = Y(t)(r\,\mathrm{d}t + \sqrt{2r}\,\mathrm{d}B(t))$ – solved by $Y(t) = xe^{\sqrt{2r}\cdot B(t)}$ – instead, and take expectation. Then discount by e^{-rT}:

a) $\mathsf{E}[1_{Y(T)>x}] = \mathsf{Pr}[e^{\sqrt{2r}\cdot B(T)} > 1] = \mathsf{Pr}[B(T) > 0] = \frac{1}{2}$, so the price is $\frac{1}{2}e^{-rT}$.

b) $\mathsf{E}[1_{\max_{t\in[0,T]} Y(t)>M}] = \mathsf{Pr}[\max_{t\in[0,T]} B(t) > \frac{1}{\sqrt{2r}} \ln\frac{M}{x}] = 2\mathsf{Pr}[B(T) > \frac{1}{\sqrt{2r}}\ln\frac{M}{x}]$ by the reflection principle, so that the price is $2e^{-rT}(1 - \Phi(\frac{1}{\sqrt{2rT}}\ln\frac{M}{x}))$.

Problem 5

a) The F distribution has density $f(x) = C\frac{x^{-1+n/2}}{(m+nx)^{(m+n)/2}}$. The limit in interest is $\lim \frac{\overline{F}(x)}{x^{-\alpha}} = \lim \frac{f(x)}{\alpha x^{-\alpha-1}}$. In order to have $\left(x^{2\alpha+n}/(m+nx)^{m+n}\right)^{1/2}$ converge to something positive, put $2\alpha = m$ and obtain $\lim(\frac{m}{x} + n)^{(m+n)/2} \in (0,\infty)$. So the limiting distribution is Fréchet$(m/2)$, i.e. GEV with $\xi = 2/m$.

b) (***Note:*** This result is given in handouts, so a reference to Embrechts et al. p. 165 (more precisely Theorem 3.4.13 (d)) will be sufficient, as long as one has identified the parameters appropriately. The following is a deduction which closely follows the proof in Embrechts et al. p. 166 (also handed out):)

$$\begin{aligned}\mathsf{Pr}[M \le m] &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (F_Y(m))^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda F_Y(m))^n}{n!} \\ &= e^{-\lambda + \lambda F_Y(m)} = e^{-\lambda(1+cm)^{-d}}\end{aligned}$$

so that for $a > 0$,

$$\begin{aligned}\ln \mathsf{Pr}[aM + b \le x] &= -\lambda(1 + c\frac{x-b}{a})^{-d} \\ &= -(1 + \frac{1}{d}\frac{x + \lambda^{-d} \cdot a/c - a/c - b}{a/cd})^{-d}\end{aligned}$$

So $\xi = 1/d$, $a = cd$ (> 0) and $b = (\lambda^{-d} - 1)d$ will do the job.

c) (i) $aM + b$ is distributed GEV$(1/d)$ with $d > 0$, and therefore by Embrechts et al. Theorem 3.4.13 (b) p. 165 (handed out), we have $R_1 < \infty$ if and only if $d > 1$, and $R_2 < \infty$ if $d > 2$. $R_3 < \infty$ always.

(ii) Since $\mathsf{E}M = \infty$ for these parameters, the reinsurer will – no matter what premium, i.e. no matter what r_3 – lose in the long run. However, with r_3 large enough, the accumulated surplus before bankruptcy will with high probability be large. Shareholders can therefore want to do this business and extract dividends which by limited liability will not be reclaimed later.

(Not attached in public version: scan from Embrechts et al.)