AST2000 Lecture Notes

Part 2A
The special theory of relativity: Basic principles

Questions to ponder before the lecture

1. You have already used the Lorentz transformations. Do you know where they come from? Which basic principles/formulas would you use if you wanted to deduce the Lorentz transformation?
2. You may have heard about the twin paradox: one of the twins is launched into space, travels with a speed close to the speed of light and returns to the Earth. After returning, which of the twins are older?
3. You have already learned how time goes slower when travelling close to the speed of light. So in principle, it is not a paradox that the two twins have different ages after the space trip. What is then the paradox of the twin paradox?

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1 Simultaneity

We all know that 'velocity' is a relative term. When you specify velocity you need to specify velocity *with respect to something*. If you sit in your car which is not moving (with respect to the ground) you say that your velocity is zero with respect to the ground. But with respect to the Sun you are moving at a speed of 30 km/s. From the point of view of an observer passing you in his car with a velocity of 100 km/h with respect to the ground, your speed is -100 km/h (see Figure 1). Even though you are not moving with respect to the ground, you are moving backwards at a speed of 100 km/h with respect to the passing car.

In the following we will use the expression *'frame of reference'* to denote a system of observers having a common velocity. All observers in the same frame of reference have zero velocity with respect to each other. An observer always has velocity zero with respect to his own frame of reference.

An observer on the ground measures the velocity of the passing car to be 100 km/h with respect to his frame of reference. On the other hand, the driver of the car measures the velocity of the ground to be moving at -100 km/h with respect to his frame of reference. We will also use the term *'rest frame'* to denote the frame of reference in which a given object has zero velocity. In our example we might say: In the rest frame of the passing car, the ground is moving backwards with 100 km/h.

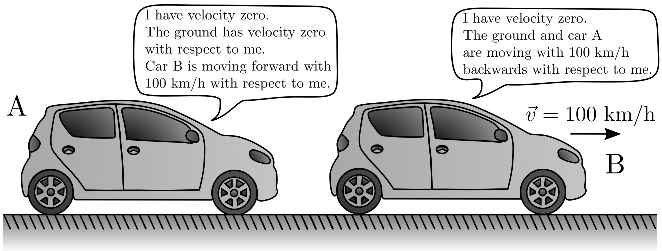


Figure 1: Velocities are relative.

You are observing a truck coming towards you with a speed of $v_{\rm truck}^{\rm ground} = -50$ km/h with respect to the ground (see Figure 2, velocities are defined to be positive to the right in the figure). From your frame of reference, which is the same frame of reference as the ground, the speed of the truck is $|v_{\rm truck}^{\rm ground}| = 50$ km/h towards you. Now you start driving your car in the direction of the truck with a speed of $v_{\rm car}^{\rm ground} = +50$ km/h with respect to the ground (see again Figure 2). From your frame of reference you observe the ground to be moving backwards with a velocity of $v_{\rm ground}^{\rm car} = -50$ km/h. Again, from your frame of reference you now observe the velocity of the approaching truck to be $v_{\rm truck}^{\rm car} = v_{\rm truck}^{\rm ground} - v_{\rm car}^{\rm ground} = (-50 \text{ km/h}) - (50 \text{ km/h}) = -100$ km/h (whereas from the frame of reference of an observer on the ground, the truck still has $v_{\rm truck}^{\rm ground} = -50$ km/h). Now you make a turn so that you drive in the opposite direction: Now your velocity is -50 km/h with respect to the ground, but now you are driving in the same direction as the truck. You are now moving in the same direction as the truck with exactly the same speed with respect to the ground. From your frame of reference (which is now the same frame of reference as the truck) the truck is not moving.

So far, so good. This was just stating some obvious facts from everyday life in a difficult way. Now, replace the truck with a beam of light (a photon) and the car with the Earth. The situation is depicted in Figure 3. You observe the speed of light from a distant star at two instants: One at the 1st of January, another at the 1st of July. In January you are moving away from the photons approaching you from the distant star. In July you are moving towards the photons arriving from the star. If the speed of light with respect to the distant star is c, then in January you expect to measure the speed of the light beam from the star to be $c - v$ where $v = 30$ km/h is the speed of the Earth with respect to the same star (we assume that the star does not move with respect to the Sun, so this is also the orbital speed of the Earth). In July you expect to measure the speed of light from the star to be $c + v$, just as for the truck in the example above: The speed of the light beam seen from your frame of reference is supposed to be different depending on whether you move towards it or away from it.

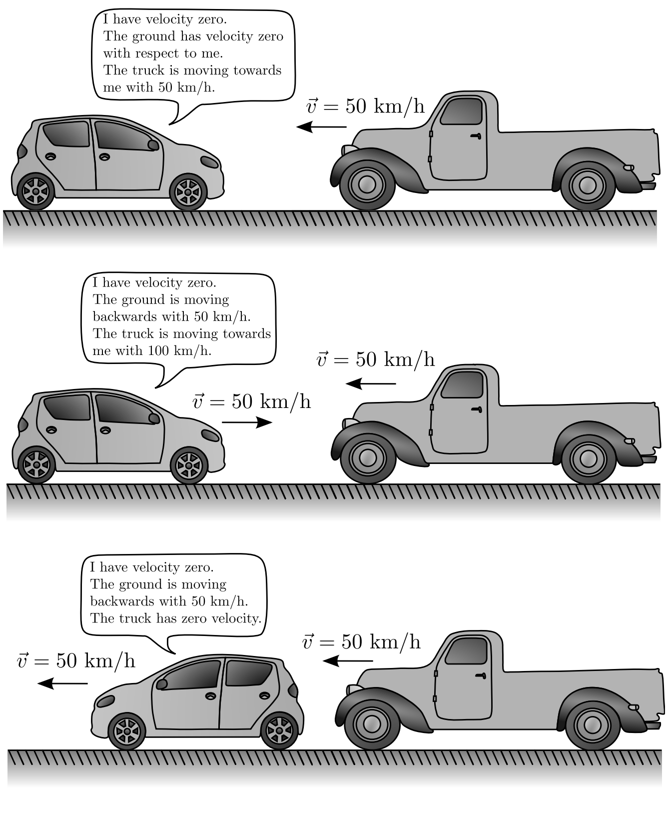


Figure 2: The velocity of the truck seen from the car depends on the velocity of the car.

In 1887 Michelson and Morley performed exactly this experiment which is now famous as the 'Michelson-Morley experiment'. The result however, was highly surprising: They measured exactly the same speed of light in both cases. The speed of light seemed to be the same independently of the frame of reference in which it is measured. This has some quite absurd consequences: Imagine that you see the truck driving at the speed of light (or very close to the speed of light, no material particle can ever travel at the speed of light). You are accelerating your car, trying to pass the truck. But no matter at which speed you drive, you see the truck moving with the speed of light with respect to your frame. Even when you reach half the speed of light, you still see the truck moving with velocity c. But how is this possible? An observer at rest with respect to the ground measures the truck moving with the speed of light as well, not with the velocity $c + c/2 = 3c/2$ as you would expect given that it moves with velocity c with respect to something moving with velocity $c/2$.

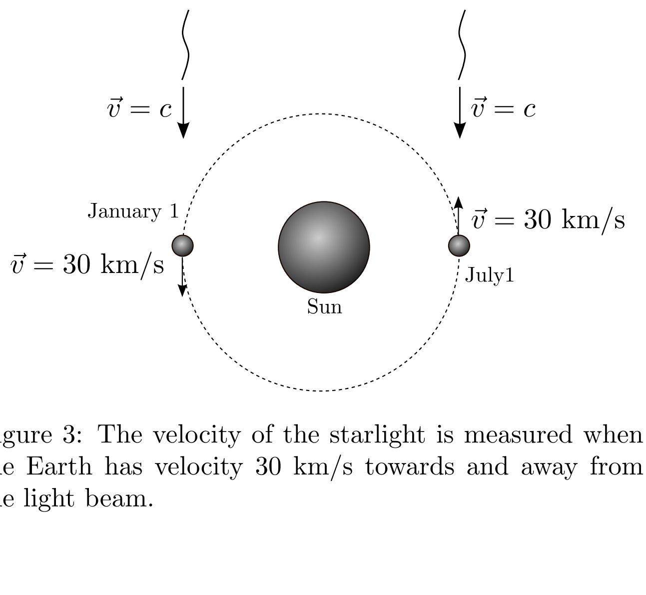


Figure 3: The velocity of the starlight is measured when the Earth has velocity 30 km/s towards and away from the light beam.

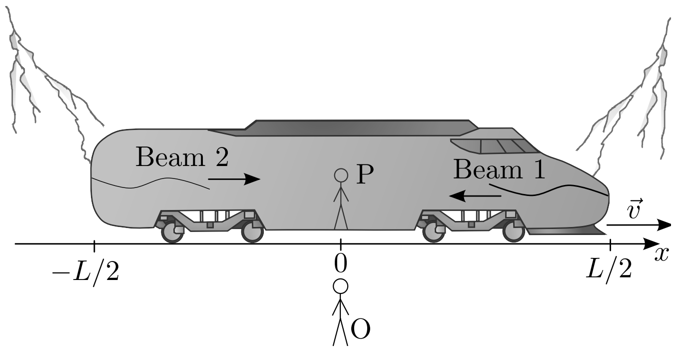


Figure 4: Event A: Lightning strikes the front part of the train. Event B: Lightning strikes the rear part of the train. These two events are observed by observer O on the ground and observer P in the train. The train has length L.

This was one of the first signs showing that something was wrong with classical physics. The fact that the speed of light seemed to be constant in all frames of reference led to several contradictions. We have already seen one example of such a contradiction. We will now look at another one which might shed some light on the underlying reason for these contradictions.

In Figure 4 we show the situation. Observer O is standing on the ground (at rest with respect to the ground), observer P is standing in the middle of a train of length L moving with velocity v with respect to the ground. Observer O sees two lightnings striking the front and the rear of the train simultaneously. We call the two events A and B (An event is a point in space and time, a point with a space and time coordinate): Event A is the lightning striking the front, event B is the lightning striking the rear. Events A and B are simultaneous.

The light from these two lightnings start traveling from the front and back end of the train towards observer P. The beam approaching observer P from the front is called beam 1 and the beam approaching from the rear is called beam 2.

Both observers had synchronized their clocks to $t = 0$ at the instant when the lightnings strike the train. Both observers have also defined their own coordinate systems x (observer on the ground) and x' (observer in the train) which is such that the position of observer P is at $x = x' = 0$ in both coordinate systems at the instant $t = 0$ when the lightenings strike. Thus the lightnings hit the train at the points $x = x' = L/2$ and $x = x' = -L/2$ as seen from both observers. We will now look how each of these observers experience these events:

From the point of view of observer O standing on the ground:

The frame of reference of observer O on the ground is often referred to as the *laboratory frame* . It is the frame of reference which we consider to be at rest. At what time $t = t_C$ does observer P see beam 1 (we call this event C)? To answer this question, we need to have an expression for the x-coordinate of observer P and the x-coordinate of beam 1 at a given time t. Observer P moves with constant velocity v so his position at time t is $x_P = vt$. Beam 1 moves in the negative x-direction with the speed of light c starting from $x_1 = L/2$ at $t = 0$. The expression thus becomes $x_1 = L/2 - ct$. Observer P sees beam 1 when $x_1 = x_P$ at time t_C. Equating these two expressions, we find

$$t_C = \frac{L/2}{c+v}. \tag{1}$$

At what time $t = t_D$ does observer P see beam 2 (we call this event D)? Following exactly the same line of thought as above, we find

$$t_D = \frac{L/2}{c-v}. \tag{2}$$

So according to observer O in the laboratory frame, $t_C < t_D$ and observer P should see the light beam from the lightning in front before the light from the back. This sounds reasonable: Observer P is moving towards beam 1 and away from beam 2 and should therefore see beam 1 first.

From the point of view of observer P standing in the train:

At what time $t = t_C$ does observer P see beam 1? We have just agreed on the fact that the speed of light is independent of the frame of reference. The result is that the speed of light is c also for the observer in the train. Seen from the frame of reference of observer P, observer P himself is at rest and the ground is moving backwards with speed v. Thus from this frame of reference, observer P is always standing at the origin $x'_P = 0$ (the coordinate system x' moves with observer P). The expression for x'_1 is the same as seen from observer

O:$x'_1 = L/2 - ct$ (convince yourself that this is the case!). Again we need to set $x'_1 = x'_P$ giving

$$t_C = \frac{L/2}{c}$$

At what time $t = t_D$ does observer P see beam 2? Again we follow the same procedure and obtain

$$t_D = \frac{L/2}{c}$$

As calculated from the frame of reference of observer P, the two beams hit observer P at exactly the same time.

So not only are the exact times t_C and t_D different as calculated from the two frames of reference, but there is also an even stronger contradiction: Observer P should be hit by the two beams simultaneously as calculated from the frame of reference of observer P himself, but as calculated from the laboratory frame, beam 1 hits observer P before beam 2. What does really happen? Do the beams hit observer P simultaneously or not? Well, let's ask observer P himself:

So observer P, two lightnings struck your train simultaneously at the front and rear end. Did you see these two lightnings simultaneously or did you see one flash before the other?
Observer P: Sorry? I think you are not well informed. The two lightnings did not happen simultaneously. There was one lightning which struck the front part and then shortly afterwards there was another one striking the rear. So clearly I saw the flash in the front first.
Observer O: No, no, listen, the lightnings did strike the train simultaneously, there was no doubt about that. But you were moving in the direction of beam 1 and therefore it appeared to you that the front was hit by the lightning first.
Observer P: So you didn't watch very carefully I see. It is impossible that the two lightnings struck at the same time. Look, I was standing exactly in the middle of the train. The speed of light is always the same, no matter from which direction it arrives. Beam 1 and beam 2 had to travel exactly the same distance $L/2$ with exactly the same speed c. If the beams were emitted simultaneously I MUST have seen the two flashes at the same time. But I didn't....beam 1 arrived before beam2, and so event A must have happened before event B

So beam 1 did indeed hit observer P before beam 2. And indeed, observer P has got a point: From observer P the two lightnings could not have occurred at the same time. Asking observer O one more time he says that he is absolutely certain that the two lightnings struck simultaneously. Who is right?

We have arrived at one of the main conclusions that Einstein reached when he was discovering the theory of relativity: *simultaneity is relative.* If two events happen at the same time or not depends on who you ask. It depends on your frame of reference. In the example above, the two lightnings were simultaneous for the observer at rest on the ground, but not for the observer moving with velocity v. This has nothing to do with the movement of the light beams, it is simply time itself which is different as seen from two different frames of reference. Simultaneity is a relative term in exactly the same way as velocity is: When you say that two events are simultaneous you need to specify that they are simultaneous with respect to some frame of reference.

In order to arrive at the conclusion of the relativity of simultaneity, Einstein excluded an alternative: Couldn't it be that the laws of physics are different in different frames of reference? If the laws of physics in the train were different from those in the laboratory frame, then simultaneity could still be absolute. The problem then is that we need to ask the question 'Physics is different in frames which move with respect to *which* frame of reference?'. In order to ask this question, velocity would need to be absolute. If velocity is relative, then we can just exchange the roles: The observer in the train is at rest and the observer on the ground is moving. Then we would need to change the laws of physics for the observer on the ground. This would lead to contradictions. In order to arrive at the theory of relativity, Einstein postulated the *Principle of Relativity* . The principle of relativity states that all laws of physics, both the mathematical form of these laws as well as the physical constants, are the same in all *free float frames.* In the lectures on general relativity we will come back to a more precise definition of the free float frame. For the moment we will take a free float frame to be a frame which is not accelerated, i.e. a frame in which we do not experience fictive forces. You can deduce the laws of physics in one free float frame and apply these in any other free float frame. Imagine two spaceships, one is moving with the velocity $v = 1/2c$ with respect to the other. If you close all windows in these spaceship there is no way, by performing experiments inside these spaceships, that you can tell which is which. All free float frames are equivalent, there is no way to tell which one is at rest and which one is moving. Each observer in a free float frame can define himself to be at rest.

2 Invariance of the spacetime interval

We have seen that two events which are simultaneous in one frame of reference are not simultaneous in another frame. We may conclude that time itself is relative. In the same way as we needed two coordinate systems x and x' to specify the position in space relative to two different frames, we need two time coordinates t and t' to specify the time of an event as seen from two different frames. We are used to think of time as a quantity which has the same value for all observers but we now realize that each frame of reference has its own measure of time. Clocks are not running at the same pace in all frames of reference. Observers which are moving with respect to each other will measure different time intervals between the same events. Time is not absolute and for this reason simultaneity is not absolute.

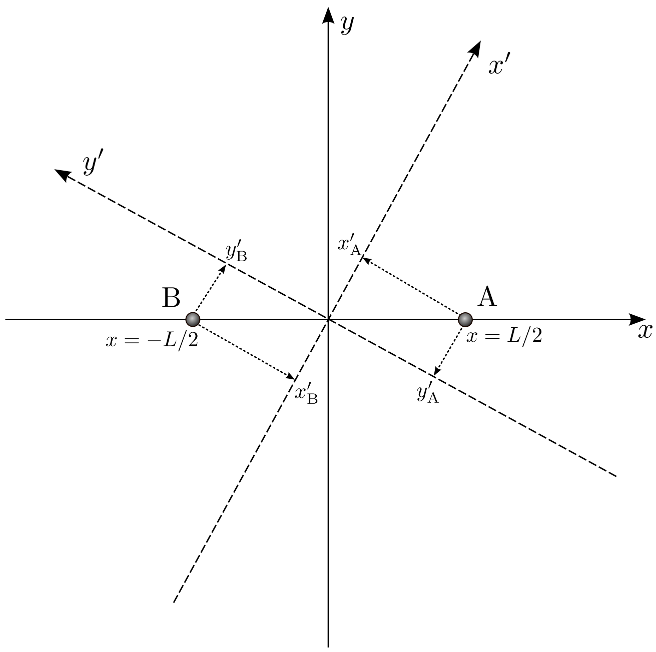


Figure 5: The position of two points A and B measured in two different coordinate systems rotated with respect to each other.

Look at Figure 5. It shows two points A and B and two coordinate systems (x, y) and (x', y') rotated with respect to each other. The two points A and B are situated at a distance $\Delta x_{AB} = L$ and at the same y-coordinate $\Delta y_{AB} = 0$ in the (x, y) system. In the rotated (x', y') system however, there is a non-zero difference in the y-coordinate, $\Delta y_{AB} \neq 0$. Now, replace y with t. Do you see the analogy with the example of the train above?

If we replace y with t and y' with t', then the two points A and B are the events A and B in *spacetime*. Our diagram is now a *spacetime diagram* showing the position of events in space x and time t, rather than a coordinate system showing the position of a point in space (x, y). Consider the two coordinate systems (x, t) and (x', t') as measurements in two different frames of reference, the lab frame and the frame of observer P. We see that in the (x, t) system, the two events are simultaneous $\Delta t_{AB} = 0$ whereas in the (x', t') system, the events take place at two different points in time.

We are now entering deep into the heart of the special theory of relativity: We need to consider time as the *fourth dimension*. And moreover, we need to treat this fourth dimension similar (but not identical) to the three spatial dimensions. That is, we need to talk about distances in space and distances in time. But, you might object, we measure distances in space in meters and time intervals in seconds. Can they really be similar? Yes they can, and you will soon get rid of the bad habit of measuring space and time in different units. From now on you will either measure both space *and* time in meters, or both time *and* space in seconds. By the time you have finished this course you will, without thinking about it, ask the lecturer how many meters the exam lasts or complain to your friends about how small your room in the dormitory is, giving them the size in square seconds.

How do you convert from meters to seconds and vice versa? The conversion factor is given by the universal factor c, the speed of light. If you have a time interval measured in seconds, multiply it by c and you have the time interval in meters. If you have a distance in space measured in meters, divide it by c and you obtain the distance measured in seconds:

$$x = ct, \quad t = x/c.$$

From now on we will drop the factor c and suppose that distances in space and time are measured in the same units. When you put numbers in your equations you need to take care that you always add quantities with the same units, if you need to add two quantities with different units, the conversion factor is always a power of c.

Measuring time in meters might seem strange, but physically you can think about it this way: Since the conversion factor is the speed of light, a time interval measured in meters is simply the distance that light travels in the given time interval. If the time interval between two events is 2 meters, it means that the time interval between these events equals the time it takes for light to travel 2 meters. We might say that the time interval between these events is 2 meters of light travel time. Similarly for measuring distances in seconds: If the spatial distance between two events is 10 seconds, it means that the distance equals the distance that light travels in 10 seconds. The distance is 10 light seconds. Actually you are already accustomed to measure distances in time units: You say that a star is 4 light years away, meaning that the distance equals the distance that light travels in four years. Note also one more effect of measuring space and time in the same units: Velocities will be dimensionless. Velocity is simply distance divided by time, if both are measured in meters, velocity becomes dimensionless. We can write this as $v_{\text{dimensionless}} = dx/(cdt) = v/c$ (to convert dt to units of length we need to multiply it by c, thus cdt). If the velocity $v = dx/dt = c$ is just the speed of light, we get $v_{\text{dimensionless}} = 1$. From now on we will just write v for $v_{\text{dimensionless}}$. Note that some books use β to denote dimensionless velocity, here we will use v since we will always use dimensionless velocities when working with the theory of relativity. The absolute value of velocity v is now a factor in the range $v = [0, 1]$ being the velocity relative to the velocity of light.

This was the first step in order to understand the foundations of special relativity. Here comes the second: Let us, for a moment, return to the spatial coordinate systems (x, y) and (x', y') in Figure 5. Clearly the coordinates of the points A and B are different in the two coordinate systems. But there is one thing which is identical in all coordinate systems: The distance between points A and B. If we call this distance Δs_{AB} we can write this distance in the two coordinate systems as

$$\begin{aligned}(\Delta s_{AB})^2 &= (\Delta x_{AB})^2 + (\Delta y_{AB})^2 \\ (\Delta s'_{AB})^2 &= (\Delta x'_{AB})^2 + (\Delta y'_{AB})^2\end{aligned}$$

(check that you understand why!). The distance between A and B has to be equal in the two coordinate systems, so

$$(\Delta s_{AB})^2 = (\Delta s'_{AB})^2.$$

Is this also the case in spacetime? Can we measure intervals between events in spacetime? This is now, at least in theory, possible since we measure space and time separations in the same units. In a spatial (x, y, z) system we know the geometrical relation,

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2,$$

from Euclidean geometry: The square of the distance between two points (called the *line element*) is simply the sum of the squares of the coordinate distances between these two points. But do the rules of Euclidean geometry apply to spacetime? No, not entirely. The geometry of spacetime is called *Lorentz geometry*. The distance between two events (line element) in Lorentz spacetime Δs^2, is given by

The spacetime interval

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2).$$

Note the minus sign. This minus sign is the only thing which distinguishes space from time. The square of the spacetime distance between two events equals the square of the time separation between these events *minus* the square of the spatial separations between the events. And in the same way as the distance between two points in space is the same in all coordinate systems, the distance in spacetime, *the spacetime interval* is the same in all frames of reference. We say that the spacetime interval is *invariant*. A quantity is invariant if it has the same value in all frames of reference. We already know another invariant quantity: the speed of light.

So far in this section, we've determined that two simultaneous events in one frame of reference are not simultaneous in another. The question now is, does this imply that there exists a frame of reference in which an event does not occur at all? The answer to this question is a resounding **NO**! Take the following example into consideration: imagine there exists a bomb that is detonated when two separate laser detectors are activated simultaneously; it is nonsensical to imagine that the bomb would both detonate and remain undetonated at the same time, since that would lead to paradoxical situations in which a person in its proximity could be both alive and dead. As a result, the event must occur in *all* frames of reference, though the time and place may differ. This is summed up in the following axiom:

An event which happens in one frame of reference must happen in all frames of reference. The time and position might differ but the event will happen.

So, that was it. We're done. Now you know what the special theory of relativity is all about. Congratulations! You now see that we may write the special theory of relativity in two sentences: Measuring space and time intervals in the same units, you can calculate the spacetime interval between two events using the formula for the line element in Lorentz geometry. This spacetime interval between two events is invariant, it has the same value as measured from all frames of reference. We will now see what this means in practice. But before you continue, take a walk, go for a coffee or simply take half an hour in fresh air. Your brain will need time to get accustomed to this new concept.

3 An example

A train is moving along the x-axis of the laboratory frame. The coordinate system of the laboratory frame is (x, y) and of the train, (x', y'). In the train a light signal is emitted directly upwards along the y-axis (event A). Three meters above, it is reflected in a mirror (event B) and finally returns to the point where it was emitted (event C). In the train frame it takes the light beam 3 meters of time to reach the mirror and 3 meters of time to return to the point where it was emitted. The total up-down trip (event A to event C) took 6 meters of time in the frame of the train (light travels with a speed of $v = 1$, one meter per meter of light travel time). From event A to event C, the train had moved 8 meters along the x-axis in the laboratory frame. Because of the movement of the train, the light beam moved in a pattern as shown in Figure 6 seen from the lab frame.

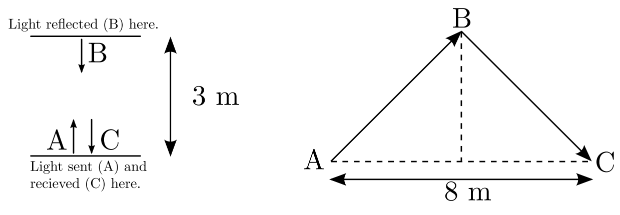


Figure 6: The light emitted (event A) upwards in the train is reflected (event B) and received (event C) at the same place (in the train frame) as it was emitted.

1. *Use the figure to find the total distance d traveled by the light beam in the laboratory frame.* Dividing the triangle into two smaller triangles (see the figure), we find from one triangle that the distance traveled from the

emission of the light beam to the mirror is $d/2 = \sqrt{(4\text{ m})^2 + (3\text{ m})^2} = 5$ m and similarly for the return path. Thus, the total distance traveled by the light beam from event A to event C is $d = 10$ m.

2. *What was the total time it took for the light beam from event A to event C in the laboratory frame?* We have just seen that in the laboratory frame, the light beam traveled 10 meters from event A to event C. Since light travels at the speed of one meter per meter of time, it took 10 meters of time from event A to event C. In the frame of the train, it took only 6 meters of time.

3. *What is the speed of the train?* The train moved 8 meters in 10 meters of time, so the speed is $v = 8/10 = 4/5$, 4/5 the speed of light.

4. *What is the spacetime interval $\Delta s'$ between event A and event C with respect to the train frame?* In the train frame, event A and event C happened at the same point, so $\Delta x' = 0$. It took 6 meters of time from event A to event C, so $\Delta t' = 6$ m. The spacetime interval is thus $\Delta s' = \sqrt{(6\text{ m})^2 - 0} = 6$ m. (check that you also got this result!)

5. *What is the spacetime interval Δs between event A and event C with respect to the laboratory frame?* In the laboratory frame, the distance between the events were $\Delta x = 8$ m and the time interval was $\Delta t = 10$ m. The spacetime interval is thus $\Delta s = \sqrt{(10\text{ m})^2 - (8\text{ m})^2} = 6$ m (check that you also got this result!), exactly the same as $\Delta s'$ in the train frame.

6. *Was there an easier way to answer the previous question?* Oh...uhm, yes, you're right, the spacetime interval is the same in all frames of reference so I should immediately had answered $\Delta s = \Delta s' = 6$ m without any calculation...much easier!

Indeed much easier...remember that this will be very useful when calculating distances and intervals with respect to frames moving close to the speed of light.

4 Observer O and P revisited

Armed with the knowledge of the invariance of the spacetime interval we now return to observer O and P in order to sort out exactly what happened for each of the observers. We know that with respect to the laboratory frame, the two lightnings struck simultaneously (events A and B were simultaneous) at points $x = \pm L/2$ at the time $t = 0$ when observer P was at the origin $x_P = 0$. But at what time did the two lightnings strike with respect to observer P in the train? We have learned that with respect to the frame of reference following the train, the events A and B were *not* simultaneous. But in the reference frame of observer P, at what time t'_A and t'_B did the two lightnings strike? The two observers exchange a signal at $t = 0$ such that their clocks are both synchronized to $t = t' = 0$ at the instant when observer P is at the origin in both coordinate systems $x_P = x'_P = 0$. Did event A and B happen before or after $t' = 0$ on observer P's wristwatch? (It is common to talk about *wristwatches* when referring to the time measured in the rest frame of a moving object, i.e. the time measured by observers moving with the object. This wristwatch time is also called *proper time*).

We know that an event is characterized by a position x and a time t in each of the frames of reference. Let's collect what we know about the position and time of event A, B and the event when observer P passes $x = x' = 0$ which we call event P:

Event P:

$$x = 0 \quad t = 0$$
$$x' = 0 \quad t' = 0$$

Event A:

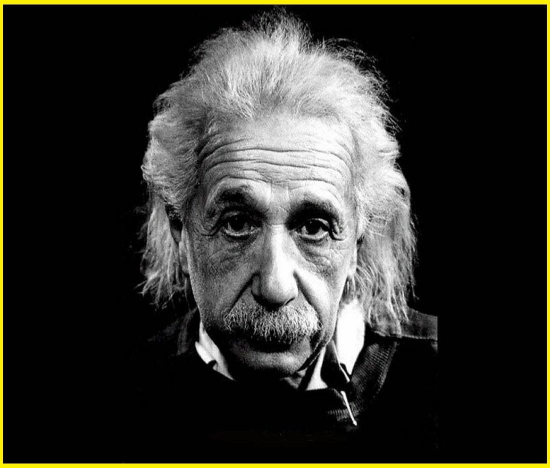
$$x = L/2 \quad t = 0$$
$$x' = L_0/2 \quad t' = t'_A$$

Event B:

$$x = -L/2 \quad t = 0$$
$$x' = -L_0/2 \quad t' = t'_B$$

Note that the length of the train is L_0 for observer P and L for observer O. We have already

Fact sheet: Near the beginning of his career, Albert Einstein (1879 - 1955) thought that Newtonian mechanics was no longer enough to reconcile the laws of classical mechanics with the laws of the electromagnetic field. This led to the development of his special theory of relativity (1905). It generalizes Galileo's principle of relativity – that all uniform motion is relative, and that there is no absolute and well-defined state of rest – from mechanics to all the laws of physics. Special relativity incorporates the principle that the speed of light is the same for all inertial observers regardless of the state of motion of the source. This theory has a wide range of consequences that have been experimentally verified, including length contraction, time dilation and relativity of simultaneity, contradicting the classical notion that the duration of the time interval between two events is equal for all observers. On the other hand, it introduces the spacetime interval, which is invariant.



seen that observers in different frames of reference only agree on the length of the spacetime interval, *not* on lengths in space or intervals in time separately. For this reason, we do expect L and L_0 to be different. Look also at Figure 5, the distance Δx_{AB} between the points A and B differ between the two coordinate systems, in the system (x, y) it is $\Delta x_{AB} = L$, but in the system (x', y') it is $\Delta x'_{AB} = x'_B - x'_A \equiv L'$. The length of the train in the rest frame of the train, L_0, is called *the proper length*. We will later come back to why it is given a particular name.

We want to find at which time t'_A and t'_B observed from the wristwatch of observer P, did events A and B happen? Did they happen before or after event P? For observer O all these events were simultaneous at $t = 0$, the moment in which the two observers exchanged a signal to synchronize their clocks. For observer P, could these events possibly had happened before they happened for observer O? Or did they happen later than for observer O?

In order to solve such problems, we need to take advantage of the fact that we know that the spacetime interval between events is invariant. Let's start with the spacetime interval between events A and B.

Spacetime interval AB: From each of the frames of reference it can be written as

$$\Delta s^2_{AB} = \Delta t^2_{AB} - \Delta x^2_{AB},$$
$$\Delta (s'_{AB})^2 = (\Delta t'_{AB})^2 - (\Delta x'_{AB})^2.$$

(note that the y and z coordinates are always 0 since $v_y = v'_y = v_z = v'_z = 0$, so $\Delta y = \Delta y' = 0$ and $\Delta z = \Delta z' = 0$). In order to calculate the spacetime interval, we need the space and time intervals Δx^2_{AB}, Δt^2_{AB}, $(\Delta x'_{AB})^2$ and $(\Delta t'_{AB})^2$ separately. In both frames, the spatial distance between the two events equals the length of the train in the given frame of reference. So $\Delta x_{AB} = L$ and $\Delta x'_{AB} = L_0$. For observer O the events were simultaneous $\Delta t_{AB} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AB} = t'_A - t'_B$. Setting the two expressions for the spacetime interval equal we obtain,

$$L^2 = L_0^2 - (t'_A - t'_B)^2. \tag{3}$$

(check that you obtain this as well!). We have arrived at one equation connecting observables in one frame with observables in the other. We need more equations to solve for t'_A and t'_B. Let's study the spacetime interval between events A and P.

Spacetime interval AP: From each of the frames of reference it can be written as

$$\Delta s^2_{AP} = \Delta t^2_{AP} - \Delta x^2_{AP}$$
$$\Delta (s'_{AP})^2 = (\Delta t'_{AP})^2 - (\Delta x'_{AP})^2$$

In order to calculate the spacetime interval, we need the space and time intervals Δx^2_{AP}, Δt^2_{AP}, $(\Delta x'_{AP})^2$ and $(\Delta t'_{AP})^2$ separately. In both frames, the spatial distance between the two events equals half the length of the train in the given frame of reference. So $\Delta x_{AP} = L/2$ and $\Delta x'_{AP} = L_0/2$. For observer O the events were simultaneous $\Delta t_{AP} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AP} = t'_A - 0 = t'_A$.

Setting the two expressions for the spacetime interval equal we obtain,

$$(L/2)^2 = (L_0/2)^2 - (t'_A)^2. \tag{4}$$

Note that we have three unknowns, t'_A, t'_B and L. We need one more equation and therefore one more spacetime interval. The spacetime interval between B and P does not give any additional information, so we need to find one more event in order to find one more spacetime interval. We will use event C, the event that beam 1 hits observer P.

Spacetime interval CP: Again, we need

$$\Delta s_{CP}^2 = \Delta t_{CP}^2 - \Delta x_{CP}^2,$$
$$\Delta(s'_{CP})^2 = (\Delta t'_{CP})^2 - (\Delta x'_{CP})^2.$$

In the first section we calculated the time t_C when beam 1 hit observer P in the frame of observer O. The results obtained in the laboratory frame were correct since the events A and B really were simultaneous in this frame. As we have seen, the results we got for observer P were wrong since we assumed that events A and B were simultaneous in the frame of observer P as well. Now we know that this was not the case. We have $\Delta t_{CP} = t_C - 0 = t_C = L/2/(v+1)$ (from equation 1, note that since we measure time and space in the same units $c = 1$). As event C happens at the position of observer P, we can find the position of event C by taking the position of observer P at time t_C giving $\Delta x_{CP} = v\Delta t_{CP} = vL/2/(v+1)$. In the frame of observer P, event C clearly happened at the same point as event P so $\Delta x'_{CP} = 0$. The time of event C was just the time t'_A of event A plus the time $L_0/2$ it took for the light to travel the distance $L_0/2$ giving $\Delta t'_{CP} = t'_A + L_0/2$. Equating the line elements we have

$$\frac{L^2/4}{(v+1)^2}(1-v^2) = (t'_A + L_0/2)^2 \tag{5}$$

Now we have three equations for the three unknowns. We eliminate L from equation (5) using equation (4). This gives a second order equation in t'_A with two solutions, $t'_A = -L_0/2$ or $t'_A = -vL_0/2$.

The first solution is unphysical: The time for event C is in this case $t'_C = t'_A + L_0/2 = 0$ so observer P sees the lightning at $t' = 0$. Remember that at $t = t' = 0$ observer O and observer P are synchronizing their clocks, so at this moment, and only this moment, their watch show the same time. This means that observer P sees flash A at the same moment as the lightening strikes for observer O. Thus at $t = t' = 0$, the lightning hits the front of the train for observer O, but at the same time he would see the light from the lightening hit observer P. The light from event A would therefore have moved instantaneously from the front of the train to the middle of the train.

Disregarding the unphysical solution we are left with

$$t'_A = -v\frac{L_0}{2}.$$

Thus event A happened for observers in the train before it happened for observers on the ground. Now we can insert this solution for t'_A in equation 4 and obtain L,

Length contraction

$$L = L_0\sqrt{1-v^2} \equiv L_0/\gamma, \tag{6}$$

with $\gamma \equiv 1/\sqrt{1-v^2}$. So the length of the train is smaller in the frame of observer O than in the rest frame of the train. We will discuss this result in detail later, first let's find t'_B. Substituting for t'_A and L in equation (3) we find

$$t'_B = v\frac{L_0}{2} = -t'_A.$$

So event B happened later for observers in the train than for observers on the ground. To summarize: Event A and B happened simultaneously at $t = t' = 0$ for observers on the ground. For observers in the train event A had already happened when they synchronize the clocks at $t = 0$, but event B happens later for the observers in the train. Note also that the time t'_A and t'_B are symmetric about $t' = 0$. If you look back at Figure 5 we see that the analogy with two coordinate systems rotated with respect to each other is quite good: If we replace y by t we see that for the events which were simultaneous $\Delta y_{AB} = 0$ in the (x, y) frame, event A happens before $y = 0$ and event B happens after $y = 0$ in the rotated system (x', y'). But we need to be careful not taking the analogy too far: The geometry of the two cases are different. The spatial (x, y) diagram has Euclidean geometry whereas the spacetime diagram (x, t) has Lorentz geometry. We have seen that this simply means that distances are measured differently in the two cases (one has a plus sign the other has a minus sign in the line element).

We have seen that for observer P event A happens before event P when they synchronize their clocks. But does he also see the lightning before event P? As discussed above, this would be unphysical, so this is a good consistency check:

$$t'_C = t'_A + \frac{L_0}{2} = -v\frac{L_0}{2} + L_0/2 = L_0/2(1-v),$$

which is always positive for $v < 1$. Thus observer P sees the flash after event P. When does observer P see the second flash (event D) measured on the wristwatch of observer P? Again we have $t'_D = t'_B + L_0/2$ giving

$$t'_D = L_0/2(1+v),$$

so the time interval between event C and D measured on the wristwatch of a passenger on the train is

$$\Delta t' = t'_D - t'_C = vL_0$$

How long is this time interval as measured on the wristwatch of observer O? We already have t_C and t_D from equations (1) and (2). Using these we get the time interval measured from the ground,

$$\Delta t = vL_0/\sqrt{1-v^2}$$

So we can relate a time interval in the rest frame of the train with a time interval on the ground as

Time dilation

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2}} = \gamma \Delta t'. \qquad (7)$$

Note that index CD has been skipped here since this result is much more general: It applies to any two events taking place at the position of observer P. This is easy to see. Look at Figure 7. We define an observer O which is at rest in the laboratory frame using coordinates (x, t) and an observer P moving with velocity v with respect to observer O. In the frame of reference of observer P we use coordinates (x', t').

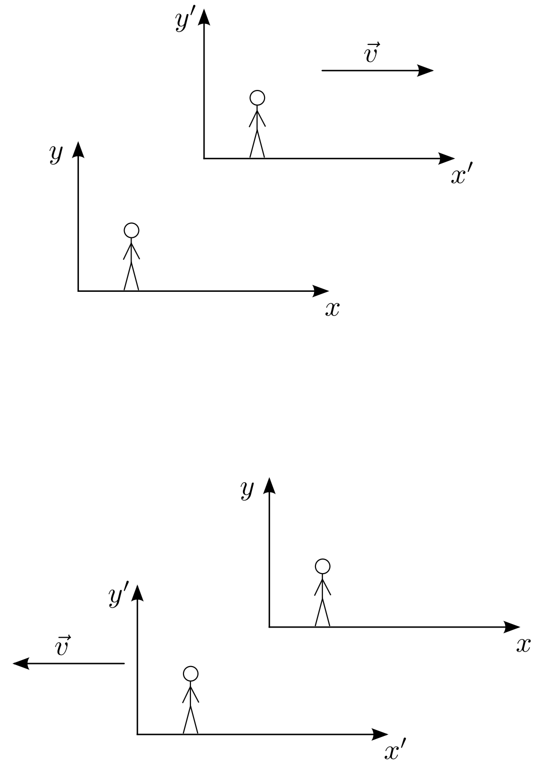


Figure 7: Two reference frames: (x, y) coordinates are used for the system defined to be at rest and (x', y') coordinates are used for the system defined to be moving. In the upper figure, observer O is in the laboratory frame with observer P in the frame moving with velocity v. In the lower figure, the two systems have exchanged roles and $v \to -v$. All equations derived in the above system will be valid for the system below by exchanging $v \to -v$.

We now look at two ticks on the wristwatch of observer P. Observer P himself measures (on his wrist watch) the time between two ticks to be $\Delta t'$ whereas observer O measures the time intervals between these two ticks on P's watch to be Δt (measured on observer O's wrist watch). In the coordinate system of observer P, the wristwatch does not move, hence the space interval between the two events (the two ticks) is $\Delta x' = 0$. For observer O, observer P and hence his wristwatch is moving with velocity v. So observer O measures a space interval of $\Delta x = v\Delta t$ between the two events. The spacetime interval in these two cases becomes

$$\begin{aligned}(\Delta s)^2 &= \Delta t^2 - \Delta x^2 = \Delta t^2 - (v\Delta t)^2 \\ &= (\Delta t)^2(1-v^2) \\ (\Delta s')^2 &= (\Delta t')^2.\end{aligned}$$

Spacetime intervals between events are always equal from all frames of reference so we can equate these two intervals and we obtain equation (7).

Going back to the example with the train: If the train moves at the speed $v = 4c/5$ then we have $\Delta t = 5/3\Delta t' \approx 1.7\Delta t'$. When observer O on the ground watches the wristwatch of observer P, he notes that it takes 1.7 hours on his own wristwatch before one hours has passed on the wristwatch of observer P. If observer P in the train is jumping up and down every second on his own wristwatch, it takes 1.7 seconds for each jump as seen from the ground. For observers on the ground it looks like everything is in slow-motion inside the train.

How does it look for the observers in the train? Remember that velocity is relative. Being inside the train, we define ourselves as being at rest. From this frame of reference it is the ground which is moving at the speed $-v$. Everything has been exchanged: Since we now define the train to be at rest, the coordinate system (x, t) is now for the train whereas the coordinate system (x', t') is for the ground which is moving at velocity $-v$ (see Figure 7). Note the minus sign: The motion of the ground with respect to the train is in the opposite direction than the motion of the train with respect to the ground.

We can now follow exactly the same calculations as above for two events happening at the position of observer O instead of observer P. For instance we watch two ticks on the clock of observer O. Then we find again formula (7) but with the meaning of Δt and $\Delta t'$ interchanges. Assuming again a speed of $v = -4c/5$ (note again the minus sign), observer P sees that it takes 1.7 hours on his wristwatch for one hour to pass on the wristwatch of observer O. It is the opposite result with respect to the above situation. While observers on the ground observe everything in the train in 'slow-motion', the observers on the train observe everything on the ground in 'slow-motion'. This is a consequence of the principle of relativity: There is no way to tell whether it is the train which is moving or the ground which is moving. We can define who is it rest and who is moving, the equations of motion that we obtain will then refer to one observer at rest and one observer in motion. When we change the definition, the roles of the observers in the equation will necessarily also change. Thus, if we define the ground to be at rest and the train to be moving and we deduce that observers on the ground will see the persons in the train in 'slow-motion', the opposite must also be true: If we define the train to be at rest and the ground to be moving, then the observers on the train will observe the observers on the ground in 'slow-motion'. Confused? Welcome to special relativity!

Consider two observers, both with their own wristwatch, one at rest in the laboratory frame (observer O) another moving with velocity v with respect to the laboratory frame (observer P). Going back to equation (7) we now know that if $\Delta t'$ is the interval between two ticks on the wristwatch of observer P, then Δt is the time interval between the same two ticks of observer P's watch measured on observer O's wristwatch. Using equation 7 we see that the shortest time interval between two ticks is always the time measured directly in the rest frame of the wristwatch producing the ticks. Any other observer moving with respect to observer P will measure a longer time interval for the ticks on observer P's wristwatch. This is of course also valid for observer O: The shortest time interval between two ticks on observer O's wristwatch is the time that observer O himself measures. The wristwatch time is called the *proper time* and is denoted τ. It is the shortest interval between these two events that can be measured.

Note that the proper time between two events (two ticks on a wristwatch) also equals the spacetime interval between these events. This is easy to see: consider again the ticks on observer P's wristwatch. In the rest frame of observer P, the wristwatch is not moving and hence the spatial distance between the two events (ticks) is $\Delta x = 0$. The time interval between these two events is just the proper time $\Delta\tau$. Consequently we have for the spacetime interval $\Delta s^2 = (\Delta t')^2 - (\Delta x')^2 = \Delta\tau^2 - 0 = \Delta\tau^2$.

Proper time

$$\Delta s^2 = \Delta\tau^2$$

in the rest frame.

Now, let's return to another result, the length of the train L as measured by observer O on the ground. Again, the result in equation 6 can be shown in a similar manner to be more general. The length L_0 can be the length of any object in the rest frame of this object. We see from equation 6 that any observer which is not at rest with respect to the object will observe the length L which is always smaller than the length L_0. The length of an object measured in the rest frame of the object is called the *proper length* of the object. An observer in any other reference frame will measure a smaller length of the object. The proper length L_0 is the longest possible length of the object. This also means that an observer in the moving train will measure the shorter length L for another identical train being at rest with respect to the ground (being measured to have length L_0 by observers on the ground).

5 The Lorentz transformations

Given the spacetime position (x, t) for an event in the laboratory frame, what are the corresponding coordinates (x', t') in a frame moving with velocity v along the x-axis with respect to the laboratory frame? So far we have found expressions to convert time intervals and distances from one frame to the other, but not coordinates. The transformation of spacetime coordinates from one frame to the other is called the Lorentz transformation. In the exercises you will deduce the expressions for the Lorentz transformations. Here we state the results. We start by the equations converting coordinates (x', y', z', t') in the frame moving along the x-axis to coordinates (x, y, z, t) in the laboratory frame,

The Lorentz transformations

$$t = v\gamma x' + \gamma t', \tag{8}$$
$$x = \gamma x' + v\gamma t', \tag{9}$$
$$y = y',$$
$$z = z'.$$

To find the inverse transformation, we have seen that we can exchange the roles of the observer at rest and the observer in motion by exchanging the coordinates and let $v \to -v$ (see Figure 7),

The Lorentz transformations (cont.)

$$t' = -v\gamma x + \gamma t, \tag{10}$$
$$x' = \gamma x - v\gamma t, \tag{11}$$
$$y' = y,$$
$$z' = z.$$

Here

$$\gamma = \frac{1}{\sqrt{1 - v^2}}.$$

6 List of expressions you should know by now

7 Introduction to MCAst

During your journey through relativity you will get acquainted with the 3D-application MCAst. MCAst will be excessively used during almost all the exercises in relativity, and is therefore necessary. Go and download the software from the link provided on the course-page. Once MCAst is downloaded grab a partner (or two) and be prepared to cooperate on the exercises. Most of the exercises are **meant and should be cooperated on**, but they can be done alone. Each student gets her/his frame of reference and is supposed to calculate what is going on in the other frame of reference. It is very important that you do not get tempted to download the videos of both frames, you will loose a significant part of the learning process by doing this: you need to agree with your partner who does which frame, then you download only your video. When you are done you will meet and look at each others videos and check your results.

To use MCAst during the exercises you will need to use the corresponding xml-file. For the standard variant of the course you will find these on the course web page with names corresponding to the exercise numbers. For the project students these xml-files will be generated through the `AST2000SolarSystem` class with your own specific solar system (seed) and planet. The information on how to generate the xml-file, your own personal seed and how to find the information regarding your solar system can be found in `AST2000SolarSystem` documentation. Each unique seed generate unique videos with different numerical answer. When generating the videos a corresponding solution for the numerical answers will be generated. For those experiencing error when trying to generate the xml-file, you can use the xml-files on the course page.

Put the xml-file for your frame in the data directory of MCAst. Then launch MCAst and you can load the xml and start the video. Note that after starting MCAst, you can use the option 'settings' where can choose between using a GPU renderer, or switch off GPU using instead your computer's CPU. The graphics will be much better when using GPU, but it requires your computer to have a powerful graphics processor. **Important:** the random generator is different between the GPU/CPU. This means that the landscape, events and the exercises in general will be different depending on whether you use CPU or GPU. It is therefore very important that you agree with you partner whether you will use GPU or CPU, both of you **must** use the same.

In the upper left corner, there will be a clock showing the time in your frame of reference. There will also be a ruler on top to measure the position of events. The numbers which always appear next to the pointer shows the position measured on this ruler. The slider on the left can be used to adjust how fast the video plays.

Note: In almost all videos you can assume that the camera receives light from everything instantaneously (it is clearly written in the exercise when this assumption does not hold). This means that we have not taken into account the time it takes for light to travel from the planets/objects/events and to the camera. In this way, all the times and positions of the events are seen immediately as they happen. You therefore see events happening when and where they *actually* happen, but since light has a limited velocity a real observer would see some of the scenes quite differently. As an example, taking into account the effect of a non-zero light travel time a planet would not appear length contracted: it would still appear spherical but you would see the back side of the planets towards the edges.

8 Exercises

Exercise 2A.1

We have seen the effects of Lorentz contractions, namely that a stick of proper length[1] L_0 moving at a speed v along the x-axis in the laboratory frame has a measured length of $L = L_0/\gamma$. What happens to the size of the stick in the y and z directions measured from the laboratory frame? Does the stick become thinner?

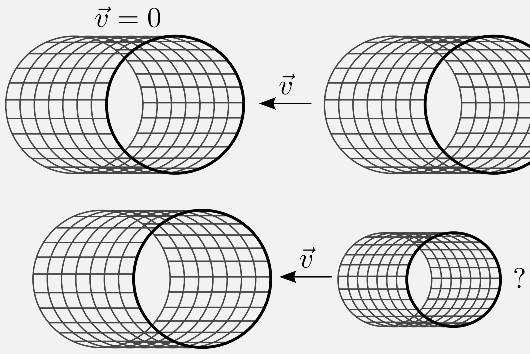


Figure 8: Does a moving cylinder become both thinner and contracted as observed from the laboratory frame? We will study this more closely in Problem 1.

To determine whether or not a contraction occurs in a direction orthogonal to the cylinder's movement, picture two identical hollow cylinders A and B. If the radius of one cylinder becomes smaller than the other, it should be able to slide into the larger cylinder (see Figure 8). The axes of both cylinders are aligned with the x-axis at $y = z = 0$. Thus both cylinders are centered around the x-axis. Cylinder A is at rest in the laboratory frame, cylinder B is moving at a velocity v along the x-axis, approaching cylinder A.

1. We know that the length of cylinder B as measured from the laboratory frame decreases. Assume that the same effect takes place in the y and z directions such that the radius of cylinder B gets smaller as measured in the laboratory frame. What happens when the two cylinders meet?

2. Now, consider the same situation from the point of view of an observer sitting on cylinder B. What happens when the two cylinders meet?

3. Using the above conclusions, explain why we must have that $y = y'$ and $z = z'$ in the Lorentz transformation[2]

Exercise 2A.2

Relevant theory: Section 1 - 4.
In this exercise we will study a spaceship's failed attempt to land on a planet. The first (rather unlucky) observer will be sitting in the spaceship, while the other will be standing on the planet's surface, and will eventually be hit by the crashing spaceship. Use MCAst to load the xml file corresponding to this exercise; there are two frames with one xml file for each. You and your partner should agree upon who is in charge of which frame. You should only look at the video for your frame until you are told otherwise.

In this exercise there are exactly two frames of reference, one for the person standing on the planet, and one for the spaceship; both are placed at the origin in their respective frames. In the left corner of the video, the time elapsed when the spaceship enters the atmosphere and then crashes will be returned with a message indicating which event is taking place. The necessary distances to perform the calculations will also be given.

Your main task will be to calculate the time elapsed starting when the ship enters the planet's atmosphere, up until it crashes; the catch here is that you must determine the time elapsed as seen from the other observer's frame of reference.

1. How much time passes (in *your* frame of reference) from the moment the spaceship enters the atmosphere until it crashes?

2. In your frame of reference, what's the distance from the ground to the beginning of the atmosphere?

3. What is the velocity of the spaceship or the planet in each frame of reference?

4. Calculate the spacetime interval between these two events (entering the atmosphere and crashing) in your frame of reference, give the answers in milliseconds.

[1]Meaning the length measured in the stick's rest frame.
[2]Note: this transformation holds strictly for movements along the x-axis. For multidimensional movement, the Lorentz transformation becomes significantly more complicated. This is outside the scope of this course.

5. Write an expression for the same spacetime interval in the other frame of reference. Use the invariance of the spacetime interval to calculate the time elapsed from when the spaceship enters the atmosphere until it crashes in **the other frame of reference**. **Hint for one of the frames:** distances can rewritten using the equation of constant motion $x = vt$ and you may need to calculate a velocity.

6. Now meet your partner and look at the videos in both frames.

When high energy cosmic ray protons collide with atoms in the upper atmosphere, so-called muon particles are produced. These muon particles have a mean lifespan of about 2 μs (2×10^{-6} s) after which they decay into other types of particles. They are typically produced about 15 km above the surface of the Earth. We will now study a cosmic ray muon approaching the surface with the velocity of $0.999c$.

7. How long time does it take for a muon to arrive at the surface of the Earth as measured from the Earth frame?

8. Ignore relativistic effects: Do you expect many muons to survive to the surface of the Earth before decaying? (compare with the mean life time)

9. Now use invariance of the space-time interval (think of the muon as the spaceship), to find the time it takes to reach the surface of the Earth in the muons frame of reference. Does it change your conclusion on the previous question?

Exercise 2A.3

Relevant theory: Section 1 - 4.
In this exercise there are a total of two frames of reference. There are therefore two xml files, one for each frame and person, and all the necessary information will be given in the upper left corner for each frame. You and your partner should agree upon who is in charge of which frame. You should only look at the video for your frame until you are told otherwise. The two frames of reference correspond to:

- The frame of reference of the spaceship with primed coordinate system (x', t'), the spaceship is always at origin $x' = 0$.
- The frame of reference of the planet with unprimed corresponding coordinate system (x, t). At time $t = 0$, the spaceship is positioned at $x = 0$. At this moment the clocks are synchronized such that also $t' = 0$.

The relative velocity between the spaceship and the planet is given in the upper left corner of the video.

The spaceship is traveling through the atmosphere of the planet and is struck by two lightenings. The two lightenings are yellow and blue, both strike the spaceship. Thus we have two events: event 'Y' which is the event of the yellow lightening striking and event 'B' which is the event of the blue lightening striking.

The purpose of this exercise is to use invariance of the space-time interval to calculate the time interval between the two events in the other frame.

1. **For both frames:**
   (a) Write a table with the times and positions of event Y and B in your frame of reference. (for the planet frame, you may need some very basic physics to calculate the positions)
   (b) What is the time interval between the two strikes in your frame?

2. **For the frame of the spaceship:**
   (a) Express the planet-frame-position of the two lightenings, x_Y and x_B, using the relative velocity v and the (for you) unknown times t_Y and t_B (the times of the events in the planet frame). **Hint:** At $t = 0$ the origin of both systems are aligned: look at the landscape just below the space ship at this moment. This point in the landscape corresponds to $x = 0$ in the frame of the planet. At any later time t, what is the position of the space ship x at that moment measured in the planet frame? Since both events take place at the position of the space ship, you can use this information to find

expressions for the planet-position of the lightenings.

(b) Express the space-time interval between event Y and B in the planet frame with the unknown variables t_Y and t_B. Thereafter express the known space-time interval in your frame using variables v, t'_Y and t'_B, not numbers.

(c) Insert Δt for the time interval in the planet frame and $\Delta t'$ for the time interval in your frame (do not insert numbers for the times). Do you find the expression for time dilation?

(d) Now insert numbers to find the time interval between the lightenings in the planet frame.

3. **For the frame of the planet:**

   (a) What are the positions x'_Y and x'_B for the two lightenings in the frame of the spaceship? **Hint:** Where is origin in the frame of the spaceship?

   (b) Express the space-time interval between event Y and B in the spaceship frame with the unknown variables t'_Y and t'_B. Thereafter express the known space-time interval in your frame using variables v, t_Y,and t_B, not numbers.

   (c) Insert Δt for the time interval in the planet frame and $\Delta t'$ for the time interval in the spaceship frame (do not insert numbers). Do you find the expression for time dilation? **Hint:** you will need to find an expression for the positions x_Y and x_B of the lightenings in your frame, as a function of velocity v and time of events t_Y and t_B.

   (d) Now insert numbers to find the time interval between the lightenings in the spaceship frame.

4. Now watch both videos with your partner and check that your numbers were correct. Look at the landscape to see the position of the spaceship at both events. Although it might be difficult to see, they should both occur above the same positions in the landscape. Imagine that the lightenings burned the landscape just below each lightening. Look at the distance between these two positions in the landscape. In which frame is the distance between these larger?

Exercise 2A.4

Relevant theory: Section 1 - 4.
In this exercise there are a total of two frames of reference. There are therefore two xml files, one for each frame, and all the necessary information will be given in the upper left corner for each frame. In this exercise you do not need a partner. In the beginning you are only supposed to look at frame 1 until further notice. The two frames of reference correspond to:

- The frame of reference of the spaceships with primed coordinate system (x', t'), the leftmost spaceship is always at origin $x' = 0$.
- The frame of reference of the planet with unprimed coordinate system (x, t).

The idea behind this exercise is to make you rediscover what Einstein discovered: that the invariance of the light speed must imply that simultaneous events are not simultaneous in all frames. To do this we use following problem:

Two spaceships are moving with equal speed with respect to the ground. In the frame of the spaceships, both spaceships simultaneously shoot a laser beam towards the other (event A and B, left spaceship shooting is event A). When the laser beams hit, the ships explode creating two more events (event C and D, leftmost explosion is event C). In this exercise, we will study these 4 events from two different frames of reference.

Event A, the emission of the laser beam from the leftmost spaceship, takes place at $x = x' = 0$ and $t = t' = 0$ in both frames.

Part 1

In part 1 until further information is given special relativity is for you an unknown concept, but you do know that the velocity of light is the same for all observers (as has been shown empirically).

The two spaceships are firing laser beams simultaneously in the spaceship frame. Stationed perfectly in the middle between the two spaceships we have observer M(iddle). Observer M, as the spaceships, is at rest in the spaceship frame.

1. Why will observer M observe events A and B simultaneously? (Note that we know that the events are simultaneous for observer M since she is in the same frame of reference as the spaceships, the question is why she will also *observe* these events simultaneously?) Check in MCast that this is really the case.

2. Now we will try to figure out what happens in the planet frame without having looked at the planet frame video: Using that

   - we know observer M sees the two light beams crossing just at her position (why must this be the case in both frames?),
   - the fact that observer M is in the middle between the spaceships,
   - that the laser beams were emitted simultaneously in the space ship frame,

   we can conclude that in the planet frame, the laser beams where not emitted simultaneously, but at two different times. Why? Try to think how the spaceship and observer M are moving while the laser beams are emitted.

3. In order for the laser beams to cross exactly at the position of observer M, which of the laser beams must have been emitted first in the planet frame? Why? Try to imagine the movements of the space ships and the laser beams in the planet frame.

4. In the spaceship frame the explosions are simultaneous, is this still the case in the planet frame? Again, in the spaceship frame, the lights from the explosions will reach observer M who is stationed in the middle, simultaneously.

5. Which explosion occurs first in the planet frame? **Hint:** think twice before answering, the correct answer might not necessarily be the first idea that comes to your mind.

6. Now order events A, B, C and D in chronological order in the planet frame. Write a short summary of why this has to be the case. Imagine how this will look. Then **only after you have really tried to imagine how this looks**, look at the video for the planet frame.

Now that you have gained some understanding in why the planet frame events must have a different order that the spaceship frame events, it is time to calculate numbers for the times and positions in the rest of part 1:

7. Make a table of the times t' and positions x' of all four events in the spaceship frame, all expressed in km. Also calculate the distance L' between the spaceships in the spaceship frame. We will call the unknown distance between the spaceships in the planet frame L.

8. Write the times and positions of these same 4 events in the planet frame, expressed in terms of the velocity v and the unknown planet frame quantities t_B, t_C, t_D and L. Use the video for the planet frame (not using numbers just qualitatively looking at what is happening) as assistance to find expressions for x_B, x_C and x_D.

9. Make a function for the position of the laser beam emitted in event A as a function of time. Use this function together with a function for the position of the rightmost spaceship to find an expression for the time t_D of event D expressed in terms of the unknown L as well as the velocity. Rewrite the expression for x_D.

10. Use invariance of the space-time interval Δs_{AC} to find a value for the time t_C of event C expressed for the moment in km. Use this to find x_C, also in km.

11. In the planet frame, make a function for the position of the laser beam emitted in event B as a function of time. Use this function together with a function for the position of the leftmost spaceship to find an expression for the time t_B of event B expressed in terms of the unknown L. Find also an expression for x_B.

12. Use invariance of the space-time interval Δs_{BD} to find the value of L. This calcula-

tion might be long and ugly if you don't do it right: Wait by inserting the expressions you have for t_B and t_D until you have simplified the expression as much as you can, remember you also have an expression for t_D.

Part 2

In the second part of this exercise, we will now again imagine that we do not know about length contraction and time dilation. Our goal now is to try to imagine being Einstein when he just discovered relativity. Using only the fact that the speed of light is the same in both frames, we will try to arrive at the expression for time dilation using the situation with the spaceships and pure reasoning. Be prepared that we might not quite arrive though.

In the following we will not use numbers, only symbols for times, positions and distances in the planet frame.

1. Write equations for the positions of (1) the leftmost spaceship, (2) the observer in the middle, and (3) the light beam emitted from the leftmost spaceship as a function of time t, velocity v, the time t_A and the distance L. Remember that at time $t = 0$ (which is the origo event), the position of the leftmost spaceship is $x = 0$.

2. Use the fact that at the time t_M (when the two beams cross at the position of the middle observer), the position of the middel observer equals the position of the beam emitted from the leftmost spaceship, to show that

$$t_A = t_M - \frac{L/2}{1-v}.$$

(is the light beam emitted before, at or after the origo event?)

3. Write also the position of the beam emitted from the rightmost spaceship as a function of time t expressed in terms of the time t_M, L and v. Use this equation and the fact that at time t_C, the position of the leftmost spaceship equals the position of the beam emitted from the rightmost spaceship to show that

$$t_C = t_M + \frac{L/2}{1+v}.$$

4. From an observer at the planet, how long time Δt does it take from the beam is emitted from the leftmost spaceship at time t_A to the time the leftmost spaceship explodes at time t_C? Express the answer in terms of L and v only. What is the corresponding time $\Delta t'$ in the frame of the spaceships?

5. Clearly these time intervals are different, which should come as a surprise given that you do not know anything about relativity. This shows that time needs to run differently in the two frames, or could there be a different solution to this discrepancy?

6. What is the ratio between $\Delta t'$ in the spaceship frame and Δt in the planet frame? Does it look similar to the expression for time dilation in special relativity? Why is it different? Having our (wrong) assumptions in mind, and knowing the real formula for time dilation, could you actually have guessed this result?

Exercise 2A.5

Relevant theory: Section 1 - 4.
In this exercise there are a total of two frames of reference. There are therefore two xml files, one for each frame, and all the necessary information will be given in the upper left corner for each frame. You do not need a partner for this exercise. In the beginning you are only supposed to look at frame 1 (spaceship frame), information will be given when you are allowed to look at frame 2. The two frames are:

- The frame of reference of the spaceships with primed coordinate system (x', t'), the leftmost spaceship is at origin.
- The frame of reference of the space station (and planet) with unprimed coordinate system (x, t), the space station (shown as a white disc) is at the origin.

In this exercise we will play cosmic ping-pong with a laser beam. Two spaceships with equal velocity moving to the left with respect to the space station are located at a fixed distance L' (spaceship frame) apart. Both spaceships are equipped with mirrors which enables them to reflect laser beams. The leftmost spaceship emits a laser beam which results in the following events:

- Event A which is the emission of the laser beam at $t = t' = 0$ at the position $x = x' = 0$.
- Event B which is the first reflection from the rightmost spaceship.
- Event D which is the when the laser reflected in event B reaches the leftmost spaceship and is reflected again.
- Event C which is a random explosion that happens on the space station simultaneously with event B in the spaceship frame.

Your task in this exercise is therefore to denote the time differences between the reflections in the space station frame, some general intuition of how the scene looks in the other frame and why, as well as the difference between the relativistic and the non-relativistic case. We will start with the visual understanding, therefore in the first questions you are only supposed to do reasoning, no calculations.

1. We start by comparing the time it takes for the laser beam to go from left to right, compared to the time it takes to go from right to left. In the spaceship frame, which time interval if any is the largest $\Delta t'_{\text{AB}}$ or $\Delta t'_{\text{BD}}$? Why?
2. Now try to imagine the whole scene in the space station frame: remember that the speed of light is invariant and that the space ships move at a speed close to the speed of light. How are the space ships and laser beam moving? Try visualizing.
3. Which time interval will therefore be the largest Δt_{AB} or Δt_{BD} in the space station frame? Why?
4. Do not yet look at frame 2. There is however a third xml-file: For visualizing how the time intervals change depending on the velocity of the spaceships watch frame 3. Here the ships will increase velocity for every second reflection until they reach a velocity of $0.8c$ (note that in this illustration video, the ships move in the opposite direction).

Let's now look at the same situation only non-relativistic. Suppose the laser beam is now a ping pong ball moving back and forth always at 80km/h with respect to the spaceships and the spaceships are moving at 50km/h with respect to the planet. As with the light beam, the ping pong ball is always moving with the same velocity with respect to the space ships. Therefore, one should think that the same argument as in the previous questions is valid in the space station system: When the ball is moving to the right it moves towards the spaceship which is approaching the ball. When it moves left, it moves towards the spaceship which moves away from the ball. As with the light beam, it should therefore take longer going left than going right.

5. It looks as if the two different observers will observe different travel times for the ball, just as for the light beam. Can this really be the case? If not, where is the error in the argument? Did we use an important principle from relativity which is not applicable in this case?
6. Going back to the relativistic case with the light beam: Decide whether event C or B happens first in the space station frame.
   Hint: To solve this exercise, imagine an object located in the middle between event B and C with velocity equal to the spaceships. How will the light from the explosion in event C and the reflection from event B pass this object? You will see that this situation corresponds to the one from exercise 2A.4.
7. Now try to visualize how the video for frame 2 will look like. In particular, think about the order of the event and the positions of the spaceships and space station during the events.
8. Look at the video for frame 2. Does it look like you imagined?

We are now done with the visualization, and from here on out we will calculate the exact times of the events.

9. Write down the time t' and position x' of all events in the spaceship frame. Use these to find the distance L' between the spaceships as well as the time intervals between

the first reflections, $\Delta t'_{\mathrm{AB}}$ and $\Delta t'_{\mathrm{BD}}$ in the spaceship frame. It is convenient to convert all numbers to time units, milliseconds will make reasonable numbers.

10. Now our task will be to find Δt_{AB} and Δt_{BD} in the space station frame: We will do this step by step in the following questions. Start by writing down the positions and times of events in the space station frame. Some positions may be expressed through the velocity and the unknown time of an event in order to reduce the number of unknowns. The only unknowns should be x_B, t_B, t_D and t_C, other unknown positions and events should be written in terms of these.
11. Write the spacetime intervals Δs_{AB} and $\Delta s'_{AB}$ between events A and B in the two frames. Show that invariance of the interval gives $x_B = t_B$ in the space station frame. Could you have guessed this using physical arguments without any calculations?
12. Write the spacetime intervals Δs_{AC} and $\Delta s'_{AC}$ between events A and C in the two frames. Show that invariance of the interval gives a number for t_C in the space station frame.
13. Write the spacetime intervals Δs_{BC} and $\Delta s'_{BC}$ between events B and C in the two frames. Show that invariance of the interval gives t_B in the space station frame.
14. Use invariance of the spacetime interval for appropriate events to find at what time t_D event D happened in the space station frame.
15. In the space station frame, how long time did it take from the light was emitted to the first reflection?
16. How long time did it take from the first reflection to the second reflection?
17. Which event happened first in the space station frame, event B or C? Is it consistent with you reasoning above?

Exercise 2A.6

Relevant theory: Section 1 - 4.
In this exercise there are a total of two frames of reference. There are therefore two xml files, one for each frame and person, and all the necessary information will be given in the upper left corner for each frame. You and your partner should agree upon who is in charge of which frame. You should only look at the video for your frame until you are told otherwise. The two frames of reference corresponds to:

- The frame of reference of the spaceship with primed coordinate system (x', t'), the spaceship is always at origin $x' = 0$.
- The frame of reference of the planet with unprimed corresponding coordinate system (x, t), the spaceship starts at $x = 0$.

The space ship starts at $x = 0$ and moves along the positive x-axis in the planet frme. In the space ship frame: note at which point in the landscape above which the space ship starts. Remember that this is the origin $x = 0$ in the planet frame. When the space ship moves, this point in the landscale moves backwards, but remember that from an observer on the ground, the space ship moves along the positive x-axis.

The goal of this exercise is to deduce the Lorentz transformation using only invariance of spacetime interval. This will be done through the following situation:

A spaceship is traveling through a planets atmosphere. In the atmosphere there are a total of four lightning strikes giving a total of four events:

- Event G which is the green light, which occurs at $t = t' = 0$ at position $x = x' = 0$.
- Event P which is the pink light.
- Event B which is the blue light, which occurs simultaneously with event P in the planet frame and at origin in the spaceship frame.
- Event Y which is the yellow light, which occurs simultaneously with event P in the spaceship frame and at origin in the planet frame.

Those who are working with the planet frame will be using event G, P and B. Those who are working with the spaceship frame will be using event G, P and Y.

Our main task here is to deduce the time and position of the pink lightening in the other frame using only information obtained from observations in our own frame as well as the invariance of the space-time interval.

1. Write a table with the space-time coordinates for the events in both frames of reference. During this exercise you are supposed to use both variables and numbers but keep them separate. The space-time coordinates of event P and the time of event B/Y in the other frame is supposed to be the only unknown variables.

2. **In the frame of the spaceship:**

   (a) Use invariance of the space-time interval Δs_{GY} and $\Delta s'_{GY}$ to find the time of event Y in the planet frame t_Y.

   (b) Use invariance of the space-time interval Δs_{GP} and $\Delta s'_{GP}$ to find an expression for the position x_P of event P in the planet frame, expressed in terms of the unknown time t_P of event P in the planet frame.

   (c) Use invariance of the space-time interval Δs_{PY} and $\Delta s'_{PY}$ to find an expression for the time t_P. What is the time of event P (use numbers)? **Tips:** You are supposed to insert the equation deduced from the previous exercise to eliminate the unknown x_P. You should not get a second order equation here. If you do, you have got one of the events wrong: look carefully at the videos again and read carefully the information given at the beginning of this exercise.

   (d) Now use the time t_P to obtain a number for the position x_P of event P in the planet frame.

3. **In the frame of the planet:**

   (a) Use invariance of the space-time interval Δs_{GB} and $\Delta s'_{GB}$ to find the time of event B in the spaceship frame t'_B.

   (b) Use invariance of the space-time interval Δs_{GP} and $\Delta s'_{GP}$ to find an expression for the position x'_P of event P in spaceship frame, expressed in terms of the unknown time t'_P of event P in the spaceship frame.

   (c) Use invariance of the space-time interval Δs_{PB} and $\Delta s'_{PB}$ to find an expression for the time t'_P. What is the time of event P (use numbers)? **Tips:** You are supposed to insert the equation deduced from the previous exercise to eliminate the unknown x'_P. You should not get a second order equation here. If you do, you have got one of the events wrong: look carefully at the videos again and read carefully the information given at the beginning of this exercise.

   (d) Now use the time t'_P to obtain a number for the position x'_P of event P in the spaceship frame.

4. With your partner, look at both videos together and observe in particular event B/Y. Discuss the differences (order of the events).

Now here is the main point of this exercises: we will now deduce the Lorentz transformations using event P: The Lorentz transformations are two equations relating x, t for an event in one frame with x', t' for the same event in another frame. Here we will use event P and thereby deduce the relation between x_P, t_P and x'_P, t'_P where the relation also contains the velocity v between the frames.

5. Your task is therefore to deduce the Lorenz transformation using the equations you already have deduced. **Tips:** You should have five variables being t_P, x_P, t'_P, x'_P and t_Y (or t'_B). The time t_Y or t'_B can be rewritten in terms of the velocity v as done in exercise 2A.3. The rest is algebraic magic.

The student in the spaceship frame should have obtained the expression for the forward Lorenz transform (equations 8 and 9), and the student in the planet frame should have obtained the expression for the backward Lorenz transform (equations 10 and 11).

Exercise 2A.7

Relevant theory: Section 5.

We will now return to the cosmic ping-pong in exercise 2A.5 and solve this using the Lorentz transformations instead of the spacetime interval. Your task is again to calculate the time intervals Δt_{AB} and Δt_{BD} in the other frame of reference. Using the Lorentz transformations we will only need events A, B and D.

1. Again, write up the coordinates (x, t) and (x', t') for these three events, some as numbers, some expressed through other coordinates. The following are unknown: x_B, t_B and t_D in the other frame of reference.
2. Use the Lorentz transformations to find t_B and t_D. You do not need to find x_B.
3. Now find Δt_{AB} and Δt_{BD} in the other frame of reference.

Exercise 2A.8

Relevant theory: Section 1 - 5.
We will finish this part on special relativity by studying the twin paradox in detail. This long and detailed exercise is very important to gain some basic understanding for the underlying physics of many of the so-called paradoxes in the theory of relativity. There are three xml files for this exercise and you should be three students doing this exercise together: you may do part 1 alone, then it is recommended that you meet starting from part 2 and do the rest together. **Please note that you will really loose many important points if you do this exercise alone, in particular it is important to be able to see the different videos at the same time without having to switch continuously between xml files.**

Astronaut Lisa is traveling from her homeplanet Homey to another planet Destiny located 200 light years away. She travels in her spaceship Apollo-Out with velocity $v = 0.99c$. Important note; the planets do **NOT** move with respect to each other and are therefore in the same frame of reference . To begin with we therefore have two frames of reference:

- The frame of reference of the planets with unprimed space-time coordinates (x, t). Homey is always at origin with Destiny located 200 light years away long the positive x-axis.
- The frame of reference of Apollo-Out with primed space-time coordinates (x', t'). Apollo-Out is always at origin in this frame.

We also have two events:

- Event A occurs at $x = x' = 0$ and $t = t' = 0$ and is when Apollo-Out is departing from Homey.
- Event B is when Apollo-Out arrives at Destiny.

Part 1

Before we can truly start on the paradox we need to get some basic math done first.

1. How long does the trip from Homey to Destiny (event A to B) take for observers on Homey? How long does it take measured on Lisa's clock (use the formula for time dilation)?
2. After arriving on Destiny, Lisa quickly starts the return flight. She travels with exactly the same velocity $v = 0.99$ back towards Homey. Use the same arguments (or symmetry arguments) to find the time Δt and $\Delta t'$ it took from Destiny and back to Homey in the two frames of reference.

If you have done your calculations correct, here is a summary of the situation, the whole trip took 404 years measured on Homey-clocks, while it took 57 years measured on Lisa's wrist watch. So while many generations have passed on Homey, Lisa returns 57 years older.

Part 2

During this part and **ONLY** this part we will switch frames, this is to uncover the paradox. The laboratory frame (x, t) is now the frame of Apollo-Out and the moving frame (x', t') is the planet frame. Because of the principle of relativity we are allowed to switch the roles and should still arrive at exactly the same result using the same laws of physics.

From Lisa's point of view, event A can be viewed as Homey departing from the spaceship with $v = 0.99c$, and event B is Destiny arriving at Apollo-Out with velocity $v = 0.99c$. Remember from part 1 where you calculated that it took Lisa 28.5 years to arrive on Destiny.

1. Use time dilation again (and make sure not to confuse Δt and $\Delta t'$, check who is the observer 'at rest' here) to show that the clocks on Homey at the moment when Destiny arrives at Lisa's position show 4 years. Above you showed that 202 years had passed. Now, this might look like a paradox, but we will show further down that it is not. No matter how strange this might sound, it is consistent. The paradox is still to come.

Quickly after Destiny arrives at Lisa's position, Destiny departs and Homey approaches you again with a velocity of $v = 0.99c$. From earlier calculations you know that this trip took 28.5 years for Lisa.

2. By using time dilation (or symmetry) how long does it take in the planet frame for Homey to reach Lisa?

If you made the last calculation correct, this is now the situation: It took Lisa 57 years from Homey departed until Homey returned. However, on Homey, the trip took 8 years. So while Lisa is 57 years older, only 8 years have passed on Homey. Above we found that 404 years had passed on Homey. Now, this is a paradox!

Clearly we made an error somewhere in the calculations. Or maybe we simply forgot some basic principles from special relativity? It appears at first sight that the two roles (traveling to Destiny and traveling from Destiny) are equal, that we can choose whether we consider the planet frame as the laboratory frame or the Apollo-Out frame as the laboratory frame.

4. Are the two roles really identical? If not what is the difference?

Don't read on until you have found an answer to the previous question. Here comes the solution: The difference is that whereas the observers on Homey always stay in the same frame of reference, Lisa changes frame of reference: Apollo-Out needs to accelerate at Destiny in order to change direction and return towards Homey. Homey does not undergo such an acceleration. The expression $\Delta t = \gamma \Delta t'$ was derived for constant velocity (look back at its derivation). It is not valid when the velocity is changing. In order to solve this problem properly one needs to either use general relativity which deals with accelerations or we can view the acceleration as an infinite number of different free float frames, frames with constant velocity, and apply special relativity to each of these frames. We will not do the exact calculation here, but we will do some considerations giving you some more understanding of what is happening.

Part 3

In this part we will study the 'paradox' in detail and see what happened when Lisa changed frame of reference. To do this, we will introduce one more planet and one more astronaut. The third planet, Beyond, is located 400 light years from Homey along the positive x-axis. The locations of the planets is illustrated in Figure 9. There is also a second spaceship, Apollo-In, traveling from Beyond with velocity $v = -0.99c$ with astronaut Peter (denoted P in the figure). There is therefore a total of three reference frames:

- The frame of reference of the planets with unprimed coordinate system (x, t). Homey is always at origin with Destiny located 200 light years away and Beyond located 400 light years away.
- The frame of reference of the spaceship Apollo-Out traveling from Homey to Destiny with primed coordinate system (x', t'). Apllo-Out with astronaut Lisa is always at origin in this frame.
- The frame of reference of the spaceship Apollo-In traveling from Beyond to Homey with double primed coordinate system (x'', t''). Apollo-In with astronaut Peter is always at origin in this frame.

Now let's introduce a new way of thinking. Instead of one spaceship traveling from Homey, we will look at it as a queue of infinite amounts of spaceships, all traveling with the same velocity in the same direction. In all the spaceships before and after Lisa there are other observers. The situation is depicted in Figure 9, in this illustration the queue is an elevator. During the rest of the exercise there will be two elevators, the elevator from Homey to Beyond will be called 'outgoing elevator' (the primed reference system using coordinates (x', t')) and the elevator from Beyond to Homey will be called 'returning elevator' (with double primed reference system using coordinates (x'', t'')). During this part we have these events:

- Event A occurs at $x_A = x'_A = 0$ and $t_A = t'_A = 0$ and is when Lisa is jumping aboard the outgoing elevator at Homey.

- Event B is when Lisa arrives at Destiny and launches herself to the returning elevator from the outgoing elevator.

- Event B' is defined in the following way: At the same time (outgoing elevator frame) as Lisa arrives at planet Destiny, another astronaut in the same elevator but in another space ship (thus in the same frame of reference with clocks synchronized with Lisa's clock, but in another elevator compartment) passes Homey at position $x_{B'} = 0$. Event B' is that he looks at the clocks on Homey as he passes by and sends a light signal from his spaceship which is observed at Homey. In short: B' takes place at the position of Homey at the same time as Lisa arrives at Destiny in her frame of reference.

In the following questions you should use Lorentz transformations to transform between the coordinate systems when necessary. During this part, write the distance between planet Homey and Destiny in the planet frame as L_0.

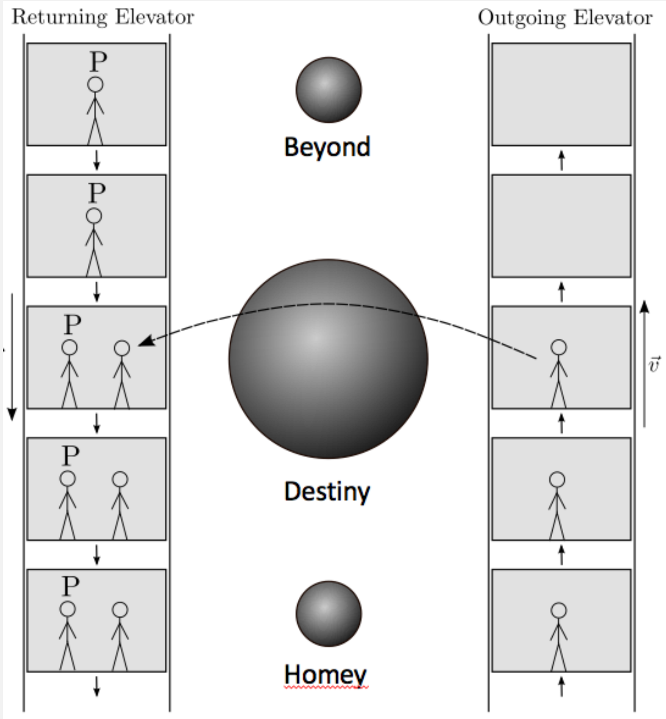


Figure 9: The elevators between planet Homey and planet Beyond.

1. At what time t_B in the planet frame does Lisa arrive at planet Destiny? (express the answer in terms of L_0 and v)
2. Use the Lorentz transformations to find an expression for t'_B, the time when Lisa arrives at Destiny measured on her wrist watch. Insert numbers and check that you still find that the trip takes 28.5 years for her.
3. Show that the time $t_{B'}$ can be written as $t_{B'} = L_0/v - vL_0$. Insert numbers. **Hint:** You first need to find the position $x'_{B'}$ of event B' in the outgoing elevator frame, to find $t'_{B'}$, which you also need, thoroughly read the event description.

The time $t_{B'}$ which you just calculated is the time when the observer in the outgoing elevator reads the time at Homey clocks at the same time (in his frame) as Lisa arrives at Destiny. At this moment, observers at Homey receives the signal from the space ship in the outgoing elevator (at the position of Homey) that Lisa has reached Destiny. Remember that in the planet frame, this trip takes 202 years so in the planet frame, Lisa has **NOT** yet reached Destiny.

4. Now is the time to look at the videos:

Apollo-Out is yellow. When arriving at Destiny, Lisa is launching herself from the yellow to a red spaceship (Apollo-In with astronaut Peter) in the incoming elevator using a spherical space capsule (event B). Then Lisa returns to Homey in the red spaceship. Can you see the blue light signal from the spaceship in frame 1? Compare the numbers you calculated from the earlier exercises with the numbers in the MCAst videos.

5. Explain the result which we found earlier when using Apollo-Out as the laboratory frame: Namely that when Destiny arrived at the spaceship, we calculated that on Homey clocks only 4 years had passed. Why is this not a surprise? Those who were surprised earlier, do you now understand which error you made when you got surprised? Which basic principle of relativity had you forgotten?

We learned in the previous questions that even if Homey clocks were observed at the same moment as the spaceship/elevator arrived at Destiny (in the outgoing frame), these two events (the observation of Homey clocks and the arrival at Destiny) were *not* simultaneous in the planet frame. For Lisa, only 4 years have passed on Homey when she arrives at Destiny. For observers on Homey on the other hand, Lisa arrived at Destiny when 202 years had passed.

Part 4a

We will now tie all the loose treads together. Especially how time passes during the change of frame at event B. This is where the paradox will be answered.

We will start this part by introducing a couple of new events:

- Event D is when Peter jumps aboard the returning elevator from Beyond. This occurs in the planet frame at time $t = 0$ and at a distance $x = 2L_0$ from Homey. In the returning elevator frame this occurs at $x'' = 0$ and $t'' = 0$. In the returning elevator frame, Peter is always at the origin. In the planet frame, event A and even D happen at the same time (Lisa and Peter start their journey simultaneously), they need to travel the same distance L_0 to Destiny with the same velocity v and therefore arrive simultaneously at Destiny at event B where Lisa is transferred to Peter's elevator and frame of reference.
- Event B" is conceptually similar to event B': Event B" takes place at the same time as event B in the frame of the returning elevator. The event is a person in the returning elevator at the position of Homey, looking at the clocks at Homey and sending a blue light signal (check the video). Thus exactly at the same time (returning elevator frame) as Lisa is arriving in the returning elevator and meets Peter, event B" takes place at the position of planet Homey.

In the following, we cannot use the Lorentz transformation because the clocks in the double primed reference system is not correctly synchronized with Homey clocks. We therefore need to use the space time interval.

1. We will in the following try to find the time t''_B at Peter's wrist watch when he arrives at Destiny. Write down the space and time intervals Δx_{BD}, Δt_{BD}, $\Delta x''_{BD}$ and $\Delta t''_{BD}$. Show that invariance of the spacetime interval gives
$$\frac{L_0^2}{v^2} - L_0^2 = (t''_B)^2,$$
which gives $t''_B = L_0/(v\gamma)$. Compare with your expression for t'_B.
2. By using intuition you should be able to deduce that the spacecrafts from event A and D to event B use equal amount of time in their respective frame. The reason for this is that both have equal velocity and no acceleration. Now check the result comparing the videos of the frame of the outgoing (yellow) spaceship and the incoming (red) spaceship: when both meet at Destiny, what is the time in each of the spaceships? Compare with the numbers you have calculated.

We will now try to find the time on Homey at the moment when Peter is reaching Destiny in the returning elevator frame. We will use the same 'trick' as earlier with event B', and use an observer in an elevator compartment positioned at Homey in the returning elevator at the same time as event B occurs in the returning elevator frame. This means that an observer will be sending a light signal at Homey in the returning elevator frame to an observer at Homey in the planet frame to tell that Lisa now has been transferred to the returning elevator and has met Peter. (note that the person in the returning elevator frame which is positioned next to Homey and is sending the signal can not know for sure that Lisa actually managed to meet Peter, he can only infer this from looking at his clock and calculating the time at which this should happen in his frame).

We found that only 4 years had passed on Homey when Lisa arrived at Destiny (seen from outgoing elevator frame). We will now make the same check from the returning elevator. We will now try to find out what time $t_{B''}$ the observer in the returning elevator saw when looking at Homey clocks (and sending the signal) at event B". For this we will use the space-time interval $\Delta s_{DB''}$.

3. Show that the space and time intervals from each frame are the following:

$$\begin{aligned}\Delta x_{DB''} &= 2L_0 \\ \Delta t_{DB''} &= t_{B''} \\ \Delta x''_{DB''} &= L_0/\gamma \\ \Delta t''_{DB''} &= L_0/(\gamma v)\end{aligned}$$

   You might be a bit surprised by one of these results, but if you have doubts, do the following: Make one drawing for event D and one for event B". Show the position of the zero-point (the position of Peter is the zero point of the x'' axis) of each of the x-axes in both drawings and find the distances between events.

4. Use invariance of the space-time interval (event D and B") to show that

$$t_{B''} = \frac{L_0}{v} + L_0 v$$

Inserting numbers should give $t_{B''} = 400$ years. Use the video of the planet frame to check at which time the astronaut in the incoming spaceship reads the clocks on Homey and as sends the blue light signal. Surprised? What has happened?

Lisa is still at event B, she made a very fast transfer so almost no time has passed since she was in the outgoing elevator. But just before the transfer, only 4 years had passed on Homey since she started her journey. Now, less than the fraction of a second later, 400 years have passed on Homey. So in the short time that the transfer lasted, 396 years passed on Homey! This is were the solution to the twin paradox is hidden: When she makes the transfer, she changes reference frame: She is accelerated. Special relativity is not valid for accelerated frames (actually one could solve this looking at the acceleration as an infinite sum of reference frames with different constant velocities). When she is accelerated, she experiences fictive forces. This does not happen on Homey, the planet does not experience the same acceleration. This is the reason for the asymmetry: If her speed had been constant, she and Homey could exchange roles and you would get consistent results. But since she is accelerated during transfer while Homey is not, there is no symmetry here, her frame and the planet frame cannot switch roles.

Let's summerize the situation: In the planet frame, Lisa started her journey at $t = t' = 0$ and arrived on Destiny after $t' = 28.5$ years. In the planet frame she arrived on Destiny after 202 years of travel. In her frame, the clocks on Homey show 4 years when she arrives on Destiny. Only 4 years have passed on Homey at the time she arrives at Destiny, seen from her frame. Then she is launched to the returning elevator. Her watch still shows $t' = t'' = 28.5$ years. But now she has switched frame of reference. Now suddenly 400 years have passed on Homey, Homey clocks went from 4 years to 400 years during the time she launched herself from one elevator to the other, in her frame. In the planet frame, the clock showed 202 years during her transfer.

Seen from Homey, she also needs 202 years to return, so the total time of her travel measured in the frame of reference of the planets is $t = 404$ years. In her own frame, the return trip took 28.5 years (by symmetry to the outgoing trip), so her total travel time was 57 years. But according to her frame of reference, Homey clocks again aged 4 years during her return trip (by symmetry to the outgoing trip). When she was at Destiny, the observer in her frame of reference saw that Homey clocks showed 400 years. In her frame, 4 years passed on Homey during her return trip. So consistently she finds Homey clocks to show 404 years when she sets her feet on Homey again. This is also what we find making the calculation in the planet frame $202 \times 2 = 404$. But hundreds of generations have passed, and she has only aged 57 years. But after all these strange findings I'm sure you find this pretty normal by now. Everything clear? Now check the clocks in the video of the returning (red) spaceship as well as in the planet frame at the moment you return to Homey. Is everything consistent? Read through one more time.

4. **Now comes the most important part of this exercise:** Use the videos, and only the videos, not the text, to tell this story together, explaining every detail and every possible paradox.

Part 4b

We will end part 4 with a very different view on the solution to the twin paradox. You have by now probably learned about spacetime diagrams from the next lecture, you will need spacetime diagrams here:

We will now look at an analogy to explain the 'paradox'. Look at Point Q in Figure 10: It is located at $(2, 2)$ in the (x, y) coordinate system. Your friend has for some reason been able to convince you that the (x', y') coordinate system is better so you decide to switch coordinate systems. Unfortunately you didn't finish 10$^{\text{th}}$ grade, which is why you forgot to change the value of your (x, y) coordinates as you switched to the new coordinate system. The result is you actually changed your original point Q to a new, uterlly unrelated point B.

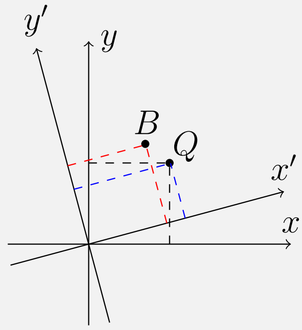


Figure 10: An illustration of a fundamental mistake: forgetting to change your coordinates when changing the coordinate system. You wish to go from the black dashed lines to the blue dashed lines, but fail to change the values of the coordinates, which results in the red dashed lines.

The mistake we made in Figure 10 may seem blatantly obvious, but it is actually identical to the resolution of the twin paradox. As a visual representation, Figure 10 serves as a great tool. However, we did not confuse Euclidean space coordinates in our paradox; we confused Minkowski spacetime coordinates. The problem arose as Lisa reached Destiny and changed her velocity. By doing so she effectively changed her coordinate system, but we forgot to change her spacetime coordinates! In particular, her time coordinate when she arrives on Destiny is *not* the same as the time coordinate as she leaves Destiny. Treating time as pure coordinate is a strange experience at first, but you need to get used to it!

Let's review simultaneity using two frames of reference: S (planet frame) and S' (Lisa's frame). Assume two events, M and N. Both frames observe events M and N, but only S' observes the events simultaneously.

1. What is the value of $\Delta t'_{MN}$?.

The Lorentz transformations are:

$$\Delta t = \gamma\big(\Delta t' + v\Delta x'\big) \tag{12}$$
$$\Delta x = \gamma\big(\Delta x' + v\Delta t'\big) \tag{13}$$
$$\Delta t' = \gamma\big(\Delta t - v\Delta x\big) \tag{14}$$
$$\Delta x' = \gamma\big(\Delta x - v\Delta t\big) \tag{15}$$

2. Use the Lorentz transformations in order to find Δt_{MN} as a function of Δx_{MN}.

What does your new-found expression mean? Well, it actually allows us to draw a *line of si-*

multaneity in a spacetime digram. Lines of simultaneity are exactly what they sound like: lines that show the time t and position x in frame S of events which are simultaneous $\Delta t' = 0$ in frame S'. Let's return to the Twin Paradox:

3. Plot Lisa's worldline as seen from the planet frame of reference when $v = 0.99$ and $L = 200$ ly.

Recall the failure of our preconcieved notion that the time coordinates of Lisa's arrival and departure from Destiny were the same. Instead of claiming her coordinates must be the same, let's now investigate the lines of simultaneity.

4. Assuming Lisa is arriving on Destiny with $v = 0.99$, use your expression from question 2 in order to plot her line of simultaneity through event B in the outoing elevator system on top of your plot from question 3. **Hint**: You want to draw the line of simultaneity through event B, thus you want to find the line showing all times t and positions x such that $\Delta t = t - t_B$ and $\Delta x = x - x_B$ in frame S for which $\Delta t' = 0$ in frame S'. In particular look at where the line of simultaneity crosses the position $x = 0$, the position of the planet and event B'.

5. Assuming now that Lisa is departing Destiny with $v = -0.99$, use your expression from question 2 in order to plot her line of simultaneity through event B in the returning elevator system on top of your plot from question 4. In particular look at where the line of simultaneity crosses the position $x = 0$, the position of the planet and event B".

6. Use your results to explain in a new way why the Twin paradox is *not* a paradox.

You may feel as though this answer is incomplete, but it is really not. If you had a spaceship capable of instantaneously changing its velocity from v to $-v$, you would actually experience this mind-boggling effect. Obviously such spaceships don't exist in reality, there will always be an accelerated phase which takes much more than zero time.