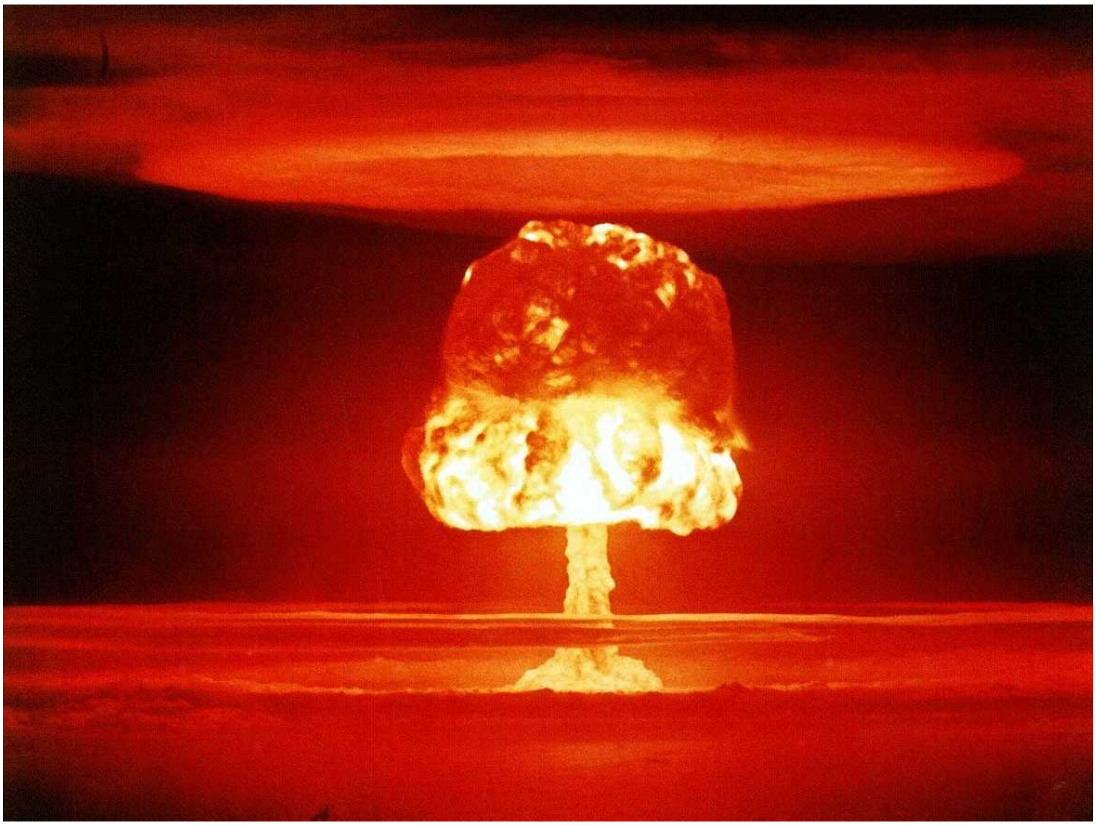
AST2000 Lecture Notes

Part 2B
Four vectors and relativistic dynamics

Questions to ponder before the lecture

1. A position vector is a vector pointing to a position in 3 dimensions. In relativity it could be useful to include the position in time and make a four dimensional position vector. Would such a vector obey the usual rules for vector aritmetics? (try to think about some simple examples, i.e. of adding position vectors)
2. We have seen that in the special theory of relativity, also the pace of time changes when you move. Could this be interpreted as you having a four-dimensional velocity including a time component of your velocity vector? How could you define such a 4 dimensional velocity?
3. The velocity of an object changes when you change your frame of reference. Does this mean that also momentum and energy are relative quantities? What happens in this case to the law of conservation of energy?



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1 Worldlines

In the spacetime diagram in figure 1 we see the path of a particle (or any object) through spacetime. We see the different positions (x, t) in space and time that the particle has passed through. Such a path showing the points in spacetime that an object passed is called a *worldline*. We will now study two events A and B (on the worldline of a particle) which are separated by a small spacetime interval Δs. These events could be the particle emitting two flashes of light or the particle passing through two specific points in space. The corresponding space and time intervals between these two events in the laboratory frame are called Δt and Δx. From the figure you see that $\Delta t > \Delta x$. You can see that this also holds for every small spacetime interval along the path. This has to be this way: The speed of the particle at a given instant is $v = \Delta x/\Delta t$. If $\Delta x = \Delta t$ then $v = 1$ and the particle travels at the speed of light. That $\Delta t > \Delta x$ simply means that the particle travels at a speed $v < c$ which it must. The worldline of a photon would thus be a line at 45° with the coordinate axes. The worldline of any material particle will therefore always make less than 45° with the time axis.

Events which are separated by spacetime distances such that $\Delta t > \Delta x$ are called *timelike events*. Timelike events may be causally connected since a particle with velocity $v < c$ would have the possibility to travel from one of the events to the other event. There is a possibility that the second event could have been caused by the first event since it is possible for a signal to travel between the events. Timelike events have positive line elements,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 > 0.$$

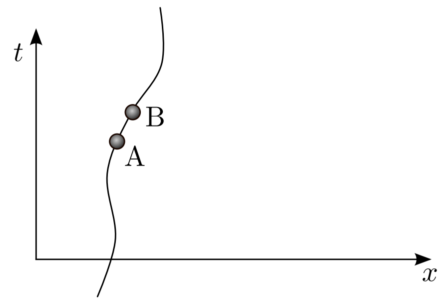


Figure 1: The worldline, the trajectory of a particle in a spacetime diagram. Two events A and B along the path of the particle have been marked.

Events for which $\Delta t = \Delta x$ are called *lightlike events*. Only a particle traveling at the speed of light ($v = \Delta x/\Delta t = 1$) could travel from the first event to the second. Lightlike events have zero spacetime interval,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = 0.$$

Note one consequence of this: Remember that the proper time interval $\Delta\tau^2$ equals the spacetime interval Δs^2. Thus, photons always have $\Delta\tau = 0$, the wristwatch attached to a photon would not change. Photons and other particles traveling at the speed of light do not feel the effect of time.

Events for which $\Delta x > \Delta t$ are called *spacelike events*. For these events, the spatial component of the distance is larger than the time component. No worldline could ever connect two spacelike events as it would require a particle to travel faster than light. Thus, spacelike events are not causally connected. The first event could not have caused the second. The spacetime interval for spacelike events is negative,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 < 0.$$

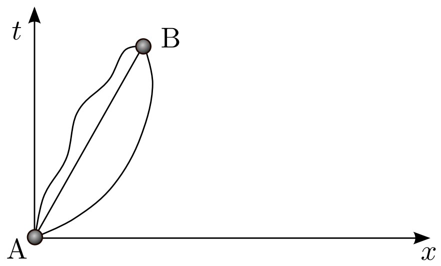


Figure 2: Different worldlines connecting the two events A and B.

In figure 2 we see two events A and B and three different worldlines between these events. These events could be a car passing position x_A and position x_B in the laboratory frame. In the spacetime diagram we see three worldlines each corresponding to a car. The straight worldline must correspond to a car driving with constant speed $v = \Delta x/\Delta t =$ constant. The two other worldlines must correspond to cars accelerating (changing their speed and thereby changing the slope of the worldline) along the way from x_A to x_B, but all cars reach point x_B at the same time (event B). All cars also passed point x_A at the same time (event A). Same time here means 'same time' for all frames of reference: all the cars meet at event A and B, so if they meet simultaneously in one frame of reference they must meet simultaneously in all other frames of reference (did you get this? If not, read the sentences again!).

We will now ask a question which answer may seem obvious in this case, but which might not be so obvious in other situations. The question is: Given a particle (or a car) going from event A to event B. If this particle is in free float (in special relativity this means that no forces act on the particle), which worldline will the particle take between event A and event B? Looking back at figure 2 we see three possible worldlines, but in fact there is an infinite number of possible worldlines connecting the two events. The obvious answer in this case is that it will follow a straight line in spacetime, i.e. the straight worldline corresponding to constant velocity. This is just a modern way of saying Newton's first law: A body which is not under the influence of external forces will continue moving with constant velocity. But is there a deeper principle behind? In the theory of relativity there is, and this principle is called the *principle of maximal aging*. This is a fundamental principle in the special as well as in the general theory of relativity.

The principle of maximal aging says that a particle in free float (no forces act on the particle) will follow the worldline which corresponds to the longest possible proper time interval between the two events. We remember that proper time is the wristwatch time, the time measured on the clock attached to the particle. So let different particles take different paths in spacetime between the two events. Attach a wristwatch to each of the particles. At event B, you look at the time interval between event A and B measured on the wristwatch of each of the particles. The particle which measures the longest proper time, i.e. the particle with the wristwatch which made most ticks during the trip from event A to event B, is the particle taking the path that a particle in free-float would take.

How do we calculate the proper time interval that a given particle takes from event A to event B? The clue is to remember that the proper time interval $\Delta\tau$ between two events equals the spacetime interval, or the total length of the path in spacetime Δs taken between the two events. For the worldline of a particle with constant velocity, we know that the distance in spacetime traveled from event A to event B is just $\Delta s = \sqrt{\Delta t^2 - \Delta x^2}$ where Δx and Δt are space and time intervals measured in an arbitrary frame of reference. To measure the total spacetime interval along the worldline of a particle which does not move with constant velocity, we need to break the path up into small path lengths ds. This path length is so small that we can assume the velocity to be constant during the time it takes to travel this interval in spacetime. We can thus write $ds = \sqrt{dt^2 - dx^2}$ where dx and dt are the corresponding small space and time displacement measured in the arbitrary frame of reference. To obtain the total length of the path in spacetime traveled between two events A and B, we need to integrate all these tiny spacetime intervals ds

giving

$$\Delta s = \int_A^B \sqrt{dt^2 - dx^2}. \qquad (1)$$

This equals measuring the length s of a curved path between two points A and B in the x-y plane:

$$\Delta s = \int_A^B \sqrt{dx^2 + dy^2}.$$

Note again a huge difference here: The minus sign in the spacetime interval. We know from Euclidean geometry that the shortest path s between two points A and B in the plane, is the straight line. The minus sign in the line element for Lorentz geometry gives rise to the opposite result (which we will not derive here): The *longest* path s between two events A and B in spacetime is the straight worldline. Therefore, if we measure the length of the spacetime path for all the three worldlines in figure 2 using the integral in (1), we find that the longest path in spacetime is the straight worldline, i.e. the worldline of the car driving with constant velocity. Remember again that the length of the spacetime interval Δs equals the total proper time $\Delta\tau$ measured on the wristwatch of the particle. So the longest proper time interval between two events is measured on the particle taking the straight line in spacetime, i.e. the particle which has constant velocity. We have just deduced Newton's first law from the principle of maximal aging. When we come to the general theory of relativity, we will see that the spacetime geometry and hence the form of the line elements Δs is different in a gravitational field. We will need the principle of maximal aging to tell us how a free float particle is moving in this case.

2 Four-vectors

So far we have used three dimensional vectors to determine a position in space. A generalized way way of writing a vector is as follows

$$\vec{x} = (x_1, x_2, x_3),$$

which can potentially be the three spatial dimensions (x, y, z). A general 4-vector is similarly defined

$$\mathbf{x} = (x_0, x_1, x_2, x_3),$$

where the components may potentially be the position (t, x, y, z) of an *event* in four dimensional spacetime. In the latter case, the four-vector points to an event in spacetime for a given frame of reference. We have already learned that in order to transform spacetime coordinates from one frame of reference to another, we need the Lorentz transformations. Thus, we may write the transformation of a four-vector $\mathbf{x}$ in one frame of reference to $\mathbf{x}'$ in another frame of reference by a matrix multiplication,

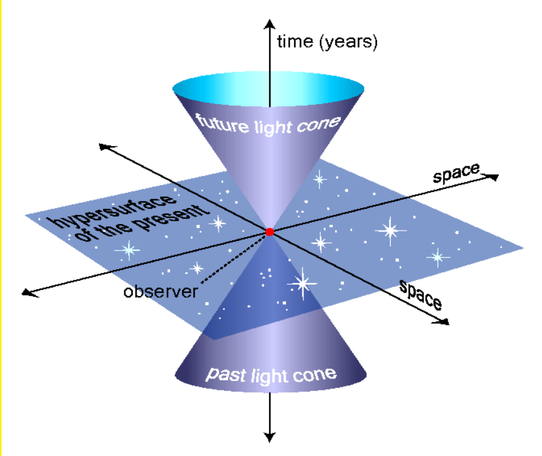
$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the case where this 4-vector is indeed the spacetime position of an event (t, x, y, z), compare with the expression for the Lorentz transformation in the previous lecture notes. Check that the matrix multiplication gives you the correct equations. For those of you liking linear algebra, the matrix multiplication can be thought of as a type of coordinate mapping between different coordinate systems (or reference frames) using the Lorentz transformation (Compare the upper part of this equation with matrices which are used to rotate between coordinate systems in two spatial dimensions, do you see a similarity? Remember the analogy used in the previous lecture notes between a coordinate change in the (x, y) plane and the (x, t) diagram).

A relativistic 4-vector cannot be any collection of four numbers, in order for a four dimensional vector to be a relativistic 4-vector, the components need to be (1) physical quantities and (2) these physical quantities need to transform from one frame of reference to another by the Lorentz transformation. If these two conditions are not fullfilled, the vector is **not** a 4-vector. We have so far only seen one example of a 4-vector: the position of an event in spacetime. We will soon see more examples, but first we need to learn some notation.

For components of a normal three dimensional vector, we use Latin letters, typically i and j, for the indices: The components of $\vec{x}$ are x_i where i goes from 1 to 3. For the components of a 4-

Fact sheet: An example of a light cone, the three-dimensional surface of all possible light rays arriving at and departing from a point in spacetime. Here it is depicted with one spatial dimension suppressed. In general, there are three types of curves in spacetime: 1) Time-like curves, with a speed less than the speed of light. These curves must fall within a cone defined by light-like curves. 2) Light-like curves, having at each point the speed of light. They form a cone in spacetime, dividing it into two parts. 3) Space-like curves, falling outside the light cone. (Figure: Wikipedia)



vector, we use Greek indices, typically μ and ν. The components of a four-vector $\mathbf{x}$ are x_μ where μ run from 0 to 3, 0 being the time component. If we wish to separate the time and space part of a four-vector we might also write it as $\mathbf{x} = (t, x_i)$ where x_i refers to all three spatial components.

The matrix multiplication (Lorentz transformation) introduced earlier can be written as

$$x'_\mu = \sum_{\nu=0}^{3} c_{\mu\nu} x_\nu,$$

where $c_{\mu\nu}$ is the matrix above. This is the equation which transforms any four-vector from one frame of reference to another. We will now write this equation using the so-called Einstein conventions which will be covered more thoroughly in future courses, but for now will save you from a lot of writing. Instead of writing the sum symbol, the Einstein conventions say that when two factors in a term contain the same index, there is an implicit sum over this index. If the index is Latin, then there is a sum over the three spatial dimensions, if the index is Greek, there is a sum over the three spatial dimensions plus time. Using this convention we can write the previous equation simply as

$$\boxed{x'_\mu = c_{\mu\nu} x_\nu \qquad (2)}$$

which is the formal mathematical definition of a 4-vector: as mentioned above, this equation, saying that a 4-vector transforms from one frame of reference to another, needs to hold for the vector to be a 4-vector.

It can be shown that four-vectors follow the normal rules for summations and subtractions (see exercise 2B.2). We will now look at the scalar product. For three dimensional vectors, the usual scalar product can be written as,

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x_i y_i = x_i y_i,$$

where the Einstein convention was used in the last expression. We can also define a scalar product for four-vectors. Instead of writing a dot between the vectors, one usually writes the scalar product with one upper index and one lower index,

$$x^\mu y_\mu = x_0 y_0 - x_i y_i.$$

One index μ is written high and the other low to show that this is the scalar product and *not* a normal sum. Note that the scalar product is defined with a minus sign in front of the spatial part. If we had written both indices low, this would mean,

$$x_\mu y_\mu = x_0 y_0 + x_i y_i,$$

using the Einstein summation convention. This is different from the scalar product. It should be clear where the minus sign comes from, consider a spacetime interval Δx_μ (a spacetime interval is an interval between two points x^1_μ and x^2_μ in time and space such that $\Delta x_\mu = x^1_\mu - x^2_\mu = (\Delta t, \Delta x, \Delta y, \Delta z)$). The scalar product of a spacetime interval with itself gives,

$$\Delta x^\mu \Delta x_\mu = \Delta t^2 - \Delta x^2 = \Delta s^2$$

(assuming $\Delta y = \Delta z = 0$). The result is the *scalar* Δs^2. A scalar is a quantity which is invariant, which has the same value in all frames of reference. We already knew that the spacetime interval Δs^2 is a scalar (where did we learn this?). For infinitesimal distances between events, we may write this as,

$$ds^2 = dx^\mu dx_\mu.$$

We learned above that a four vector is a vector which transforms according to the Lorentz transformation (equation 2) when changing from one frame of reference to another frame of reference having velocity v with respect to the first. This has an important consequence: You cannot choose 4 numbers on random, put them together and call it a 4-vector! The numbers entering in a four-vector need to be physical quantities which are such that the 4-vector transforms accoring to equation 2. We thus need to take care when performing mathematical operations with 4-vectors: The result may not necessarily be a 4-vector.

As an example we will now investigate what happens with a 4-vector when multiplying it with some random physical quantity. Say that you for some reason need to multiply a spacetime distance $\Delta x_\mu = (\Delta t, \Delta x, \Delta y, \Delta z)$ with the corresponding time interval Δt forming

$$\Delta u_\mu = \Delta t \Delta x_\mu.$$

Is Δu_μ a 4-vector? We can easily check this by checking whether it transforms according to equation 2 when changing frame of reference. We therefore need to find $\Delta u'_\mu$ as

$$\Delta u'_\mu = \Delta t' \Delta x'_\mu$$

and test if equation 2 is satisfied.

We know that Δx_μ follows this transformation. We also now that $\Delta t' = (1/\gamma)\Delta t$ when changing frame of reference. We thus have for $\Delta u'_\mu$ in a new frame of reference

$$\Delta u'_\mu = \Delta t' \Delta x'_\mu = (1/\gamma)\Delta t c_{\mu\nu} \Delta x_\nu = (1/\gamma) c_{\mu\nu} \Delta u_\nu.$$

Because of the factor $1/\gamma$ we see that Δu_μ does not transform according to equation 2 and Δu_μ is therefore NOT a 4-vector. We thus cannot multiply a 4-vector with a time interval and obtain a 4-vector.

A four-vector which is multiplied by a scalar however, is itself a four-vector. If instead of multiplying Δx_μ with Δt, we multiply it with the corresponding spacetime interval Δs we get

$$\Delta u_\mu = \Delta s \Delta x_\mu.$$

Transforming to a different frame of reference we have again $\Delta x'_\mu = c_{\mu\nu}\Delta x_\nu$ since Δx_μ is a four-vector and $\Delta s' = \Delta s$ since Δs is a scalar. We thus have

$$\Delta u'_\mu = \Delta s' \Delta x'_\mu = \Delta s c_{\mu\nu} \Delta x_\nu = c_{\mu\nu} \Delta u_\nu$$

which does follow equation 2. In this case Δu_μ is a four-vector. We thus have generally that when A_μ is a four vector and f is a scalar, the product

$$B_\mu = f A_\mu,$$

is a 4-vector.

3 Four-velocity

Can we define a four dimensional velocity V_μ, that is, a four dimensional vector showing the direction of motion in spacetime of a particle with coordinates x_μ? By analogy to normal three dimensional velocity, the four-velocity V_μ should be the the rate of change of the position vector x_μ. A natural choice would be dx_μ/dt, but this is not a four-vector: As we discussed above, Δt or dt is not a scalar, it has different values in different frames of reference. Thus dx_μ/dt does not transform as a 4-vector, i.e. you cannot use the Lorentz transformation to transform it from one frame of reference to another. But in order to have velocity, we need the rate of change with respect to some time interval Δt. Which measure of time can we use?

Remember that proper time τ is a scalar, it is defined as the time observed on the wristwatch of an observer. All observers will measure the same time interval $\Delta\tau$ between two events (how do they measure $\Delta\tau$?). Consider the example with the train and observer P who is jumping up and down. Measured on the wrist watch of observer P, one jump takes one second, thus one second of proper time for the frame of reference of the train. According to observer O's wristwatch, the jump takes 1.7 seconds, but this is not the proper time for the train (remember the definition of proper time!). But observer O can take his binoculars and read of the time between each jump on observer P's wristwatch. He will then find, in agreement with observer P, that in proper time units for the train, each jump takes one second.

Note that proper time needs to be defined with respect to some frame of reference (in this case the train), but once this is defined, everybody agrees on the proper time interval between two events taking place at the same spot in that frame. In the case of four-velocity, there is no doubt about which proper time we are speaking about: Four-velocity is the velocity of a particle or an object (for instance a train) and the proper time $\Delta\tau$ which we use to define four velocity is the time measured in the rest frame of this object. So four-velocity can be defined as

$$V_\mu = \frac{dx_\mu}{d\tau}.$$

Let us find the length (absolute value) of the four-velocity (the square root of the scalar product of the vector with itself). The square of the length is (as for normal vectors) given by

$$V_\mu V^\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{dx_\mu dx^\mu}{d\tau^2} = \frac{ds^2}{d\tau^2} = \frac{d\tau^2}{d\tau^2} = 1.$$

(did you understand every step here?) Taking the square root of this we still get 1. The length of the four-velocity is thus always one. Remember that a velocity of one means the velocity of light. All particles move with the velocity of light in space-time! For each proper time interval $\Delta\tau$ a particle moves an equal interval Δs in spacetime.

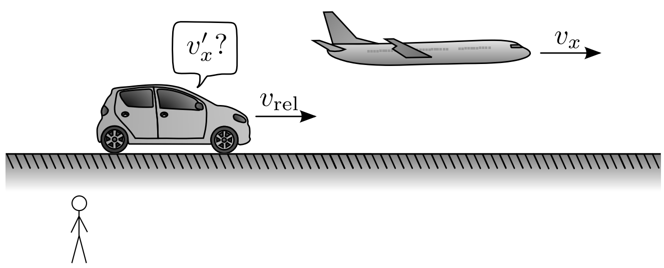


Figure 3: The observer on the ground measuring a velocity v_x for the airplane, wondering which velocity v_x' the driver of the car measures for the same airplane.

We can write the four-velocity in terms of normal 3-velocity as

$$\begin{aligned} V_\mu &= (\frac{dt}{d\tau}, \frac{dx_i}{d\tau}) \\ &= (\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{dx_i}{dt}) = \frac{dt}{d\tau}(1, \vec{v}) = \gamma(1, \vec{v}) \end{aligned}$$

where we have used the formula for time dilation $\Delta t/\Delta\tau = dt/d\tau = \gamma$ from the previous lecture notes (go back and check how you derived this, it is important!). Now we are ready to answer a question that has bothered us all the time since we learned about the Lorentz transformations: We know how to transform between coordinates (x, t) and (x', t') in two different frames of reference. But how do you transform a velocity v_x from one frame to the other? Say that you stand on the ground and look at a passing airplane. You measure the velocity of the airplane along the x-axis to be v_x. A car is passing you on the street with velocity $v_{\rm rel}$ along the same x-axis and you note that the driver is also watching the airplane. You start to wonder which velocity v_x' that the driver is measuring for the airplane. The situation is depicted in figure 3. In normal non-relativistic physics you know that the answer should read $v_x' = v_x - v_{\rm rel}$, but we have learned that this does not work for velocities close to the velocities of light (for instance, look back at the Michelson-Morley experiment). Assuming that there are no motions in the y and z direction, we can now write the four velocity of the airplane from our laboratory frame as $V_\mu = \gamma(1, v_x)$ and from the car as $V_\mu' = \gamma'(1, v_x')$ where $\gamma = 1/\sqrt{1 - v_x^2}$ and $\gamma' = 1/\sqrt{1 - (v_x')^2}$. We know that four-velocity is a four-vector and that four-vectors by definition transform from one frame of reference to the other under the Lorentz transformation,

$$V_\mu' = c_{\mu\nu} V_\nu,$$

or written in terms of matrices as

$$\begin{pmatrix} \gamma' \\ \gamma' v'_x \\ \gamma' v'_y \\ \gamma' v'_z \end{pmatrix} = \begin{pmatrix} \gamma_{\rm rel} & -v_{\rm rel}\gamma_{\rm rel} & 0 & 0 \\ -v_{\rm rel}\gamma_{\rm rel} & \gamma_{\rm rel} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

where $\gamma_{\rm rel} = 1/\sqrt{1 - v_{\rm rel}^2}$.

From this matrix equation, we obtain two equations for the velocity v_x and v'_x,

$$\gamma' = (\gamma_{\rm rel} - v_{\rm rel}\gamma_{\rm rel} v_x)\gamma$$
$$\gamma' v'_x = (-v_{\rm rel}\gamma_{\rm rel} + \gamma_{\rm rel} v_x)\gamma.$$

Dividing the second equation by the first, we obtain

$$v'_x = \frac{v_x - v_{\rm rel}}{1 - v_{\rm rel} v_x}, \tag{3}$$

which is the Lorentz transformation for velocities. Note that when the speed of the airplane approaches the speed of light, $v_x \to 1$ then $v'_x \to 1$ showing that the laboratory observer and the observer in the car will both measure the speed of light for the airplane. This solves the weird result obtained by Michelson and Moreley: The speed of light is the same from all frames of reference.

4 Relativistic momentum and energy

What about momentum and energy? We have learned that the velocity v of an object as measured from two different frames of reference transform according to the Lorentz transformation (equation 3). This must necessarily have consequences for how we measure momentum $p = mv$ and energy $E = 1/2mv^2$ from two different frames of reference. There must be some corresponding Lorentz transformations for momentum and energy. We have learned a simple and easy recipe for finding the transformation equations between different frames: Construct a four-vector and use the transformation properties for four-vectors. This worked for velocity so let's try with momentum and energy.

We start with momentum. In order to construct a four-vector P_μ for momentum, let's try a form which is as similar as possible to the Newtonian form $\vec{p} = m\vec{v}$. Rest mass (the mass measured in the rest frame of the object) is a scalar quantity, so

$$P_\mu = mV_\mu$$

is a four-vector. Using that $V_\mu = \gamma(1, \vec{v})$, we can write momentum as

$$P_\mu = m\gamma(1, \vec{v}) = \gamma(m, \vec{p}),$$

where $\vec{p}$ is the Newtonian momentum. Taking the spatial part of this equation we see that relativistic momentum can be written in three dimensions simply as

$$\vec{p}_{\rm relativistic} = \gamma m\vec{v}, \tag{4}$$

where $\vec{v}$ is the normal 3-velocity of an object. What is the meaning of the time component $P_0 = \gamma m$ of the momentum 4-vector? In order to investigate this let us write it in the Newtonian limit. For $v << 1$ (velocity much lower than the velocity of light) we can make a Taylor expansion in v,

$$P_0(v) = P_0(v=0) + \frac{dP_0}{dv}(v=0)v + \frac{1}{2}\frac{d^2P_0}{dv^2}(v=0)v^2,$$

where the derivatives taken at $v = 0$ are (check it!) $P_0(v = 0) = m$, $dP_0/dv(v = 0) = 0$ and $d^2P_0/dv^2(v = 0) = m$. We get

$$P_0 = m + \frac{1}{2}mv^2.$$

The last term is just the expression for Newtonian kinetic energy. The first term is the *rest energy* of a particle, converted to normal units it can be written as the more well known $E = mc^2$. The rest energy is the energy of a particle at rest, it is the energy in the mass of the particle. Thus, the time component of the momentum four-vector is relativistic energy,

$$E_{\rm relativistic} = m\gamma, \tag{5}$$

which in the Newtonian limit reduces to the Newtonian kinetic energy plus an energy term which did not exist in Newtonian physics, the energy of the mass of the particle. So the 4-vector P_μ is not just a momentum 4-vector, it is *the momentum-energy 4-vector* which time component is energy and space component is momentum. It means that energy and momentum are related in the same way as space and time are. In the same manner as we talk about spacetime, indicating that space and time are basically two aspects of the same thing, we can call energy and momentum collectively as *momenergy*. The four-vector P_μ is simply the *momenergy four-vector*.

What is the length of the momenergy four-vector? Using that $P_\mu = mV_\mu$ we have for the square of the length

$$P_\mu P^\mu = m^2 V_\mu V^\mu = m^2.$$

The length is the square root of m^2 which is m. The length of the momenergy four-vector is an invariant and it is thus simply the mass. We have seen that we can write $P_\mu = \gamma(m, \vec{p})$ giving (using equations 4 and 5)

$$P_\mu = (E_{\text{relativistic}}, \vec{p}_{\text{relativistic}}).$$

From now on we will drop the subscript 'relativistic' and always refer to the relativistic energy and relativistic momentum using E and p. But how can we be so sure? How can we know that this is the correct expression for energy and momentum? What is the criterion for a quantity to be energy or momentum? We know that energy and momentum are conserved quantities. The total energy and momentum of particles after a collision should always be the same as the total energy and momentum before the collision. So this is easy to check: Measure the total energy and momentum of particles before and after a collision, if they are the same we have found the correct expressions for momenergy. This has been tested thousands of times in particle accelerators with particles moving close to the speed of light. It turns out that the Newtonian energy and momentum are *not* conserved in these collisions. The relativistic energy and momentum defined as we have done above however, *are* conserved.

By now we have got used to measure time and space in the same units and therefore we have also got used to add these quantities $\Delta x + \Delta t$ without hesitating. We see that the result of measuring time and space in the same units is that momentum and energy are also measured in the same units, the units of mass. We remember that since space and time are measured in the same units, the speed v is a dimensionless number. The factor γ is clearly also dimensionless, so the momentum $p = m\gamma v$ can be measured in the units of mass (kg). The same goes for energy $E = m\gamma$, which also has dimension mass. So both energy and momentum are measured in kg and these quantities can therefore be added, just as we can add intervals in time and distances in space. The momenergy four-vector is $P_\mu = (E, \vec{p})$, taking the scalar product we have (remembering the result above that the length of P_μ is just m),

$$P_\mu P^\mu = E^2 - p^2 = m^2,$$

we can thus write energy in terms of momentum as

$$E = \sqrt{m^2 + p^2}.$$

A photon is massless, so for photons this relation is just

$$E = p,$$

or by using normal units $E = pc$ which is a more known form of this expression (In SI units, the energy of the photon can also be written in terms of the frequency ν or wavelength λ of the radiation as $E = h\nu = h/\lambda$).

We return to the above example with the airplane and the passing car. You measure the relativistic energy and momentum of the airplane from the laboratory frame (the ground) and you wonder what energy and momentum the driver of the car measures for the same airplane. The momenergy four-vector is a four-vector which means that it can be transformed from one frame of reference to the other by the Lorentz transformation,

$$\boxed{P'_\mu = c_{\mu\nu} P_\nu,}$$

or in matrix form (remember that there were no movements in the y and z direction)

$$\boxed{\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma_{\text{rel}} & -v_{\text{rel}}\gamma_{\text{rel}} & 0 & 0 \\ -v_{\text{rel}}\gamma_{\text{rel}} & \gamma_{\text{rel}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}}$$

Giving the following transformation equations for momentum and energy

$$E' = \gamma_{\text{rel}} E - v_{\text{rel}}\gamma_{\text{rel}} p_x$$
$$p'_x = \gamma_{\text{rel}} p_x - v_{\text{rel}}\gamma_{\text{rel}} E$$

where $v_{\rm rel}$ is the relative velocity between the two frames of reference, the observer on the ground and the car (see figure 4).

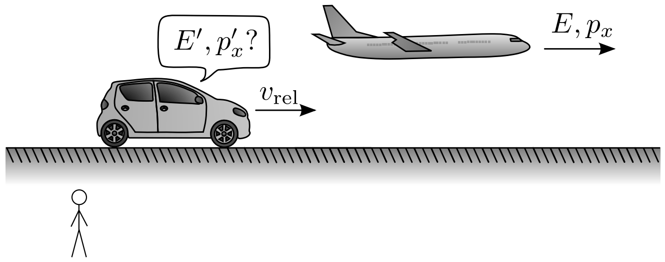


Figure 4: The observer on the ground measuring a velocity v_x for the airplane, wondering which velocity v'_x the driver of the car measures for the same airplane.

We will now use these equations to answer the following question: What energy and momentum (E', p'_x) does a person passing you in his car with a velocity v (relative to you) measure that you have? From your frame of reference in which you are at rest, your momentum is by definition zero $p = 0$ and you energy equals your mass $E = m$. We will now transform these quantities to the driver of the car measuring your energy and momentum to be E' and p'. The relative velocity of the car with respect to you is simply $v_{\rm rel} = v$. Then the energy and momentum that the driver in the car measures that you have is simply (using the equations above, check that you get the same result),

$$E' = \gamma E \qquad p'_x = -v\gamma E$$

Note that $\gamma > 1$ so the driver in the car measures, not only a larger absolute momentum, but also larger energy.

From the point of view of Newtonian mechanics this was to be expected: with respect to the driver you have a non-zero velocity and kinetic energy, thus both your momentum and energy are clearly larger with respect to him than with respect to your rest frame. But from the point of view of geometry it might seem strange: In your rest frame the four-vector P_μ only has a time component and no space component. In the frame of the driver, both the time and space component of the vector are larger than in your frame. But the length of the momenergy vector P_μ is always the same, equal to m. Going back to normal 3D geometry this would not be possible. Imagine a vector $\vec{a} = (f, g, 0)$ and another vector $\vec{b} = (2f, h, 0)$. If the length of these vectors are the same, then we have that $h < g$. We see that from normal geometry you would expect that if the length of a vector is constant, then if you increase one component of the vector the other should decrease. The reason for this discrepancy with normal geometry is that spacetime has Lorentz geometry whereas 3D space has Euclidean geometry. Lorentz geometry has a minus sign in the definition of the scalar product (which also defines the length of the vector) making such an effect possible.

Now you know the basics of the special theory of relativity and you have got the necessary preparations to start studying the general theory of relativity. In the general theory of relativity we will study how masses curve spacetime, making the expression for the line element Δs different close to a large mass. This change in the line element changes the dynamics and gives rise to what we in Newtonian terms call the force of gravity.

5 List of expressions you should know by now

Worldline	→	page 2
Timelike	→	page 2
Lightlike	→	page 2
Spacelike	→	page 2
Principle of maximal aging	→	page 3
Wristwatch time	→	page 3
Scalar	→	page 6
Four vector	→	page 6
Four velocity	→	page 7
Momenergy	→	page 9

6 Exercises

Exercise 2B.1

Relevant theory: Section 1.
Go to MCAst and load the xml corresponding to this exercise. In this exercise it is recommended to be three students working togehter: There are three frames with one xml for each frame and student. Choose who does which frame, and only look at the video for your frame!

In this exercise there are three spaceships traveling with different velocities with respect to a space station. The different frames of reference in the videos correspond to the frame of the space station, ship 1 and ship 2. The ships 1 and 2 both travel with constant velocity while ship 3 accelerates as seen from the space station. We are not interested in exact numbers in this exercise, only roughly correct relative distances and slopes on the worldlines showing that you have understood the basic principles.

1. **Looking only at the video for your frame of reference** try to imagine how the ships and space station move in the frames of the other two students.
2. Still without looking at the other videos, draw 3 spacetime diagrams: One for each frame of reference, your frame as well as the frames of your two fellow students. In these three diagrams, draw the worldlines of the space station as well as ship 1, 2 and 3 (the accelerated ship).
3. Now meet with your fellow students and compare the diagrams. Do they agree?
4. Look at all the videos togehter and check if the other videos look as you imagined: discuss why you were right/wrong.
5. Draw a spacetime diagram in the reference frame of ship 3 (no video here) with worldline for all objects.

Return to the spacetime diagram for the space station frame, we will only work with this diagram for the rest of the exercise. We now define two events:

- Event 1 occurs at $x = 0$ and $t = 0$ is when all the spaceship are aligned.
- Event 2 is defined as when spaceship 3 catches up with ship 2 reaching the same position.

Measured on the clock in the space station it takes 10 milliseconds between the two events, on the clock in the frame of ship 2, it takes 8 milliseconds. Assume that the clocks make a tick every millisecond. The first tick happens at event 1 and the last tick happens at event 2.

6. Draw dots on the time axis between event 1 and 2 which represents the ticks in the space station frame.
7. Draw dots on the worldline of ship 2 based on the ticks which occurs in the frame of ship 2. The important point here is to have correct relative spacings between each tick.
8. Spaceship 3 has also been equipped with a clock identical to those in the space station and ship 2. Use the principle of maximal aging to judge whether an astronaut in ship 3 will experience more or less ticks on the clock from event 1 to event 2 compared to the astronaut in ship 2.
9. Draw dots on the worldline of ship 3 at the positions where the clock ticks in this frame. The exact position is not important, but the relative distances between the dots should be correct. **Hint:** For each dot you draw, look at the slope of the worldline.

Exercise 2B.2

Relevant theory: Section 2.
In this exercise we will take a closer look at 4-vectors. More specifically we will prove that a 4-vector follows the regular rules of addition and subtraction.

1. Explain with your own words what a 4-vector is.
2. What transformation must be fulfilled for a four vector to be called a 4-vector? Write down the mathematical definition.
3. What is the criterion for the transformation to be successful (what needs to be trans-

formed)?

4. Assume that A_μ and B_μ are 4-vectors. Prove, by using the mathematical definition and the criterion, that D_μ which is the sum of the two 4-vectors, $D_\mu = A_\mu + B_\mu$ is also a 4-vector. **Hint:** Use the transformation properties of A_μ and B_μ to obtain these vectors in a different frame A'_μ and B'_μ. Find an expression for the sum of the two vectors, D'_μ, in the other frame expressed by D_μ in the laboratory frame and show that D_μ is indeed a four vector.

Exercise 2B.3

Relevant theory: Section 1 - 3.
Go to MCAst and load the xml corresponding to this exercise, you and your partner should agree on who does which frame.

A spaceship is moving with a speed close to the speed of light and emitting two laser beams. This is seen from the frame of reference of the planet and the spaceship.

1. **For planet frame student:** use the information given at the top of the video to find the velocity of the space ship and the light beam. Check that the beam has indeed the speed of light. Given that the light beams are emitted from the spaceship, what speed would you have expected the light beam to have in your frame if you rely on classical physics? (remember how you transform velocities between frames in classical physics.) **For the spaceship frame student:** use the information given at the top of the video to find the velocity of the light beams and your velocity with respect to the planet. Check that the beam has indeed the speed of light. Which velocity would you expect observers on the planet to measure for the light beams if you rely on classical physics? (remember how you transform velocities between frames in classical physics.)

2. Make a space-time diagram of the space ship and the two laser beams in your frame of reference as well as in the other frame of reference.

3. Imagine how the scene look like in the other frame. Focus in particular on the relative velocity and distances between beams and space ship: imagine and describe how the movements look in the other frame.

4. Use the formula for relativistic transformation of velocities, to calculate the velocity of the laser beam in the other frame. Is the result as expected?

5. Now you should meet and compare videos. Does the other video look as expected? Why? Why not?

6. Calculate the distance between the two light beams in your frame and compare with the distance calculated by your partner. Find the ratio of the distances between the two frames.

7. Imagine a stick with it's start and end point at the position of the two beams: Compare the ratio you found between the two distances/stick lengths to what you expect from the formula for length contraction ($L = L_0/\gamma$). Does the formula for length contraction apply in this case? Why not?

Exercise 2B.4

Relevant theory: Section 1-4.
A free neutron has a mean life time of about 12 minutes after which it disintegrates into a proton, an electron and a neutrino. We will ignore the neutrino here, assuming that the only products of disintegration are a proton and an electron. A neutron moves along the positive x-axis in the laboratory frame with a velocity close to the velocity of light. It disintegrates spontaneously and a proton and an electron is seen to continue in the same direction as the neutron. We will try to calculate the speed of the proton and the electron in the lab-frame. The easiest way to do this is in the rest frame of the neutron where the neutron has a very simple expression for energy and momentum. In the lab frame this would have been a lot more work since all three particles have velocities. On our way we will discover a surprising fact!

Go to MCAst and load the xml files corresponding to this exercise. In this exercise you may work alone if you wish and contrary to other exercises you are supposed to look at both frames now before starting to calculate. The videos will show the velocity of the neutron seen from the lab(planet) frame as well as the masses of the particles. **Important:** It is necessary to use all the given decimals of the particle masses when making calculations in this exercise.

1. Write an expression for the momenergy four-vector $P'_\mu(e)$ of the electron in the frame of the neutron expressed in terms of m_e and the unknown velocity v'_e. You may define $\gamma'_e = 1/\sqrt{1-(v'_e)^2}$.
2. Write an expression for the momenergy four-vector $P'_\mu(p)$ of the proton in the frame of the neutron expressed in terms of m_p and the unknown velocity v'_p. You may define $\gamma'_p = 1/\sqrt{1-(v'_p)^2}$.
3. Write an expression for the momenergy four-vector $P'_\mu(n)$ of the neutron in the frame of the neutron expressed in terms of m_n and v'_n.
4. Use conservation of momenergy
$$P'_\mu(n) = P'_\mu(p) + P'_\mu(e),$$
to find expressions for the velocities v'_e and v'_p (the velocities of the electron and proton in the neutron frame). If you are observant you will discover that there are two sets of possible soultions (why do you think this is the case?), in the rest of the exercises choose one of the solutions. **Hint:** The algebra in this exercise can be extremely ugly if done wrong, here are some tips:
   - Solve the equations for γ'_e or γ'_p **NOT** for v'_e or v'_p.
   - You often insert $\gamma'^2 = 1/\left(1-v'^2\right)$ but you should instead insert $v'^2 = 1-1/\gamma'^2$.
   - If you end up with something looking like it will be a quadratic equation try writing it out, you should end up with a first order equation.
   - If you have an equation $a\sqrt{b}+c\sqrt{d}=0$, show that it may be written as $a^2b = c^2d$.

   You should arrive at:
$$\gamma'_p = \frac{m_n^2+m_p^2-m_e^2}{2m_pm_n}. \qquad (6)$$
   Find a number for γ'_p and use this to obtain numbers for v'_p, γ'_e and thereby v'_e.
5. Use the transformation properties for four-vectors
$$P'_\mu = c_{\mu\nu}P_\nu$$
to find the energy and momentum of the electron and the proton in the lab (planet) frame. (insert numbers: what units do your results have if you keep $c=1$?).
6. Use the numbers you have obtained for energy or momentum to obtain the speed of the electron and proton in the planet frame.
7. You might have observed that $m_n \neq m_p + m_e$, ie. that mass is not conserved in this process. Rather, parts of the mass has been converted to energy. In the lectures on nucelar reactions, this result will become very important. We will now show that it is impossible to conserve mass in this process, similar to most other nuclear processes: Now assume for a moment that mass is indeed conserved and $m_n = m_e + m_p$. Using only symbols, not numbers, insert this in the expression 6 for γ'_p above. Use this to obtain v'_p and v'_e. Explain why mass could not be conserved in this process.
8. As an independent check (and to see an alternative way of doing it), use the relativistic formula for addition of velocities to obtain the speed of the two particles in the laboratory frame, using only the speed you have obtained for the proton and electron in the neutron frame as well as the speed of the neutron in the planet frame.
9. For those who like long and ugly calculations only: Do everything from the beginning, but use only the planet frame to obtain the same results. Do you see the advantage of using 4-vectors and change of frames?

Exercise 2B.5

Relevant theory: Section 1 - 4.
Go to MCAst and load the xml corresponding to this exercise, you and your partner should agree on who does which frame.

Some crazy researchers have decided to test special relativity. They prepare two identical space ships to travel towards each other with a velocity close to the speed of light. Eventually they collide close enough to be seen from a planet. One of the spaceships is made solely from anti-matter including the researcher, therefore all matter is converted to photons in the collision. We assume all the photons have exactly the same wavelength. During this exercise we therefore have two objects:

- The leftmost spaceship is denoted by the letter A and is traveling with velocity v_A with respect to the planet frame.
- The rightmost spaceship is denoted by the letter B and is traveling with velocity $v_B = -v_A$ (same speed as A but opposite direction) in the planet frame.

To measure the wavelength of the photons there are two observers (you and your partner) both with a wavelength detector. The observers are in two different frames of reference:

- The frame 1 observer is at rest on the planet using unprimed coordinate system.
- The frame 2 observer has a small red space ship (you can see him/her in the frame 1 video) and follows just behind spaceship A in the same frame as spaceship A using primed coordinate system.

All the necessary information including the mass of the spaceships and number of photons produced are given in the upper left corner in MCAst.

The main goal of this exercise is to use the transformation of momenergy 4-vectors to deduce a relativistic formula for Doppler shift.

1. Use the velocity of spaceship A with respect to the ground to calculate the relative velocity of spaceship B observed from spaceship A (transform the velocity with your preferred method). Remember that with respect to the ground, the spaceships have equal speed (but opposite directions).
2. Write down expressions for the momenergy four-vectors $P_\mu(A)$ and $P_\mu(B)$ of the two spaceships in your frame of reference. You may use v_A, v_B, v'_A, v'_B, the corresponding gamma-factors $\gamma_A, \gamma_B, \gamma'_A, \gamma'_B$ and the mass m of the spaceships.
3. Use the transformation properties of four-vectors to transform momenergy four-vectors from your frame of reference to the other frame. You should now have $P_\mu(A)$ and $P_\mu(B)$ in the planet frame as well as $P'_\mu(A)$ and $P'_\mu(B)$ in the space ship frame.
4. Show that the momenergy four-vector of a photon traveling in the positive x-direction can be written
$$P_\mu^\gamma = (E, E, 0, 0),$$
where E is the energy of the photon.
5. Assume for the moment that all the energy in the explosion is emitted in only two photons, one emitted along the positive x-axis (same direction as the space ship) and the other in the opposite directions (negative x-direction). Use conservation of momenergy in the planet frame to argue that the two photons must have the same energy seen from the planet frame.
6. Now assume that these two photons are emitted with an angle θ off the x-axis. Write the momenery 4-vector for this photon and use again conservation of momenergy in the planet frame to argue that the two photons must have the same energy but opposite directions seen from the planet frame.
7. Use your previous results to argue that if there are photons emitted in all possible directions, there is, for all photons emitted always another photon emitted in the opposite direction with the same energy.(remember that we assume that all photons produced in this explosion have the same energy)
8. Using the assumption that all photons are emitted with the same wavelenght, and using also the number of photons measured, what is the energy of one photon (use conservation

of momenergy!) and thereby the wavelength in the planet frame? (**Hint:** To convert photon energy to wavelenght, it may be useful to first convert the energy you found to normal SI units.

9. Use the wavelength to find the color of the explosion seen in the planet frame (use the table in this Wikipedia article.)**Planet frame observer:** does it correspond to the colour you observe?

The collision between the spaceships was inspired by a process which happens in nature. An electron and a positron are corresponding antiparticles with equal mass. The particles are approaching each other with the same velocity in opposite directions in the center of mass frame of the two particles. In the collision, both particles are annihilated and **two** photons are produced. One photon travels in the positive x direction, the other in the negative x direction. In the rest of the exercise we will therefore only study the photons which move along the x-axis:

9. We will now study only the photons which move along the x-axis. Use transformation properties for four-vectors to show that the energy E' of a photon observed in the spaceship A frame moving with velocity v with respect to the planet frame (where the photon has energy E) is

$$E' = E\gamma(1 \pm v)$$

(Which sign is for the photon moving in positive x-direction and which is for the photon moving in negative x-direction?)

10. Use the derived expression for E' and the formula for the energy of a photon to derive the relativistic Doppler formula

$$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right)$$

11. Use this expression to find which colour the explosion has for the observer in the ship frame. Finally, meet with the other student and look at both videos to check your results. Did you calculate the colour in the other frame correctly?

12. Show that the relativistic Doppler formula is consistent with the normal Doppler formula for low velocities. **Hint:** Make a Taylor expansion of $f(v) = \sqrt{(1+v)/(1-v)}$ for small v. (**Remember:** how is the non-relativistic Doppler-formula using relativistic units where $c = 1$?)

13. Going back to the more realistic scenario: Assume the electron and the positron in their center of mass frame have the same velocities as your space ships in the planet frame (which is the center of mass frame of the ships). Which velocity would the center of mass of the electron and positron need to have with respect to the laboratory frame in order to observe a photon with the same colour as seen in space ship A frame in the above questions? (assume the center of mass is moving towards you)