AST 2000 - Part 8
Special Relativity

Welcome to Part 8 of the AST2000 Project. The aim of this part is to develop a firm understanding of special relativity. To do this you will address two fundamental topics in relativistic mechanics: the Twin Paradox and conservation of momenergy.

RELATIVITY EXERCISES

You have now landed on your destination planet and you start making relativity experiments on the surface of the planet. Before you continue reading, you now need to read section 7 of lecture 2A remembering that you are supposed to generate the xml files yourself. Refer to the documentation for the `ast2000tools` package to learn about the `RelativityExperiments` class that you will use to generate relativity xml files for MCAst.

For part 8 and part 9 you need to have a partner student for some of the exercises. Even if you work in a group, you need a partner group or partner student. You may do everything alone, but it requires more work. For the exercises where you need a partner and each partner has her/his own frame of reference, you are only supposed to report on the exercises relevant for your frame (for the exercises which are different depending on the frame).

You will create a set of videos of the experiments you are performing on the surface of your planet. You will need to decide between you and your partner (or your group and your partner group) on which planet you will do the experiments: the planet where you have landed or the planet where your partner has landed. You may also do some experiments on your planet and others on your partners planet, but it is very important that for the same experiment, you use the same planet for all frames of reference.

In part 8 and 9 you are not supposed to make a scientific paper. Instead your task is to make an educational text, similar to what the bloggers have been doing in previous parts. Some of the exercises (mostly those including calculations) have been marked ☺ which means that you can do these on paper like you do for the exams (no need for much text, just enough to understand what you are doing), scan these and deliver these as separate files. It is only to see that you have actually done it. The ones marked ∇ are the ones on which you mainly will be evaluated. These will normally contain a question asking you to summerize what you did in the previous questions with words or they are questions testing your understanding. **For each exercise separately, you should make a short text which includes the following sections:**

1. **Introduction:** This part should shortly introduce the problem and make the reader being able to understand the main question(s) you want to solve without having read the exercise text.
2. **The situation**: what are the objects/events in this exercise and what are the main questions you would like to test/answer (describe this with words) using these objects/events. When relevant you may also include a table with events and positions here.
3. **Method:** Use text only to describe the main physiocal principles you were using when solving this exercise and why these principles are relevant. Then, you may include some very few basic equations here with words explaining why these are relevant and how you will use these to solve the problem. (you should not do the calculations here!)
4. **Conclusions:** shortly summerize the conclusions, what did you find and in particular: what was the purpose if this exercise, what are you supposed to learn? What is the physics behind your results? Somebody who has not seen the exercise should be able to understand.
5. **Specific questions:** If they are not already included in the above points: Here the answer to the remaining ∇ questions should be included in one fluent text. In most cases, these are already naturally answered in some of the other sections above in which case there is no need for this section.

You will be evaluated in the following way:

- The scanned hand-ins (or latex-typed if you prefer) of the ☺-exercises will count 30%
- The Introduction and Conclusion sections together will count 20%
- The "situation" section will count 25%
- The Method section will count 25%
- The "specific questions" section will be included in the other relevant points.

In part 8, each exercise will be given the same weight, **except** exercise 3 on the twin paradox:

- part 1-4 of exercise 3 will together count as one exercise
- part 5 of exercise 3 will count as 2 exercises

If you do all of exercise 3, this will thus count as 3 exercises. The total score will be the mean of all 7 exercises, but weighted in the following way:

$$\text{totalscore} = \frac{x_1 + x_2 + x_{3a} + 2x_{3b} + x_4 + x_5 + x_6 + x_7}{8},$$

where x_i is the score of exercise i, x_{3a} corresponds to exercise 3, part 1-4 and x_{3b} corresponds to exercise 3, part 5. Moreover, **if you have done at least one exercise (on which you got at least 75% right) among exercise 1-5, and at least one exercise (on which you got at least 75% right) among exercise 6-7, your total score for part 8 will be multiplied with 1.1 before rounding up**.

EXERCISE 1

Relevant theory: Section 1 - 4 of lecture notes 2A.
In this exercise there are a total of two frames of reference. There are therefore two xml files, one for each frame, and all the necessary information will be given in the upper left corner for each frame. Generate the xml files by calling the `spaceship_duel` method. In this exercise you do not need a partner. In the beginning you are only supposed to look at frame 1 until further notice. The two frames of reference correspond to:

- The frame of reference of the spaceships with primed coordinate system (x', t'), the leftmost spaceship is always at origin $x' = 0$.
- The frame of reference of the planet with unprimed coordinate system (x, t).

The idea behind this exercise is to make you rediscover what Einstein discovered: that the invariance of the light speed must imply that simultaneous events are not simultaneous in all frames. To do this we use following problem:

Two spaceships are moving with equal speed with respect to the ground. In the frame of the spaceships, both spaceships simultaneously shoot a laser beam towards the other (event A and B, left spaceship shooting is event A). When the laser beams hit, the ships explode creating two more events (event C and D, leftmost explosion is event C). In this exercise, we will study these 4 events from two different frames of reference.

Event A, the emission of the laser beam from the leftmost spaceship, takes place at $x = x' = 0$ and $t = t' = 0$ in both frames.

Part 1 ▽

In part 1 until further information is given special relativity is for you an unknown concept, but you do know that the velocity of light is the same for all observers (as has been shown empirically).

The two spaceships are firing laser beams simultaneously in the spaceship frame. Stationed perfectly in the middle between the two spaceships we have observer M(iddle). Observer M, as the spaceships, is at rest in the spaceship frame.

1. Why will observer M observe events A and B simultaneously? (Note that we know that the events are simultaneous for observer M since she is in the same frame of reference as the spaceships, the question is why she will also *observe* these events simultaneously?) Check in MCast that this is really the case.
2. Now we will try to figure out what happens in the planet frame without having looked at the planet frame video: Using that
   - we know observer M sees the two light beams crossing just at her position (why must this be the case in both frames?),
   - the fact that observer M is in the middle between the spaceships,
   - that the laser beams were emitted simultaneously in the space ship frame,

   we can conclude that in the planet frame, the laser beams where not emitted simultaneously, but at two different times. Why? Try to think how the spaceship and observer M are moving while the laser beams are emitted.
3. In order for the laser beams to cross exactly at the position of observer M, which of the laser beams must have been emitted first in the planet frame? Why? Try to imagine the movements of the space ships and the laser beams in the planet frame.
4. In the spaceship frame the explosions are simultaneous, is this still the case in the planet frame? Again, in the spaceship frame, the lights from the explosions will reach observer M who is stationed in the middle, simultaneously.
5. Which explosion occurs first in the planet frame? **Hint:** think twice before answering, the correct answer might not necessarily be the first idea that comes to your mind.
6. Now order events A, B, C and D in chronological order in the planet frame. Write a short summary of why this has to be the case. Imagine how this will look. Then **only after you have really tried to imagine how this looks**, look at the video for the planet frame.

Part 2

In the second part of this exercise, we will now again imagine that we do not know about length contraction and time dilation. Our goal now is to try to imagine being Einstein when he just discovered relativity. Using only the fact that the speed of light is the same in both frames, we will try to arrive at the expression for time dilation using the situation with the spaceships and pure reasoning. Be prepared that we might not quite arrive though.

In the following we will not use numbers, only symbols for times, positions and distances in the planet frame.

1. ☺ Write equations for the positions of (1) the leftmost spaceship, (2) the observer in the middle, and (3) the light beam emitted from the leftmost spaceship as a function of time t, velocity v, the time t_A and the distance L. Remember that at time $t = 0$ (which is the origo event), the position of the leftmost spaceship is $x = 0$.

2. ☺ Use the fact that at the time t_M (when the two beams cross at the position of the middle observer), the position of the middel observer equals the position of the beam emitted from the leftmost spaceship, to show that
$$t_A = t_M - \frac{L/2}{1 - v}.$$
(is the light beam emitted before, at or after the origo event?)

3. ☺ Write also the position of the beam emitted from the rightmost spaceship as a function of time t expressed in terms of the time t_M, L and v. Use this equation and the fact that at time t_C, the position of the leftmost spaceship equals the position of the beam emitted from the rightmost spaceship to show that
$$t_C = t_M + \frac{L/2}{1 + v}.$$

4. ☺ From an observer at the planet, how long time Δt does it take from the beam is emitted from the leftmost spaceship at time t_A to the time the leftmost spaceship explodes at time t_C? Express the answer in terms of L and v only. What is the corresponding time $\Delta t'$ in the frame of the spaceships?

5. ∇ Summarize in 5-6 sentences what you just did (and why) in the ☺-questions above, then answer the following question continuing from above: Clearly these time intervals are different, which should come as a surprise given that you do not know anything about relativity. This shows that time needs to run differently in the two frames, or could there be a different solution to this discrepancy?

6. ∇ What is the ratio between $\Delta t'$ in the spaceship frame and Δt in the planet frame? Does it look similar to the expression for time dilation in special relativity? Why is it different? Having our (wrong) assumptions in mind, and knowing the real formula for time dilation, could you actually have guessed this result?

EXERCISE 2

Relevant theory: Section 1 - 4 of lecture notes 2A.
In this exercise there are a total of two frames of reference. There are therefore two xml files, one for each frame, and all the necessary information will be given in the upper left corner for each frame. Generate the xml files by calling the `cosmic_pingpong` method. You do not need a partner for this exercise. In the beginning you are only supposed to look at frame 1 (spaceship frame), information will be given when you are allowed to look at frame 2. The two frames are:

- The frame of reference of the spaceships with primed coordinate system (x', t'), the leftmost spaceship is at origin.
- The frame of reference of the space station (and planet) with unprimed coordinate system (x, t), the space station (shown as a white disc) is at the origin.

In this exercise we will play cosmic ping-pong with a laser beam. Two spaceships with equal velocity moving to the left with respect to the space station are located at a fixed distance L' (spaceship frame) apart. Both spaceships are equipped with mirrors which enables them to reflect laser beams. The leftmost spaceship emits a laser beam which results in the following events:

- Event A which is the emission of the laser beam at $t = t' = 0$ at the position $x = x' = 0$.
- Event B which is the first reflection from the rightmost spaceship.
- Event D which is the when the laser reflected in event B reaches the leftmost spaceship and is reflected again.
- Event C which is a random explosion that happens on the space station simultaneously with event B in the spaceship frame.

Your task in this exercise is therefore to denote the time differences between the reflections in the space station frame, some general intuition of how the scene looks in the other frame and why, as well as the difference between the relativistic and the non-relativistic case. We will start with the visual understanding, therefore in the first questions you are only supposed to do reasoning, no calculations.

1. ∇ We start by comparing the time it takes for the laser beam to go from left to right, compared to the time it takes to go from right to left. In the spaceship frame, which time interval if any is the largest $\Delta t'_{\mathrm{AB}}$ or $\Delta t'_{\mathrm{BD}}$? Why?

2. ∇ Now try to imagine the whole scene in the space station frame: remember that the speed of light is

invariant and that the space ships move at a speed close to the speed of light. How are the space ships and laser beam moving? Try visualizing.

3. ▽ Which time interval will therefore be the largest Δt_{AB} or Δt_{BD} in the space station frame? Why?

4. ▽ Do not yet look at frame 2. There is however a third xml-file: For visualizing how the time intervals change depending on the velocity of the spaceships watch frame 3. Here the ships will increase velocity for every second reflection until they reach a velocity of $0.8c$ (note that in this illustration video, the ships move in the opposite direction).

Let's now look at the same situation only non-relativistic. Suppose the laser beam is now a ping pong ball moving back and forth always at 80km/h with respect to the spaceships and the spaceships are moving at 50km/h with respect to the planet. As with the light beam, the ping pong ball is always moving with the same velocity with respect to the space ships. Therefore, one should think that the same argument as in the previous questions is valid in the space station system: When the ball is moving to the right it moves towards the spaceship which is approaching the ball. When it moves left, it moves towards the spaceship which moves away from the ball. As with the light beam, it should therefore take longer going left than going right.

5. ▽ It looks as if the two different observers will observe different travel times for the ball, just as for the light beam. Can this really be the case? If not, where is the error in the argument? Did we use an important principle from relativity which is not applicable in this case?

6. ▽ Going back to the relativistic case with the light beam: Decide whether event C or B happens first in the space station frame.
Hint: To solve this exercise, imagine an object located in the middle between event B and C with velocity equal to the spaceships. How will the light from the explosion in event C and the reflection from event B pass this object? You will see that this situation corresponds to the one from the previous exercise.

7. ▽ Now try to visualize how the video for frame 2 will look like. In particular, think about the order of the event and the positions of the spaceships and space station during the events.

8. ▽ Look at the video for frame 2. Does it look like you imagined?

We are now done with the visualization, and from here on out we will calculate the exact times of the events.

9. ☺ Write down the time t' and position x' of all events in the spaceship frame. Use these to find the distance L' between the spaceships as well as the time intervals between the first reflections, $\Delta t'_{\mathrm{AB}}$ and $\Delta t'_{\mathrm{BD}}$ in the spaceship frame. It is convenient to convert all numbers to time units, milliseconds will make reasonable numbers.

10. ☺ Now our task will be to find Δt_{AB} and Δt_{BD} in the space station frame: We will do this step by step in the following questions. Start by writing down the positions and times of events in the space station frame. Some positions may be expressed through the velocity and the unknown time of an event in order to reduce the number of unknowns. The only unknowns should be x_B, t_B, t_D and t_C, other unknown positions and events should be written in terms of these.

11. ☺ Write the spacetime intervals Δs_{AB} and $\Delta s'_{AB}$ between events A and B in the two frames. Show that invariance of the interval gives $x_B = t_B$ in the space station frame. Could you have guessed this using physical arguments without any calculations?

12. ☺ Write the spacetime intervals Δs_{AC} and $\Delta s'_{AC}$ between events A and C in the two frames. Show that invariance of the interval gives a number for t_C in the space station frame.

13. ☺ Write the spacetime intervals Δs_{BC} and $\Delta s'_{BC}$ between events B and C in the two frames. Show that invariance of the interval gives t_B in the space station frame.

14. ☺ Use invariance of the spacetime interval for appropriate events to find at what time t_D event D happened in the space station frame.

15. ☺ In the space station frame, how long time did it take from the light was emitted to the first reflection?

16. ☺ How long time did it take from the first reflection to the second reflection?

17. ☺ Which event happened first in the space station frame, event B or C? Is it consistent with you reasoning above?

18. ▽ Summarize in 5-10 sentences what you just did (and why) in the ☺-questions above, in particular you should describe the principles behind what you did.

EXERCISE 3

We will finish this part on special relativity by studying the twin paradox in detail. This long and detailed exercise is very important to gain some basic understanding for the underlying physics of many of the so-called paradoxes in the theory of relativity. **Note that you should**

not make the xml files for this exercise yourself, you need to use the standard ones which are common for all students and which you can find on the course web page for the standard variant of the course! There are three xml files (corresponding to exercise 2A.8) for this exercise and you should be three students doing this exercise together: you need to meet already from the beginning and do everything together. **Please note that you will really loose many important points if you do this exercise alone, in particular it is important to be able to see the different videos at the same time without having to switch continuously between xml files.**

Astronaut Lisa is traveling from her homeplanet Homey to another planet Destiny located 200 light years away. She travels in her spaceship Apollo-Out with velocity $v = 0.99c$. Important note; the planets do **NOT** move with respect to each other and are therefore in the same frame of reference . To begin with we therefore have two frames of reference:

- The frame of reference of the planets with unprimed space-time coordinates (x, t). Homey is always at origin with Destiny located 200 light years away long the positive x-axis.
- The frame of reference of Apollo-Out with primed space-time coordinates (x', t'). Apollo-Out is always at origin in this frame.

We also have two events:

- Event A occurs at $x = x' = 0$ and $t = t' = 0$ and is when Apollo-Out is departing from Homey.
- Event B is when Apollo-Out arrives at Destiny.

Part 1

Before we can truly start on the paradox we need to get some basic math done first.

1. ☺ How long does the trip from Homey to Destiny (event A to B) take for observers on Homey? How long does it take measured on Lisa's clock (use the formula for time dilation)?
2. ☺ After arriving on Destiny, Lisa quickly starts the return flight. She travels with exactly the same velocity $v = 0.99$ back towards Homey. Use the same arguments (or symmetry arguments) to find the time Δt and $\Delta t'$ it took from Destiny and back to Homey in the two frames of reference.

If you have done your calculations correct, here is a summary of the situation, the whole trip took 404 years measured on Homey-clocks, while it took 57 years measured on Lisa's wrist watch.

Part 2

During this part and **ONLY** this part we will switch frames, this is to uncover the paradox. The laboratory frame (x, t) is now the frame of Apollo-Out and the moving frame (x', t') is the planet frame. Because of the principle of relativity we are allowed to switch the roles and should still arrive at exactly the same result using the same laws of physics.

From Lisa's point of view, event A can be viewed as Homey departing from the spaceship with $v = 0.99c$, and event B is Destiny arriving at Apollo-Out with velocity $v = 0.99c$. Remember from part 1 where you calculated that it took Lisa 28.5 years to arrive on Destiny.

1. ☺ Use time dilation again (and make sure not to confuse Δt and $\Delta t'$, check who is the observer 'at rest' here) to show that the clocks on Homey at the moment when Destiny arrives at Lisa's position show 4 years. Above you showed that 202 years had passed. Now, this might look like a paradox, but we will show further down that it is not. No matter how strange this might sound, it is consistent. The paradox is still to come.

Quickly after Destiny arrives at Lisa's position, Destiny departs and Homey approaches you again with a velocity of $v = 0.99c$. From earlier calculations you know that this trip took 28.5 years for Lisa.

2. ☺ By using time dilation (or symmetry) how long does it take in the planet frame for Homey to reach the astronaut?
3. ▽ Make a short summary (a few sentences using words and numbers, no equations) of everything you did so far, which principles and assumptions did you use and what are the numbers you arrived at. Pay special attention to the consequences of your numbers, how much did Lisa age and how much did her freinds on Homey age? Expain what the paradox is which should now be evident.(that is: you should explain what is the paradox here, **not** the solution to the paradox, that's still to come)

Clearly we made an error somewhere in the calculations. Or maybe we simply forgot some basic principles from special relativity? It appears at first sight that the two roles (laboratory frame at Homey or in Apollo-Out) are equal, that we can choose whether we consider the planet frame as the laboratory frame or the Apollo-Out frame as the laboratory frame.

4. ▽ Are the two roles really identical? If not what is the difference?

Don't read on until you have found an answer to the previous question. Here comes the solution: The difference is that whereas the observers on Homey always stay in the same frame of reference, Lisa changes frame of reference: Apollo-Out needs to accelerate at Destiny in order to change direction and return towards Homey. Homey does not undergo such an acceleration. The expression $\Delta t = \gamma \Delta t'$ was derived for constant velocity (look back at its derivation). It is not valid when the velocity is changing. In order to solve this problem properly one needs to either use general relativity which deals with accelerations or we can view the acceleration as an infinite number of different free float frames, frames with constant velocity, and apply special relativity to each of these frames. We will not do the exact calculation yet, but we will do some considerations giving you some more understanding of what is happening. The in the last part of this exercise, we will calculate the actual time it takes including acceleration.

Part 3

In this part we will study the 'paradox' in detail and see what happened when Lisa changed frame of reference. To do this, we will introduce one more planet and one more astronaut. The third planet, Beyond, is located 400 light years from Homey along the positive x-axis. The locations of the planets is illustrated in Figure 1. There is also a second spaceship, Apollo-In, traveling from Beyond with velocity $v = -0.99c$ with astronaut Peter (denoted P in the figure). There is therefore a total of three reference frames:

- The frame of reference of the planets with unprimed coordinate system (x, t). Homey is always at origin with Destiny located 200 light years away and Beyond located 400 light years away.
- The frame of reference of the spaceship Apollo-Out traveling from Homey to Destiny with primed coordinate system (x', t'). Apllo-Out with astronaut Lisa is always at origin in this frame.
- The frame of reference of the spaceship Apollo-In traveling from Beyond to Homey with double primed coordinate system (x'', t''). Apollo-In with astronaut Peter is always at origin in this frame.

Now lets introduce a new way of thinking. Instead of one spaceship traveling from Homey, we will look at it as a queue of infinite amounts of spaceships, all traveling with the same velocity in the same direction. In all the spaceships before and after Lisa there are other observers. The situation is depicted in Figure 1, in this illustration the queue is an elevator. During the rest of the exercise there will be two elevators, the elevator from Homey to Beyond will be called 'outgoing elevator' (the primed reference system using coordinates (x', t')) and the elevator from Beyond to Homey will be called 'returning elevator' (with double primed reference system using coordinates (x'', t'')). During this part we have these events:

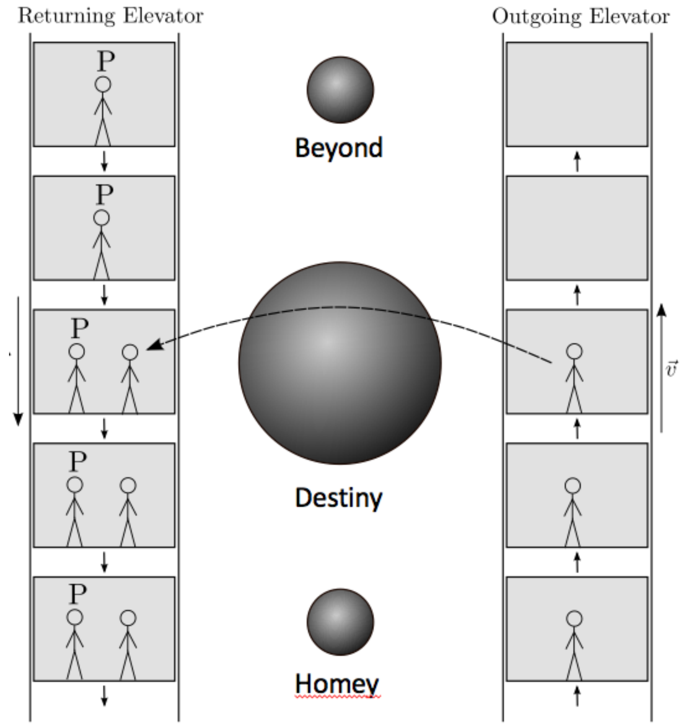

FIG. 1. The elevators between planet P1 and planet P3.

- Event A occurs at $x_A = x'_A = 0$ and $t_A = t'_A = 0$ and is when Lisa is jumping aboard the outgoing elevator at Homey.
- Event B is when Lisa arrives at Destiny and launches herself to the returning elevator from the outgoing elevator.
- Event B' is defined in the following way: At the same time (outgoing elevator frame) as Lisa arrives at planet Destiny, another astronaut in the same elevator but in another space ship (thus in the same frame of reference with clocks synchronized with Lisa's clock, but in another elevator compartment) passes Homey at position $x_{B'} = 0$. Event B' is that he looks at the clocks on Homey as he passes by and send a light signal from his spaceship which is observed at Homey. In short: B' takes place at the position of Homey at the same time as Lisa arrives at Destiny in her frame of reference.

In the following questions you should use Lorentz transformations to transform between the coordinate systems when necessary. During this part, write the distance between planet Homey and Destiny in the planet frame as L_0.

1. ☺ At what time t_B in the planet frame does Lisa arrive at planet Destiny? (express the answer in terms of L_0 and v)
2. ☺ Use the Lorentz transformations to find an expression for t'_B, the time when Lisa arrives at Des-

tiny measured on her wrist watch. Insert numbers and check that you still find that the trip takes 28.5 years for her.

3. ☺ Show that the time $t_{B'}$ can be written as $t_{B'} = L_0/v - vL_0$. Insert numbers. **Hint:** You first need to find the position $x'_{B'}$ of event B' in the outgoing elevator frame, to find $t'_{B'}$, which you also need, thoroughly read the event description.

The time $t_{B'}$ which you just calculated is the time when the observer in the outgoing elevator reads the time at Homey clocks at the same time (in his frame) as Lisa arrives at Destiny. At this moment, observers at Homey receives the signal from the space ship in the outgoing elevator (at the position of Homey) that Lisa has reached Destiny. Remember that in the planet frame, this trip takes 202 years so in the planet frame, Lisa has **NOT** yet reached Destiny.

4. ▽ Summarize (using mostly words and some numbers, not equations) what you did in Part 3 and focus on understanding the role of event B'. You should use the following two questions as an aid when writing your summary: (you are not expected to answer directly at these questions, but your summary should also include answers to these questions)
   (a) Now is the time to look at the videos: Apollo-Out is yellow. When arriving at Destiny, Lisa is launching herself from the yellow to a red spaceship (Apollo-In with astronaut Peter) in the incoming elevator using a spherical space capsule (event B). Then Lisa returns to Homey in the red spaceship. Can you see the blue light signal from the spaceship in frame 1? Compare the numbers you calculated from the earlier exercises with the numbers in the MCAst videos.
   (b) Explain the result which we found earlier when using Apollo-Out as the laboratory frame: Namely that when Destiny arrived at the spaceship, we calculated that on Homey clocks only 4 years had passed. Why is this not a surprise? Those who were surprised earlier, do you now understand which error you made when you got surprised? Which basic principle of relativity had you forgotten?

We learned in the previous questions that even if Homey clocks were observed at the same moment as the spaceship/elevator arrived at Destiny (in the outgoing frame), these two events (the observation of Homey clocks and the arrival at Destiny) were *not* simultaneous in the planet frame. For Lisa, only 4 years have passed on Homey when she arrives at Destiny. For observers on Homey on the other hand, Lisa arrived at Destiny when 202 years had passed.

Part 4

We will now tie all the loose treads together. Especially how time passes during the change of frame at event B. This is where the paradox will be answered.

We will start this part by introducing a couple of new events:

- Event D is when Peter jumps aboard the returning elevator from Beyond. This occurs in the planet frame at time $t = 0$ and at a distance $x = 2L_0$ from Homey. In the returning elevator frame this occurs at $x'' = 0$ and $t'' = 0$. In the returning elevator frame, Peter is always at the origin. In the planet frame, event A and even D happen at the same time (Lisa and Peter start their journey simultaneously), they need to travel the same distance L_0 to Destiny with the same velocity v and therefore arrive simultaneously at Destiny at event B where Lisa is transferred to Peter's elevator and frame of reference.
- Event B" is conceptually similar to event B': Event B" takes place at the same time as event B in the frame of the returning elevator. The event is a person in the returning elevator at the position of Homey, looking at the clocks at Homey and sending a blue light signal (check the video). Thus exactly at the same time (returning elevator frame) as Lisa is arriving in the returning elevator and meets Peter, event B" takes place at the position of planet Homey.

In the following, we cannot use the Lorentz transformation because the clocks in the double primed reference system is not correctly synchronized with Homey clocks. We therefore need to use the space time interval.

1. ☺ We will in the following try to find the time t''_B at Peter's wrist watch when he arrives at Destiny. Write down the space and time intervals Δx_{BD}, Δt_{BD}, $\Delta x''_{BD}$ and $\Delta t''_{BD}$. Show that invariance of the spacetime interval gives
$$\frac{L_0^2}{v^2} - L_0^2 = (t''_B)^2,$$
which gives $t''_B = L_0/(v\gamma)$. Compare with your expression for t'_B.
2. ☺ By using intuition you should be able to deduce that the spacecrafts from event A and D to event B use equal amount of time in their respective frame. The reason for this is that both have equal velocity and no acceleration. Now check the result comparing the videos of the frame of the outgoing (yellow) spaceship and the incoming (red) spaceship: when both meet at Desitny, what is the time in each of the spaceships? Compare with the numbers you have calculated.

We will now try to find the time on Homey at the moment when Peter is reaching Destiny in the returning elevator frame. We will use the same 'trick' as earlier with event B', and use an observer in an elevator compartment positioned at Homey in the returning elevator at the same time as event B occurs in the returning elevator frame. This means that an observer will be sending a light signal at Homey in the returning elevator frame to an observer at Homey in the planet frame to tell that Lisa now has been transferred to the returning elevator and has met Peter. (note that the person in the returning elevator frame which is positioned next to Homey and is sending the signal can not know for sure that Lisa actually managed to meet Peter, he can only infer this from looking at his clock and calculating the time at which this should happen in his frame).

We found that only 4 years had passed on Homey when Lisa arrived at Destiny (seen from outgoing elevator frame). We will now make the same check from the returning elevator. We will now try to find out what time $t_{B''}$ the observer in the returning elevator saw when looking at Homey clocks (and sending the signal) at event B". For this we will use the space-time interval $\Delta s_{DB''}$.

3. ☺ Show that the space and time intervals from each frame are the following:

$$\begin{aligned}\Delta x_{DB''} &= 2L_0\\ \Delta t_{DB''} &= t_{B''}\\ \Delta x''_{DB''} &= L_0/\gamma\\ \Delta t''_{DB''} &= L_0/(\gamma v)\end{aligned}$$

   You might be a bit surprised by one of these results, but if you have doubts, do the following: Make one drawing for event D and one for event B". Show the position of the zero-point (the position of Peter is the zero point of the x'' axis) of each of the x-axes in both drawings and find the distances between events.

4. ☺ Use invariance of the space-time interval (event D and B") to show that

$$t_{B''} = \frac{L_0}{v} + L_0 v$$

Inserting numbers should give $t_{B''} = 400$ years. Use the video of the planet frame to check at which time the astronaut in the incoming spaceship reads the clocks on Homey and as sends the blue light signal. Surprised? What has happened?

5. ▽ **Before reading on**, summarize what you have done and the results you have arrived at in part 4 and finally explain the solution to the twin paradox in some detail using your findings. Then, after you have written your summary and explanation, read on and, if necessary, adjust/correct you summary after reading the following text. **Please do not get tempted to read on before writing the summary and the explanation**.

Lisa is still at event B, she made a very fast transfer so almost no time has passed since she was in the outgoing elevator. But just before the transfer, only 4 years had passed on Homey since she started her journey. Now, less than the fraction of a second later, 400 years have passed on Homey. So in the short time that the transfer lasted, 396 years passed on Homey! This is were the solution to the twin paradox is hidden: When she makes the transfer, she changes reference frame: She is accelerated. Special relativity is not valid for accelerated frames (actually one could solve this looking at the acceleration as an infinite sum of reference frames with different constant velocities, this is what you will do in Part 5). When she is accelerated, she experiences fictive forces. This does not happen on Homey, the planet does not experience the same acceleration. This is the reason for the asymmetry: If her speed had been constant, she and Homey could exchange roles and you would get consistent results. But since she is accelerated during transfer while Homey is not, there is no symmetry here, her frame and the planet frame cannot switch roles.

6. ▽ **Before reading on**, merge all your answers in this challenge to one fluent text. Then, only after your complete summary of the twin paradox exercise has been written (merged), look at the following short summary. If you missed some points, go back and correct/adjust your complete text. **Please do not get tempted to read on before writing the summary**

Let's summerize the situation: In the planet frame, Lisa started her journey at $t = t' = 0$ and arrived on Destiny after $t' = 28.5$ years. In the planet frame she arrived on Destiny after 202 years of travel. In his frame, the clocks on Homey show 4 years when she arrives on Destiny. Only 4 years have passed on Homey at the time she arrives at Destiny, seen from her frame. Then she is launched to the returning elevator. Her watch still shows $t' = t'' = 28.5$ years. But now she has switched frame of reference. Now suddenly 400 years have passed on Homey, Homey clocks went from 4 years to 400 years during the time she launched herself from one elevator to the other, in his frame. In the planet frame, the clock showed 202 years during her transfer.

Seen from Homey, she also needs 202 years to return, so the total time of her travel measured in the frame of reference of the planets is $t = 404$ years. In her own frame, the return trip took 28.5 years (by symmetry to the outgoing trip), so her total travel time was 57 years. But according to her frame of reference, Homey clocks again aged 4 years during her return trip (by symmetry to the outgoing trip). When she was at Destiny, the observer in her frame of reference saw that Homey clocks showed 400 years. In her frame, 4 years passed on Homey during her return trip. So consistently she finds Homey clocks to show 404 years when she sets her feet on Homey again. This is also what we find making the calculation in the planet frame $202 \times 2 = 404$. But hundreds of generations have passed, and she has only aged 57 years. But after all these strange findings I'm sure you find this pretty normal by now. Everything clear? Now check the clocks in the video of the returning (red) spaceship as well as in the planet frame at the moment you return to Homey. Is everything consistent? Read through one more time.

Part 5

We will now try to go even deeper in understanding the twin paradox and look in detail to what happens with time and space during the accelerated phase. At the end we will connect the accelerated phase to gravity and general relativity.

1. ☻ We now want to repeat the calculation where you showed that $t_{B'} = L_0/v - L_0 v = L_0/(v\gamma^2)$ (make sure you can do the last transition here), **but** now you need to use the invariance of the space time interval $\Delta s_{\mathrm{BB'}}$ instead of the Lorentz transformation. It is important that you do this now since it will make calculations further down easier.

You showed above that just after Lisa arrived in the returning elevator, 396 years had now passed on Homey. Time on Homey, as measured by an elevator satellite at the position of Homey, suddenly made a huge jump when shifting from one system of reference to the other. Clearly the change of reference system is not instantaneous as we assumed above. We will now assume that Apollo-Out, starting at Destiny, starts accelerating with a constant **negative** acceleration (decceleration) g measured in the planet frame. At time $t_{\mathrm{turningpoint}}$ measured on planet system clocks, the space ship has reached the turning point where the velocity has reached 0 and will start returning to Destiny.

2. ☻ Show that $t_{\mathrm{turningpoint}} = t_{\mathrm{B}} - v_0/g$ where v_0 is the velocity of the space ship just before starting to accelerate.

3. ☻ What is the velocity of the space ship when it returns to Destiny? (assuming the space ship continues accelerating with the same constant negative acceleration (in the planet frame) also after reaching $v = 0$). Note that no calculation is needed here, only symmetry arguments.

We will now calculate how time on Homey runs for the space ship system during the accelerated phase. Assume that during the accelerated phase, there are constantly events similar to event B and B' happening. We will call these series of events Y and Y'. At some time t_Y, event Y refers to an event happening at the current position of the space ship and Y' is an event happening simultaneously with event Y in the space ship frame. Event Y' happens at the position of Homey: an astronaut in an outgoing space ship elevator having the same velocity of our space ship at time t_Y is just passing Homey checking the time on Homey clocks.

4. ☻ Show that the times and positions of an event Y and Y' can be written as

$$x_Y = L_0 + v_0(t_Y - t_{\mathrm{B}}) + \frac{1}{2}g(t_Y - t_{\mathrm{B}})^2 \qquad x'_Y = 0$$

$$t_Y = t_Y \qquad t'_Y = t'_Y$$

$$x_{Y'} = 0 \qquad x'_{Y'} = -\frac{x_Y}{\gamma(t_Y)}$$

$$t_{Y'} = t_{Y'} \qquad t'_{Y'} = t'_Y$$

Note that $\gamma(t_Y)$ here refers to γ taken with the velocity $v(t_Y)$ of the space ship at time t_Y.

5. ☻ Use again invariance of the space time interval $\Delta s_{\mathrm{YY'}}$ to obtain an expression for $t_{Y'}$ in terms of t_Y. **Note:** if you make this right, the equations will not be ugly. Do not insert the full expression for x_Y untill the very end, keep it as x_Y. Make sure to have the square containing $t_{Y'}$ on the left handside and everything else on the right. Note that the full expression on the right can be written as a square. Taking the square root on both sides and making sure to use the correct sign, you will easily get a result.

6. ☻ Now assume a constant acceleration (decceleration) of $g = -0.1\mathrm{m/s}^2$ in the planet frame . What is g written in units where distance and time are both measured in seconds?

7. ☻ What is the time in the planet frame $t_{\mathrm{turningpoint}}$ at the moment when the velocity has reached $v = 0$ and the spaceship starts returning to Destiny?

8. ☻ Use the formula you have found to find the time $t_{Y'}$ at the moment when $t_Y = t_{\mathrm{turningpoint}}$.

9. ▽ You should find that t_Y and $t_{Y'}$ are equal at the turning point, why must this be the case?

10. ▽ Plot the time $t_{Y'}$ elapsed on Homey measured in the space ship frame as a function of time t_Y elapsed on Homey in the planet frame in the full range $t_Y = 0$ to $t_Y = t_{\mathrm{turningpoint}}$, i.e. during both the constant velocity as well as the decceleration phase. Does the Homey clock suddenly start to run quicker (as observed from the space ship frame) during the decceleration phase?

11. ▽ Use numbers derived above together with symmetry arguments to show that the Homey clock shows 588 years (seen from the returning elevator frame) as the space ship again reaches Destiny.

12. ▽ Summarize in 5-10 sentences what you have done so far in this part without the use of any equations. Explain what you did and in particular why.

We have now seen how Homey time could go from 4 years to 588 years during the acceleration phase. We will finally calculate how much Lisa aged during the acceleration phase. We will need to add this time to the total of 57 years she aged during the two constant velocity phases. In order to make the calculation easier, we will do this for the returning phase, starting at the moment when the velocity reached 0 (called event E) and the space ship starts returning to Destiny accelerating. We will assume an infinite number of space ship elevators going back towards Destiny (and beyond), all with different velocities from 0 to v_0. Lisa stays in one elevator during a short time interval $\Delta t'$ and then accelerates step by step by jumping to the neighbouring elevator having a slightly larger velocity than the previous.

For convenience, we will restart our clocks at event E for this calculation: In order to clearly show when we use a time measure which is restarted (set to zero) at event E, these will be denoted with capital T in the planet frame and T' in the space ship frames. These are the same as the time measures t and t' which we have used so far, but they are both synchronized and set to zero at event E, making it easier to do the calculations.

12. ☺ Show that at a time T, after event E measured on planet clocks, the relation between a time interval in the space ship frame and a time interval in the planet frame can be written as

$$\Delta T' = \sqrt{1 - g^2T^2}\Delta T \tag{1}$$

Thus, when a time T has passed after event E and Lisa is in the elevator corresponding to current velocity, the short time interval $\Delta T'$ which she remains in the current elevator corresponds to a short time interval ΔT on planet clocks. **hint:** the usual formula for time dilation, but make sure you use it properly.

13. ☺ Show that the time Lisa ages from event E till she reaches Destiny again can be written as

$$T' = \int_0^{v_0/g} \sqrt{1 - g^2T^2}dT$$

14. ☺ Using i.e. The Integrator, or tables of integrals, you can show that this integral can be written as

$$T' = \frac{v_0\sqrt{1 - v_0^2} + \arcsin(v_0)}{2g}$$

Insert numbers and find the total time Lisa has aged when returning to Destiny.

15. ▽ Again summerize in a very few sentences what you just did and why. Then summerize your results. The results should also include:

- **measured in planet frame clock:** The time that the different phases take (going to Destiny, accelerating from Destiny, returning to Destiny and returning to Homey) as well as the total time.
- **mesured on Lisa's clock:** The time that the different phases take (going to Destiny, accelerating from Destiny, returning to Destiny and returning to Homey) as well as the total time.
- **The time that the observer in Lisa's frame but positioned next to Homey sees that Homey clocks show at the same time (Lisa's frame) as:** (1) Lisa reaches Destiny, (2) Lisa reaches zero velocity, (3) Lisa is back on Destiny, (4) Lisa is back on Homey.

Explain how all numbers are now consistent.

Finally we will link this to gravitation: We will call the distance from event E for r such that event E happens at $r = 0$ and the space ship accelerates in the positive r direction towards Destiny.

15. ☺ Show that after time T, the position of the space ship can be written as $r = \frac{1}{2}gT^2$.

16. ☺ Show that the time elapsed when the space ship has reached distance r from event E can be written as $T = \sqrt{\frac{2r}{g}}$.

17. ☺ Show that eq.1 above can thus be written as

$$\Delta T' = \sqrt{1 - 2gr}\Delta T$$

18. ☺ The equivalence principle states that you cannot judge whether you are in an accelerated frame or in a gravitational field. We will now see one way to show this, by observing that clocks in an acclerated frame ticks with the same rate as in a gravitational field. We know that the gravitational acceleration is given by $g = \frac{GM}{r^2}$. Insert this expression for acceleration to show that

$$\Delta T' = \sqrt{1 - \frac{2GM}{r}}\Delta T$$

In 2 weeks we will show that this is identical to the formula for time dilation in the gravitational field. We see that the form of the expressions for time dilation in an accelerated frame, or in a gravitational field take the same form. We would have got the same results if the astronaut had been in a gravitational field for some time instead of being accelerated. In both cases, time runs slower for the astronaut being either accelerated or in a gravitational field compared to observers in the frame not being accelerated or being outside of a gravitational field. This was one of Einstein's starting points when deducing the general theory of relativity.

Note that in our example, the acceleration is constant in the planet frame which means that the acceleration is not constant for the space ship observer. In order to make a direct comparison to a person in a constant gravitational field which we will consider in the lectures on general relativity (the so-called shell-observer), we should have used constant acceleration for the space ship which would have given a time dependent acceleration in the planet frame. We have avoided this due to the much uglyer calculations for the case of constant space ship acceleration.

EXERCISE 4

Relevant theory: Section 1 of lecture 2B.
Go to MCAst and load the xml files that you generate by calling the `spaceship_race` method. In this exercise it is recommended to be three students working togehter: There are three frames with one xml for each frame and student. Choose who does which frame, and only look at the video for your frame!

In this exercise there are three spaceships traveling with different velocities with respect to a space station. The different frames of reference in the videos correspond to the frame of the space station, ship 1 and ship 2. The ships 1 and 2 both travel with constant velocity while ship 3 accelerates as seen from the space station. We are not interested in exact numbers in this exercise, only roughly correct relative distances and slopes on the worldlines showing that you have understood the basic principles.

1. **Looking only at the video for your frame of reference** try to imagine how the ships and space station move in the frames of the other two students.
2. ☺ Still without looking at the other videos, draw 3 spacetime diagrams: One for each frame of reference, your frame as well as the frames of your two fellow students. In these three diagrams, draw the worldlines of the space station as well as ship 1, 2 and 3 (the accelerated ship).
3. Now meet with your fellow students and compare the diagrams. Do they agree?
4. Look at all the videos togehter and check if the other videos look as you imagined: discuss why you were right/wrong.
5. ☺ Draw a spacetime diagram in the reference frame of ship 3 (no video here) with worldline for all objects.

Return to the spacetime diagram for the space station frame, we will only work with this diagram for the rest of the exercise. We now define two events:

- Event 1 occurs at $x = 0$ and $t = 0$ is when all the spaceship are aligned.
- Event 2 is defined as when spaceship 3 catches up with ship 2 reaching the same position.

Measured on the clock in the space station it takes 10 milliseconds between the two events, on the clock in the frame of ship 2, it takes 8 milliseconds. Assume that the clocks make a tick every millisecond. The first tick happens at event 1 and the last tick happens at event 2.

6. Draw dots on the time axis between event 1 and 2 which represents the ticks in the space station frame.
7. Draw dots on the worldline of ship 2 based on the ticks which occurs in the frame of ship 2. The important point here is to have correct relative spacings between each tick.
8. Spaceship 3 has also been equipped with a clock identical to those in the space station and ship 2. Use the principle of maximal aging to judge whether an astronaut in ship 3 will experience more or less ticks on the clock from event 1 to event 2 compared to the astronaut in ship 2.
9. Draw dots on the worldline of ship 3 at the positions where the clock ticks in this frame. The exact position is not important, but the relative distances between the dots should be correct. **Hint:** For each dot you draw, look at the slope of the worldline.
10. ▽ Use the scanned drawing which you have drawn for the space station frame and explain what the world lines and dots represent. The text is aimed at somebody who has not seen this exercise text. Focus on explaining which information is contained in the position, direction and tilt of the worldlines and in the spacing between the dots. In particular, explain how you applied the principle of maximum aging.

EXERCISE 5

Relevant theory: Section 1 - 3 of lecture 2B.
Go to MCAst and load the xml files that you generate by calling the `laser_chase` method. You and your partner should agree on who does which frame.

A spaceship is moving with a speed close to the speed of light and emitting two laser beams. This is seen from the frame of reference of the planet and the spaceship.

1. ▽ **For planet frame student:** use the information given at the top of the video to find the velocity of the space ship and the light beam. Check that the beam has indeed the speed of light. Given that the light beams are emitted from the spaceship, what speed would you have expected the light beam to have in your frame if you rely on classical physics? (remember how you transform velocities between frames in classical physics.) **For the**

spaceship frame student: use the information given at the top of the video to find the velocity of the light beams and your velocity with respect to the planet. Check that the beam has indeed the speed of light. Which velocity would you expect observers on the planet to measure for the light beams if you rely on classical physics? (remember how you transform velocities between frames in classical physics.)

2. ☺ Make a space-time diagram of the space ship and the two laser beams in your frame of reference as well as in the other frame of reference.

3. ▽ **Imagine** how the scene look like in the other frame. Focus in particular on the relative velocity and distances between beams and space ship: imagine and describe how you think the movements look in the other frame, **without looking at your partner's video!**.

4. ▽ Use the formula for relativistic transformation of velocities to calculate the velocity of the laser beam in the other frame. Is the result as expected?

5. ▽ Now you should meet and compare videos. Does the other video look as expected? Why? Why not? If you did it wrong before looking at the video, please **do not correct the text above** (you will not lose points!), but explain here which mistake you made and why it looks like it does.

6. ☺ Calculate the distance between the two light beams in your frame and compare with the distance calculated by your partner. Find the ratio of the distances between the two frames.

7. ▽ Imagine a stick with it's start and end point at the position of the two beams: Compare the ratio you found between the two distances/stick lengths to what you expect from the formula for length contraction ($L = L_0/\gamma$). Does the formula for length contraction apply in this case? Why not?

8. ☺ We will now use the distance between the laser beams in the two frames to derive how the wavelength of light transforms from one frame to the other. We can imagine that the two beams correspond to two wave tops of an electromagnetic wave and in this way arrive at the relativistic Doppler formula, the formula which connects the wavelength λ of light in one frame to the wavelength λ' observed in another frame. We will do this in three steps:

   (a) We will call the distance between the two beams in the planet frame λ and in the space ship frame λ'. You should now find an expression for λ' expressed only in terms of λ and v by using the invariance of the spacetime interval. It is up to you to define the apropriate events for the spacetime interval.

   (b) Having obtained the expression for λ', find an expression for the ratio λ/λ' in terms of v, insert numbers and compare to the number you measured above.

   (c) Now use your results to derive the relativistic formula for Doppler shift:

   $$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right)$$

   In a coming exercise, you will derive this formula again using a completely different approach.

EXERCISE 6

Relevant theory: Section 1 - 4 of lecture 2B.
A free neutron has a mean life time of about 12 minutes after which it disintegrates into a proton, an electron and a neutrino. We will ignore the neutrino here, assuming that the only products of disintegration are a proton and an electron. A neutron moves along the positive x-axis in the laboratory frame with a velocity close to the velocity of light. It disintegrates spontaneously and a proton and an electron is seen to continue in the same direction as the neutron. We will try to calculate the speed of the proton and the electron in the lab-frame. The easiest way to do this is in the rest frame of the neutron where the neutron has a very simple expression for energy and momentum. In the lab frame this would have been a lot more work since all three particles have velocities. On our way we will discover a surprising fact!

Go to MCAst and load the xml files that you generate by calling the `neutron_decay` method. In this exercise you may work alone if you wish and contrary to other exercises you are supposed to look at both frames now before starting to calculate. The videos will show the velocity of the neutron seen from the lab(planet) frame as well as the masses of the particles. **Important:** It is necessary to use all the given decimals of the particle masses when making calculations in this exercise.

1. ▽ Shortly summerize the situation and write an expression for the momenergy four-vector $P'_\mu(e)$ of the electron in the frame of the neutron expressed in terms of m_e and the unknown velocity v'_e. You may define $\gamma'_e = 1/\sqrt{1-(v'_e)^2}$. Explain how you arrived at your expression.

2. ☺ Write an expression for the momenergy four-vector $P'_\mu(p)$ of the proton in the frame of the neutron expressed in terms of m_p and the unknown velocity v'_p. You may define $\gamma'_p = 1/\sqrt{1-(v'_p)^2}$.

3. ▽ Write an expression for the momenergy four-vector $P'_\mu(n)$ of the neutron in the frame of the neutron expressed in terms of m_n and v'_n and explain how you arrived at this expression.

4. ☺ Use conservation of momenergy

$$P'_\mu(n) = P'_\mu(p) + P'_\mu(e),$$

to find expressions for the velocities v'_e and v'_p (the velocities of the electron and proton in the neutron frame). If you are observant you will discover that there are two sets of possible soultions (why do you think this is the case?), in the rest of the exercises choose one of the solutions. **Hint:** The algebra in this exercise can be extremely ugly if done wrong, here are some tips:

- Solve the equations for γ'_e or γ'_p **NOT** for v'_e or v'_p.
- You often insert ${\gamma'}^2 = 1/\left(1 - {v'}^2\right)$ but you should instead insert ${v'}^2 = 1 - 1/{\gamma'}^2$.
- If you end up with something looking like it will be a quadratic equation try writing it out, you should end up with a first order equation.
- If you have an equation $a\sqrt{b} + c\sqrt{d} = 0$, show that it may be written as $a^2 b = c^2 d$.

You should arrive at:

$$\gamma'_p = \frac{m_n^2 + m_p^2 - m_e^2}{2 m_p m_n}. \qquad (2)$$

Find a number for γ'_p and use this to obtain numbers for v'_p, γ'_e and thereby v'_e.

5. ☺ Use the transformation properties for four-vectors

$$P'_\mu = c_{\mu\nu} P_\nu$$

to find the energy and momentum of the electron and the proton in the lab (planet) frame. (insert numbers: what units do your results have if you keep $c = 1$?).

6. ☺ Use the numbers you have obtained for energy or momentum to obtain the speed of the electron and proton in the planet frame.

7. ▽ You might have observed that $m_n \neq m_p + m_e$, ie. that mass is not conserved in this process. Rather, parts of the mass has been converted to energy. In the lectures on nucelar reactions, this result will become very important. We will now show that it is impossible to conserve mass in this process, similar to most other nuclear processes: Now assume for a moment that mass is indeed conserved and $m_n = m_e + m_p$. Using only symbols, not numbers, insert this in the expression 2 for γ'_p above. Use this to obtain v'_p and v'_e. Explain why mass could not be conserved in this process.

8. ☺ As an independent check (and to see an alternative way of doing it), use the relativistic formula for addition of velocities to obtain the speed of the two particles in the laboratory frame, using only the speed you have obtained for the proton and electron in the neutron frame as well as the speed of the neutron in the planet frame.

9. For those who like long and ugly calculations only: Do everything from the beginning, but use only the planet frame to obtain the same results. Do you see the advantage of using 4-vectors and change of frames?

EXERCISE 7

Relevant theory: Section 1 - 4 of lecture 2B.
Go to MCAst and load the xml files that you generate by calling the `antimatter_spaceship` method. You and your partner should agree on who does which frame.

Some crazy researchers have decided to test special relativity. They prepare two identical space ships to travel towards each other with a velocity close to the speed of light. Eventually they collide close enough to be seen from a planet. One of the spaceships is made solely from anti-matter including the researcher, therefore all matter is converted to photons in the collision. We assume all the photons have exactly the same wavelength. During this exercise we therefore have two objects:

- The leftmost spaceship is denoted by the letter A and is traveling with velocity v_A with respect to the planet frame.
- The rightmost spaceship is denoted by the letter B and is traveling with velocity $v_B = -v_A$ (same speed as A but opposite direction) in the planet frame.

To measure the wavelength of the photons there are two observers (you and your partner) both with a wavelength detector. The observers are in two different frames of reference:

- The frame 1 observer is at rest on the planet using unprimed coordinate system.
- The frame 2 observer has a small red space ship (you can see him/her in the frame 1 video) and follows just behind spaceship A in the same frame as spaceship A using primed coordinate system.

All the necessary information including the mass of the spaceships and number of photons produced are given in the upper left corner in MCAst.

The main goal of this exercise is to use the transformation of momenergy 4-vectors to deduce a relativistic formula for Doppler shift.

1. ☺ Use the velocity of spaceship A with respect to the ground to calculate the relative velocity of spaceship B observed from spaceship A (transform the velocity with your preferred method). Remember that with respect to the ground, the spaceships have equal speed (but opposite directions).

2. ☺ Write down expressions for the momenergy four-vectors $P_\mu(A)$ and $P_\mu(B)$ of the two spaceships in your frame of reference. You may use v_A, v_B, v'_A, v'_B, the corresponding gamma-factors $\gamma_A, \gamma_B, \gamma'_A, \gamma'_B$ and the mass m of the spaceships.

3. ☺ Use the transformation properties of four-vectors to transform momenergy four-vectors from your frame of reference to the other frame. You should now have $P_\mu(A)$ and $P_\mu(B)$ in the planet frame as well as $P'_\mu(A)$ and $P'_\mu(B)$ in the space ship frame.

4. ▽ Show that the momenergy four-vector of a photon traveling in the positive x-direction can be written

$$P^\gamma_\mu = (E, E, 0, 0),$$

where E is the energy of the photon.

5. ▽ Assume for the moment that all the energy in the explosion is emitted in only two photons, one emitted along the positive x-axis (same direction as the space ship) and the other in the opposite directions (negative x-direction). Use conservation of momenergy in the planet frame to argue (no calculations necessary) that the two photons must have the same energy seen from the planet frame.

6. ▽ Now assume that these two photons are emitted with an angle θ off the x-axis. Write the momenergy 4-vector for this photon and use again conservation of momenergy in the planet frame to argue (no calculations necessary) that the two photons must have the same energy but opposite directions seen from the planet frame.

7. ▽ Use your previous results to argue that if there are photons emitted in all possible directions, there is, for all photons emitted always another photon emitted in the opposite direction with the same energy.(remember that we assume that all photons produced in this explosion have the same energy)

8. ☺ Using the assumption that all photons are emitted with the same wavelenght, and using also the number of photons measured, what is the energy of one photon and thereby the wavelength in the planet frame? (**Hint:** To convert photon energy to wavelenght, it may be useful to first convert the energy you found to normal SI units.

9. ☺ Use the wavelength to find the color of the explosion seen in the planet frame (use the table in this Wikipedia article.)**Planet frame observer:** does it correspond to the colour you observe?

The collision between the spaceships was inspired by a process which happens in nature. An electron and a positron are corresponding antiparticles with equal mass. The particles are approaching each other with the same velocity in opposite directions in the center of mass frame of the two particles. In the collision, both particles are annihilated and **two** photons are produced. One photon travels in the positive x direction, the other in the negative x direction. In the rest of the exercise we will therefore only study the photons which move along the x-axis:

10. ☺ We will now study only the photons which move along the x-axis. Use transformation properties for four-vectors to show that the energy E' of a photon observed in the spaceship A frame moving with velocity v with respect to the planet frame (where the photon has energy E) is

$$E' = E\gamma(1 \pm v)$$

(Which sign is for the photon moving in positive x-direction and which is for the photon moving in negative x-direction?)

11. ☺ Use the derived expression for E' and the formula for the energy of a photon to derive the relativistic Doppler formula

$$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right)$$

12. ☺ Use this expression to find which colour the explosion has for the observer in the ship frame. Finally, meet with the other student and look at both videos to check your results. Did you calculate the colour in the other frame correctly?

13. ☺ Show that the relativistic Doppler formula is consistent with the normal Doppler formula for low velocities. **Hint:** Make a Taylor expansion of $f(v) = \sqrt{(1+v)/(1-v)}$ for small v. (**Remember:** how is the non-relativistic Doppler-formula using relativistic units where $c = 1$?)

FIG. 2. Relativistic, orbitally oriented, motivationally atmospherical, trajectory-planning duck. Launched & landed.