AST 2000 - Part 9
General Relativity

Welcome to Part 9 of the AST2000 Project. The goal of this part is to address several consequences of general relativity and Schwarzschild geometry such as gravitational time-dilation, motion in strong gravitational fields and the effect of gravitation on light.

RELATIVITY EXERCISES

You have now landed on your destination planet and you start making relativity experiments on the surface of the planet. Before you continue reading, you now need to read section 7 of lecture 2A remembering that you are supposed to generate the xml files yourself. Refer to the documentation for the `ast2000tools` package to learn about the `RelativityExperiments` class that you will use to generate relativity xml files for MCAst.

For part 8 and part 9 you need to have a partner student for some of the exercises. Even if you work in a group, you need a partner group or partner student. You may do everything alone, but it requires more work. For the exercises where you need a partner and each partner has her/his own frame of reference, you are only supposed to report on the exercises relevant for your frame (for the exercises which are different depending on the frame).

You will create a set of videos of the experiments you are performing on the surface of your planet. You will need to decide between you and your partner (or your group and your partner group) on which planet you will do the experiments: the planet where you have landed or the planet where your partner has landed. You may also do some experiments on your planet and others on your partners planet, but it is very important that for the same experiment, you use the same planet for all frames of reference.

In part 8 and 9 you are not supposed to make a scientific paper. Instead your task is to make an educational text, similar to what the bloggers have been doing in previous parts. Some of the exercises (mostly those including calculations) have been marked ☺ which means that you can do these on paper like you do for the exams (no need for much text, just enough to understand what you are doing), scan these and deliver these as separate files. It is only to see that you have actually done it. The ones marked ▽ are the ones on which you mainly will be evaluated. These will normally contain a question asking you to summerize what you did in the previous questions with words or they are questions testing your understanding. **For each exercise separately, you should make a short text which includes the following sections:**

1. **Introduction:** This part should shortly introduce the problem and make the reader being able to understand the main question(s) you want to solve without having read the exercise text.
2. **The situation**: what are the objects/events in this exercise and what are the main questions you would like to test/answer (describe this with words) using these objects/events. When relevant you may also include a table with events and positions here.
3. **Method:** Use text only to describe the main physiocal principles you were using when solving this exercise and why these principles are relevant. Then, you may include some very few basic equations here with words explaining why these are relevant and how you will use these to solve the problem. (you should not do the calculations here, unless the ▽ question asks for it!)
4. **Conclusions:** shortly summerize the conclusions, what did you find and in particular: what was the purpose if this exercise, what are you supposed to learn? What is the physics behind your results? Somebody who has not seen the exercise should be able to understand.
5. **Specific questions:** If they are not already included in the above points: Here the answer to the remaining ▽ questions should be included in one fluent text. In most cases, these are already naturally answered in some of the other sections above in which case there is no need for this section.

You will be evaluated in the following way:

- The scanned hand-ins (or latex-typed if you prefer) of the ☺-exercises will count 30%
- The Introduction and Conclusion sections together will count 20%
- The "situation" section will count 25%
- The Method section will count 25%
- The "specific questions" section will be included in the other relevant points.

In part 9, each exercise will be given the same weight, **except** exercise 3 on the spaceship falling into the black hole. In this exercise, each of the two parts will count as an entire exercise. The total score will be the mean of all 8 exercises, but weighted in the following way:

$$\text{totalscore} = \frac{x_1 + x_2 + x_{3a} + x_{3b} + x_4 + x_5 + x_6 + x_7 + x_8}{8},$$

where x_i is the score of exercise i, x_{3a} corresponds to exercise 3, part 1 and x_{3b} corresponds to exercise 3, part

2. Moreover, **if you have completed exercises 5, 6 and 8, and got at least 75% right, your total score for part 9 will be multiplied with 1.1 before rounding up**.

EXERCISE 1

Relevant theory: sections 1-4 of lecture note 2C.
Imagine a shell observer at shell r, pointing a laser pen radially outwards from the central mass. The beam has wavelength λ_{shell}. Here we will try to find the wavelength λ observed by the far-away observer.
The frequency of the light emitted by the laser pen is $\nu_{\text{shell}} = 1/\Delta t_{\text{shell}}$. The frequency of the light received by the far-away observer is $\nu = 1/\Delta t$. Here Δt_{shell} and Δt is the time interval between two peaks of electromagnetic waves.

1. ▽ Show that the difference in time interval measured by the two observers is given by

$$\Delta t = \frac{\Delta t_{\text{shell}}}{\sqrt{\left(1 - \frac{2M}{r}\right)}}.$$

Hints:

- Imagine that a clock situated at shell r ticks each time a peak of the electromagnetic wave passes.
- We have already derived this equation in the lecture notes. We are asking you to repeat the deduction for our special case.

2. ▽ Use this fact to show that the gravitational 'Doppler' formula, i.e. the formula which gives you the wavelength observed by the far-away observer for light emitted close to the central mass, is given by

$$\frac{\Delta\lambda}{\lambda_{\text{shell}}} = \frac{\lambda - \lambda_{\text{shell}}}{\lambda_{\text{shell}}} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1$$

3. ▽ Show that for distances $r \gg 2M$ this can be written as

$$\frac{\Delta\lambda}{\lambda_{\text{shell}}} = \frac{M}{r}$$

Hint: Do you see what order your Taylor expansion should be?

4. We will now study what wavelength of light an observer far away from the Sun will observe for the light with wavelength $\lambda_{\text{max}} = 500$ nm emitted from the solar surface.

   (a) Find the mass of the Sun in meters. Then find the ratio M/r for the surface of the Sun.(can you see now why we made the Doppler expansion for small M/r?)

   (b) Find the redshift $\Delta\lambda/\lambda_{\text{shell}}$ measured by a far-away observer. Does the apparent color of the Sun change due to the gravitational redshift?

   For light coming from far away and entering the gravitational field of the Earth, an opposite effect is taking place. The light is blue shifted.

   (c) Find the ratio M/r for the surface of the Earth.

   (d) Find the gravitational blue shift for light arriving at Earth. Does this change the apparent color of the Sun?
   Note 1: Did you notice that the two observers changed roles?
   Note 2: Is it nessesary to use the answer from 4b in 4d?

A quasar is one of the most powerful sources of energy in the universe. The quasars are thought to be powered by a so-called accretion disc: Hot gas circling and falling into a black hole. The gas reaches velocities close to the speed of light as it approaches the horizon.
Assume that we observe an emission line at $\lambda = 2150$ nm in the radiation from a quasar. Assume also that we recognize this emission line to be a line which in the laboratory is measured to occur at $\lambda = 600$ nm.

5. Find from which distance r (expressed in terms of the black hole mass M) from the center the radiation is emerging. Give some arguments explaining why this observation supports the hypothesis of quasars having a black hole in the center. We assume that the Doppler effect due to the quasar's movement with respect to us has been subtracted.

6. Imagine you are a shell observer living at a shell at $r = 2.01M$ very close to the horizon of a black hole of mass M. Can you use optical telescopes to observe the stars around you? Which part of the electromagnetic spectrum does your telescope (or your eyes) need to detect?

EXERCISE 2

Relevant theory: section 5 of lecture note 2C.
In this exercise we will use the principle of maximal aging to deduce the law of conservation of angular momentum in general relativity. In the text you have seen three examples of this kind of derivation and we will follow exactly the same procedure here. Before embarking on this exercise, please read the examples in the text carefully. **Note: In this exercise we should in principle take the derivative with respect to all space (x,y and z), but we have already found in the lecture notes that if we only have radial movement, we**

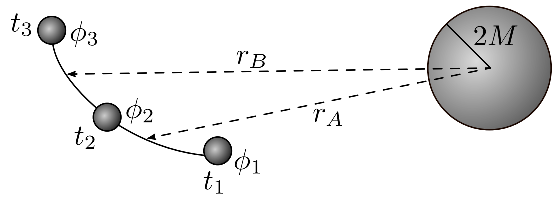

FIG. 1: A sketch of exercise 2

can deduce the expression for energy per mass. We are therefore only interested in the angular derivative here..

Use figure 1 in this exercise: We will study the motion of an object which passes through the three points (r_1, ϕ_1), (r_2, ϕ_2) and (r_3, ϕ_3) at times t_1, t_2 and t_3. We fix t_1, t_2, t_3, r_1, r_2 and r_3 as well as ϕ_1 and ϕ_3. The free parameter here is ϕ_2. We assume that between (r_1, ϕ_1) and (r_2, ϕ_2) the radius is $r = r_A$ (we assume the distance between these two points to be so small that r is constant) and between (r_2, ϕ_2) and (r_3, ϕ_3) we have $r = r_B$ (see again figure 1).

1. ☺ Use the Schwarzschild line element to show that the proper time interval from t_1 to t_3 can be written as

$$\Delta\tau_{13} = \tau_3 - \tau_1 = \Delta\tau_{12} + \Delta\tau_{23} = \sqrt{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2} + \sqrt{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2}$$

2. ☺ Use the principle of maximal aging to show that

$$\frac{r_A^2 \Delta\phi_{12}}{\Delta\tau_{12}} = \frac{r_B^2 \Delta\phi_{23}}{\Delta\tau_{23}},$$

and use this to argue that

$$r^2 \frac{d\phi}{d\tau}$$

is a conserved quantity.

3. ☺ Show that his quantity can be written as

$$\gamma_{\text{shell}} r v_{\phi,\text{shell}}$$

using shell observer speed and tangensial velocity $v_{\phi,\text{shell}}$.

4. ☺ Show that this is equivalent to classical spin per mass, L/m, in the limit where velocities are small.

5. ▽ Summerize what you did in the exercise: Make sure to explain the purpose, the main principles, the main idea, the way you solved it, the results and the significance of the results. **You are not allowed to use any equations, not even one! Figures however, if they can help you explain better, are welcome.**

EXERCISE 3

The first part of this exercise is based on lecture note 2C, the second part is also based on lecture note 2E.

Part 1

Relevant theory: sections 1-6 of lecture note 2C.
Go to MCAst and load the xml files that you generate by calling the `black_hole_descent` method with `consider_light_travel=False`. You and your partner should agree on who does which frame. Make sure you both use the same seed, agree on which of your seeds you will be using. **In the planet frame** you are positioned in a satellite close to a planet which orbits a black hole at a distance of 1AU. The satellite is not moving with respect to the planet. Another space ship is falling freely radially inwards towards the black hole having a velocity v at the moment when it is passing you. It is emitting blue light signals (seen from the falling space ship frame) with a fixed time interval (in the falling space ship frame) between each signal.
In the other frame, the falling frame, you are positioned in the falling space ship, looking at your friend positioned close to the planet who is sending red light signals (seen from the planet frame) with a fixed time interval (in the planet frame) between each signal.

The mass of the black hole, the locally observed shell speed of the falling space ship at 1 AU from the black hole as well as the time interval between each signal in the frame which emits the signal is given in the upper left corner.

Important: As in most other xml-videos in this course, the light travel time from the objects to the camera is not considered, meaning that you see all events instantaneously. This would correspond to an infinite light speed for light travelling from the objects/events to the camera. This effect is more visibile here than in most other videos. In part 2 of this exercise, you will correct the answers and the video from part 1 taking into account the real light speed.

1. ▽ Characterize each of the two observers as either far away observer, shell observer or freely falling observer.

2. ▽ Use the general expression for E/m as well as some known transformation relations between far-away and shell quantities to show that the energy of the falling space ship can be written as

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{r}} \frac{dt_{\text{shell}}}{d\tau}$$

and find physical interpretations of the quantities r, dt_{shell} and $d\tau$. Explain well the physics of what you do and why you can do this.

3. ▽ Show that for a shell observer at position r the above expression becomes

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{r}}\gamma_{\text{shell}}$$

where $\gamma_{\text{shell}} = 1/\sqrt{1 - v_{\text{shell}}^2}$ where v_{shell} corresponds to the locally observed shell velocity at a distance r from the black hole. Remember that you may use a local intertial frame, and thereby special relativity, during a short moment when the space ship passes the shell observer. (How would we write $dt_{\text{shell}}/d\tau$ in special relativity? If the shell observer is standing on the ground and the spaceship is a train passing, can you see the analogy to the exercises in special relativity?) Explain well the physics of what you do and why you can do this.

4. ☺ Calculate a number for the energy per mass E/m of the falling space ship.

5. ☺ Use the expression for energy to show that the relation between a time interval $\Delta\tau$ in the falling space ship and a time interval Δt on the far-away-clock, can be written as

$$\Delta\tau = \frac{1 - \frac{2M}{r}}{E/m}\Delta t$$

6. ☺ Use the relation between far-away-time and shell time to find a relation between a time interval in the falling space ship, $\Delta\tau$, and a time interval in the planet frame, Δt_{shell}.

We will now assume that the time interval between two emitted signals are short, so short that we can approximate the distance r between the space ship and the black hole as constant during this time interval. We will in the following use our expression above to find the (assumed constant) distance r between the space ship and the centre of the black hole during the time between two emitted signals. Remember that in both frames, the light signals are emitted with a constant known time interval between each signal in the frame of reference of the emitter.

7. ☺ No matter which frame is yours, you should now use the light signal you receive from the other observer to find the distance r from the black hole to the falling space ship in the time period between the two first signals, as well as between the two last signals received. Give the answer in units of AU and in units of Schwarzschild radii. Use both numbers to judge whether your answer may be reasonable or not (give arguments).

8. ▽ Summarize shortly (5-10 sentences) without any equations what you have been doing so far and which results you got.

9. ▽ Before you meet to compare videos, can you imagine how this looks from the other observer? How do you think the other observers sees your light signals? Focus in particular on the frequency and color of the signals. Use the equations that you already found to argue. Now meet to compare.

10. ▽ After meeting, you should discuss the result seen from the space ship frame: the time interval between each received light interval seems to grow shorter and shorter as you approach the horizon. Play the frame 2 video at very slow speed during the last light signals. Could this really be? What will happen as you hit the horizon? Is the video correct? If not, what is wrong?

Part 2

In order to do this exercise, you need to have read sections 1 and 2 of lecture note 2E. This time you will use the xml files generated by calling the `black_hole_descent` method the with `consider_light_travel=True`. These videos are the same as the videos used in part 1, with one important difference: now the light travel time has been included. In part 1, you assumed the light to travel at infinite speed such that you saw the light signals immediately as they were emitted. Now you see the light signal when it actually reaches you, this is what you would really see. As the light signals travel through a strong gravitational field, effects which we have learned about in this lecture will be at play. Please do not watch the videos for this exercise yet.

You do not need your partner for this part of the exericse.

1. ▽ Watch again the frame 1 video (observing the space craft from the shell) from part 1, **do not** look at the corresponding video for part 2 yet. Use equation for radial light speed v_r deduced in lecture note 2E to judge what you **think** will change (no calculations, just considerations) in the new video and how.

2. ▽ Now watch the frame 1 video for part 2 which takes into account light travel time. What differences do you see by eye? (if any?)

3. ▽ In addition to the xml-files, some text files are also generated for the cases with and without light travel time considered. These contain the same information which is printed during the video: the light signal number and the time when you receive the signal. Use the numpy function `x,y = np.loadtxt('name_of_txt_file.txt')` to load the information into arrays, where x is the light signal number and y is the corresponding time of reception. **Here is your task:** Plot the time **differences** (you need to convert to time differences)

between the reception of each light signal as a function of signal number for the case with and without light travel time in the same plot. Explain the difference beteween the two curves. Was this what you expected?

4. ☺ Watch the video corresponding to frame 2 (from the space craft) without light travel included (part 1), **do not** yet look at the video that includes light travel. Show that the distance Δr_γ that a photon approaching the space craft travels during a time interval $\Delta\tau$ on the space craft clock is given by

$$\Delta r_\gamma = -\frac{1-\frac{2M}{r_\gamma}}{1-\frac{2M}{r}}\frac{E}{m}\Delta\tau$$

where r_γ is the position of the photon and r is the position of the space craft. Use this equation to judge what you **think** will change (no calculations, just considerations) in the new video (part 2) and how.

5. ▽ Watch the video corresponding to frame 2, with both light travel included and not included. Do you see a difference and was this difference as expected?

6. ▽ Now use the text files for frame 2 to plot the time intervals for part 1 and part 2 in one plot as above. Can you explain the difference between the curves?

EXERCISE 4

Relevant theory: sections 1-6 of lecture note 2C.
In figure 2 we show a spaceship at position (r, ϕ, t) in Schwarzschild coordinates around a black hole of mass M. The spaceship has used all its fuel and can therefore not use its engine, it is falling freely. We will now study the motion of the spaceship step by step. We will ask the question, what is the new position (r, ϕ, t) in Schwarzschild coordinates of the spaceship after a time interval $\Delta\tau$ has passed on the wrist watches of the astronauts? By increasing $\Delta\tau$ and thereby the other coordinates step by step, we will be able to follow the motion (r, ϕ) of the spaceship.

1. ☺ We will start by finding an expression for the increase in far-away time Δt when the time on the astronauts wrist watch increases by $\Delta\tau$. Show that

$$\Delta t = \frac{E/m}{\left(1-\frac{2M}{r}\right)}\Delta\tau.$$

where E/m is energy per mass of the space ship.

2. ☺ Show that after a proper time interval $\Delta\tau$, the space ship has moved an angle

$$\Delta\phi = \frac{L/m}{r^2}\Delta\tau.$$

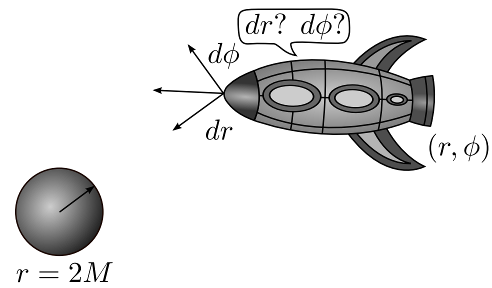

FIG. 2: For exercise 4: The spaceship is out of fuel. The engines stop. What will be the next movement in r and ϕ direction?

where L/m is the total angular momentum per mass of the space ship.

We have already obtained the displacements $\Delta\phi$ and Δt per proper time interval $\Delta\tau$. Now we need to find the radial displacement Δr.

3. ☺ Using the two previous expressions, the relation between proper time and space time interval as well as an appropriate expression for Δs, show that

$$\Delta r = \pm\sqrt{\left(\frac{E}{m}\right)^2 - \left[1+\left(\frac{L/m}{r}\right)^2\right]\left(1-\frac{2M}{r}\right)}\Delta\tau.$$

EXERCISE 5

Relevant theory: section 7 of part 2C.
Now open the xml file generated by calling the `gps` method.

In the video, you are situated at a fixed point, somewhere at the equator of a planet. The mass and radius of the planet is given in the upper left corner of the video. Two GPS satellites are passing above you in the sky, continously sending messages about the (x,y) position and time (measured on the satellite clock) specifying when and where the signal was sent. The satellites go in a circular orbit around equator. Since all positions, both for the satellite and observer are at the equator, we will use a 2-dimensional (x,y) system to denote all positions. The origin of the system is the center of the planet. During the video, the camera is fixed at your position, but the camera angle changes so that it follows the two satellites. Note that you even receive signals from the satellite when they are below the planet.

We assume that your planet clock and the satellite clocks are synchronized at the beginning of the video. Your main task in this exercise is to use the signals sent from the satelittes to determine your (x,y) position on the planet.

In this exercise, precision is of high importance. In order to get consistent results, you need to use the values of constants which were used to create this video. In the `constants` module, look up the value `c_km_pr_s` for the speed of light in km/s and `G` for the gravitational constant in SI units. **In all your calculations you need to use all digits given. This also applies for the times and positions which you find in the video.** If you omit some digits you loose the necessary precision in order to see the small effects of general relativity. We will assume that the planet is not rotating, meaning that your (x,y) position is fixed.

1. ☺ Use information given in the video to find the height of the orbit of the satellites.

2. ☺ Use information from the video as well as some celestial mechanics to fint the orbital velocity of the satellites.

3. ☺ Now choose a very early moment in the video, one of the very first frames which are displayed: Write down the position and time signals which you receive from both satellites at this moment. You must also write down current time at the planet at this moment when you receive the signals. Use this information to infer your (x,y) position.
   Hints:
   - We have already done some of the work in this lecture note. Remember that the time sent from the satellite is the time when the satellite sent the signal (on the satellite clock), whereas the planet time is the time when you receive the signal. For the moment, please ignore all relativistic effects.
   - Assume the position of the satellite is given by $\vec{r}_{\text{sat}}$ and your position is $\vec{r}$. Then you know how to write $|\vec{r}_{\text{sat}} - \vec{r}|$ in terms of Δt. You also know how to write $|\vec{r}_{\text{sat}} - \vec{r}|$ in terms of the angle α between $\vec{r}_{\text{sat}}$ and $\vec{r}$.
   - Use angles and some vector properties (the law of cosine) to find the solution.
   - You **will** need both satellites to find your position. You end up with two possible solutions if you only use one.

4. ☺ Now you should take into account relativistic effects: both the gravitational and special relativistic effects should be included. You know that the clocks onboard the satellites tick with a different rate than your clock. In order to get your correct position, you need to derive the time when the signals were sent measured on the planet clock.

5. ☺ Now use your new times to find your position. With how many meters did you miss your position?

6. ☺ Pick a moment towards the end of the video, preferably one of the very last frames of the video. Repeat all the previous exercises in order to find your position with and without correction for relativistic effects.

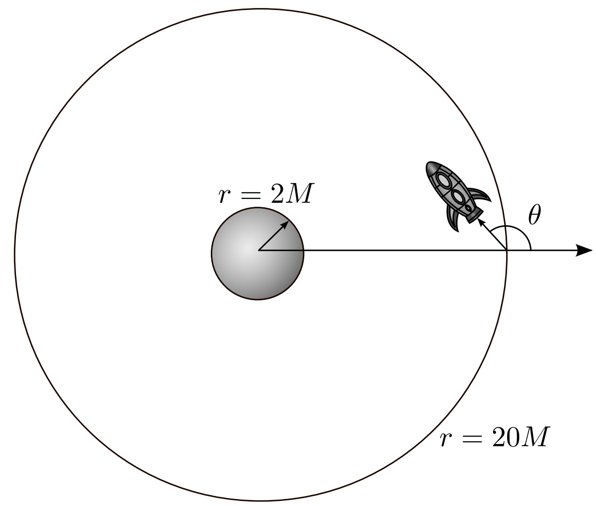

FIG. 3: For exercise 6: Rocket launched from shell $r = 20M$ inwards at an angle θ. Note: figure not to scale.

7. ▽ You should have found a considerably larger deviation in the latter case. Why? What would happen if you repeated your position estimate in a few days? Would GPS still be useful? Before answering these questions, make a short summary (a very few sentences) of you results in this section.

EXERCISE 6

Relevant theory: All sections of part 2D.

A rocket is launched from shell at $r = R = 20M$ around a black hole with mass M. The rocket has the velocity $v_{\text{shell}} = 0.993$ and is launched with $\theta = 167°$ where θ is defined as the angle from the radial vector to the velocity vector as seen in figure 3.

Just after launch the engines stop working. The astronaut therefore fears that his fate may lie within the black hole. In this exercise we will examine whether the rocket will be captured by the black hole or not. The rocket's angular momentum is L and the mass of the rocket is m. A relation that may come in handy is

$$\frac{dx}{d\tau} = \frac{dx}{dt_{\text{shell}}} \frac{dt_{\text{shell}}}{d\tau},$$

where x can be any quantity.

1. ▽ Sketch a typical gravitational potential for a black hole. What criterion needs to be fulfilled for an object in free fall to avoid being absorbed by the black hole? What quantities do you need to calculate in order to check this criterion?

2. ☺ You should have found that one of the quantities necessary to check the criterion is the energy

per mass of the rocket. Use the general relativistic expression for E/m to show that the total energy per mass of the rocket can be written as

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{R}}\gamma_{\text{shell}}$$

where $\gamma_{\text{shell}} = 1/\sqrt{1 - v_{\text{shell}}^2}$. **Hint:** Remember that for short time intervals dt_{shell}, the shell observers can use special relativity. How would we write $dt_{\text{shell}}/d\tau$ in special relativity?

3. ☺ Another quantity needed to check the criterion is the value of the potential at the maximum: Use the general relativistic expression for the effective potential to show that the minimum and the maximum of the effective potential are located at the following distances(measured in Schwarzschild coordinates) from the black hole

$$r_{\text{extremum}} = \frac{(L/m)^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{(L/m)^2}}\right).$$

Use your earlier draft to determine which of the two extrema has to be the maximum.

4. ☺ Clearly, in order to calculate the latter quantity, we need angular momentum per mass: Show that the angular momentum per mass for the rocket can be written as

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} = R\gamma_{\text{shell}}v_{\text{shell}}\sin\theta.$$

5. ▽ Insert numbers in the expression for L/m. Thereafter plot the potential for the rocket using r in units of M on the x-axis and V_{eff}/m on the y-axis. Mark the maximum effective potential and draw a horizontal line for the total energy per mass. Use the information from the plot to tell if the rocket is captured by the black hole and explain how you use the plot.

6. ☺ If he is captured by the black hole, how long does it take on the wristwatch of the astronauts to reach the singularity from the moment he enters the horizon? For simplicity ignore the angular momentum of the rocket $L/m = 0$, use the black hole in the center of the Milky way with mass $M \approx 4 \times 10^6 M_\odot$ and give the answer in seconds.

   Important hint: In this exercise $E/m \neq 1$ and you can therefore not copy the result in the lecture note. Remember that you can use 'physics math' and exchange Δ with infinitesimals in an equation linking Δr and $\Delta\tau$. For the integral, do yourself a favor and use an online calculator.

7. ▽ What will happen with the astronaut just before entering the singularity? Draw the gravitational forces on the astronaut (you can't really use forces but they are easier to draw and visualize than spacetime geometry). Which shape will he/she have just before reaching the center?

EXERCISE 7

This exercise is optional for those working alone
Relevant theory: All sections of part 2D.
The rocket has entered the horizon and is falling towards the singularity, miraculously the engine starts working, is all hope truly lost or is there a way to escape? To study the possibility of escape the astronaut emits two light beams, one towards the central singularity and one backwards away from the singularity.

In order to study how these beams of light are moving we need to write the Schwarzschild line element in terms of our wristwatch time t' instead of Schwarzschild time t. We will make this change of coordinates already before entering the horizon as this allows us to use shell frames as a help. Assume in the following that we have velocity only in the radial direction. Assume also that we started falling freely with velocity $v = 0$ far away from the black hole.

1. ☺ Use the Lorentz transformations to show that time intervals measured on the wristwatch of the astronaut are related to time and space intervals measured by shell observers as

$$dt' = -v_{\text{shell}}\gamma_{\text{shell}}dr_{\text{shell}} + \gamma_{\text{shell}}dt_{\text{shell}},$$

where v_{shell} and γ_{shell} are based on the local velocity of the astronaut measured by the shell observer at the shell which the spaceship passes.

2. ☺ Use the expressions relating shell coordinates and Schwarzschild coordinates to show that

$$dt' = -\frac{v_{\text{shell}}\gamma_{\text{shell}}dr}{\sqrt{(1 - \frac{2M}{r})}} + \gamma_{\text{shell}}\sqrt{\left(1 - \frac{2M}{r}\right)}dt.$$

3. ☺ At the end of lecture note 2D we deduced the shell velocity v_{shell} of a falling spaceship starting with $v = 0$ far from the black hole. Go back and check this expression. Insert it in the previous expression and show that

$$dt = dt' - \frac{\sqrt{2M/r}dr}{(1 - \frac{2M}{r})}.$$

4. ☺ Use this to substitute dt with dt' in the normal Schwarzschild line element and show that the Schwarzschild line element can be written

$$ds^2 = d\tau^2 =$$

$$\left(1 - \frac{2M}{r}\right)(dt')^2 - 2\sqrt{\frac{2M}{r}}dt'dr - dr^2 - r^2d\phi^2.$$

Note that this form of the Schwarzschild line element does not have a singularity at $r = 2M$.

5. ☺ We will now study the motion of the two light beams that we emit, one forwards and one backwards. We know that for light, proper time is not moving and $d\tau = 0$. The light beams in this case are moving only radially so $d\phi = 0$. Show that the speed of the two beams can be written as

$$\frac{dr}{dt'} = -\sqrt{\frac{2M}{r}} \pm 1. \quad (1)$$

6. ▽ What is the physical interpretation of this equation, and more importantly can this speed be measured by any observers?

7. ▽ Based on equation 1 how does the speed of the light beams change depending on the position of the rocket both outside and inside the horizon?

8. ▽ What speed would observers inside the horizon being at the position of the light beams actually measure for the light beams? (explain your reasoning well!)

We now have all the information needed to disclose whenever the astronaut have any possibilities to escape. So let's determine his/her fate by first studying the fate of the two emitted light beams.

9. ▽ Use equation 1 to determine the direction of the two emitted light beams inside the horizon.

10. ▽ The world line of the spaceship and the direction of the world lines for the two emitted light beams (arrows) has been plotted in figure 4. Use your previous results and equation 1 to explain why the world lines for the light beams have the direction they have for each point in the diagram.

11. ▽ Looking at equation 1 and figure 4, can the light beams exit the horizon?

12. ▽ By looking at the worldlines of light as well as the fact that light cannot escape the black hole. Does the astronaut with a rocket that can reach a velocity close to the speed of light escape the black hole?

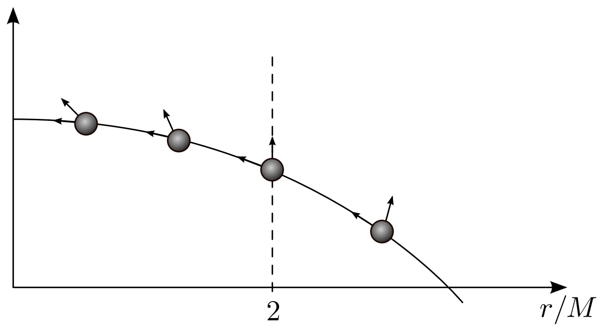

FIG. 4: For exercise 7: Worldline of the rocket (marked by a balls) and parts of the worldlines of the forward and backward light beam (arrows) at several points during the free fall into the black hole.

EXERCISE 8

Relevant theory: Section 2 of part 2E.

1. ☺ Use the equations of motion for a photon written in terms of the impact paramteter b (from lecture note 2E) to show that the radial light speed $dr_{\text{shell}}/dt_{\text{shell}}$ observed by the shell observer can be written as

$$\frac{1}{b^2}\left(\frac{dr_{\text{shell}}}{dt_{\text{shell}}}\right)^2 = \frac{1}{b^2} - \frac{\left(1-\frac{2M}{r}\right)}{r^2}. \quad (2)$$

2. ☺ Look at equation (7) and (8) from part 2D and show that we can define an effective potential for light (based on the shell velocity rather than the velocity dr/dt) as

$$V(r) = \sqrt{\frac{\left(1-\frac{2M}{r}\right)}{r^2}}.$$

3. ▽ Sketch the potential (if you wish, by hand) and explain if light may move in a stable orbit or not, justify your answer using your drawing.