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NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know

Partial solutions to problems: Part 1G

Problem 1G.1

In this exercise, we're asked to derive the expression for the mean kinetic energy of a particle in a degenerate gas. This gas no longer follows the normal M.B-distribution, which we have used in earlier exercises.

1. Let's summarize: We have a relation between $n(\vec{p})$(the number density per volume per momentum space volume for particles with momentum $\vec{p}$) and $n(p)$ (the number density per real space volume for particles with absolute momentum p). This relation is given by $n(p)dp = 4\pi p^2 n(\vec{p})dp$, where we obtain the real-space volume element by integrating a sphere over the momentum-space for a fixed absolute momentum. We're now asked to find a relation between $n(p)$ and $n(E)$. We know that

$$E = \frac{p^2}{2m}$$

such that

$$p = \sqrt{2mE}$$

and

$$dp = \frac{1}{2\sqrt{2mE}} \cdot 2mdE = \sqrt{\frac{m}{2E}}dE$$
$$\frac{dp}{dE} = \sqrt{\frac{m}{2E}}$$

Now, we switch from $n(p)$ to $n(E)$ using the chain rule:

$$n(E) = n(p)\frac{dp}{dE} = n(p)\sqrt{\frac{m}{2E}}$$

and insert for $n(p) = 4\pi p^2 n(\vec{p})$:

$$n(E) = 4\pi p^2 n(\vec{p})\sqrt{\frac{m}{2E}} = 4\sqrt{2}\pi m^{3/2}\sqrt{E}n(\vec{p})$$

where we substituted $p^2 = (2mE)$. We now insert for

$$n(\vec{p}) = \frac{2}{h^3}\frac{1}{e^{(p^2-p_F^2)/(2mkT)}+1}$$
$$n(E) = \frac{8\sqrt{2}\pi m^{3/2}}{h^3}\sqrt{E}\frac{1}{e^{(p^2-p_F^2)/(2mkT)}+1}$$

Rewriting, we find

$$n(E) = 4\pi\left(\frac{2m}{h^2}\right)^{3/2}\sqrt{E}\frac{1}{e^{(p^2-p_F^2)/(2mkT)}+1} = \frac{g(E)}{e^{(p^2-p_F^2)/(2mkT)}+1}$$

2. We continue by finding the mean kinetic energy of a particle in a degenerate gas:

$$\langle E\rangle = \int_0^\infty P(E)EdE$$

First, remember that the probability distribution is given by $n(E)$, but a probability distribution needs to be normalized such that

$$P(E) = Nn(E)$$

where N is found by

$$\int_0^\infty P(E)dE = N\int_0^{E_f} n(E)dE = 1$$

where the E_f-limit is because $n(E) = 0$ for $E > E_F$. The next thing we do is an approximation: in this energy range, the $e^{(p^2-p_F^2)/(2mkT)}$ in $n(E)$ is much less than 1. We can then approximate $n(E) \approx g(E)$, and the integral becomes surprisingly simple. But first we need to normalize the distribution. For simplicity we define $K = 4\pi\left(\frac{2m}{h^2}\right)^{3/2}$. Then

$$1 = N\int_0^{E_f} g(E)dE = NK\int_0^{E_f} E^{1/2}dE = NK\frac{2}{3}E_F^{3/2} = 1$$

such that $N = 3/2(KE_F^{3/2})$. The expectation value is thus

$$\langle E\rangle = N\int_0^{E_f} g(E)EdE = \frac{3}{2}E_F^{-3/2}\int_0^{E_f} E^{3/2}dE$$

$$\langle E\rangle = \frac{3}{2}E^{-3/2}\frac{2}{5}E_F^{5/2} = \frac{3}{5}E_F$$