The following people have participated in creating these solutions: Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger
NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know.

Partial solutions to problems: Part 3D

Exercise 3D.1

1. Full range given in following answer
2. Full range given in following answer
3. Using the HR-diagram (figure 1 in the lecture notes), the luminosity of a $G0$ star ranges between 0.8 and 60 L_{sun}, but these numbers are all approximate. In the same range, the absolute magnitude M would be between 2 and 6.
4. Recall that it is possible to decide the distance r to a star from the difference between apparent (m) and absolute (M) magnitude:

$$M - m = -5\log_{10}(\frac{r}{10pc})$$

solving for r gives

$$r = 10pc \cdot 10^{\frac{m-M}{5}}$$

With an apparent magnitude $m = 1$, we find that the range of distance for a $G0$ star becomes

$$r_{min} = 10pc \cdot 10^{\frac{1-6}{5}} = 1pc$$

$$r_{max} = 10pc \cdot 10^{\frac{1-2}{5}} = 6pc$$

which isn't very accurate.

Exercise 3D.2

1. The volume V of a sphere as function of radius r is given as $V(r) = \frac{4}{3}\pi r^3$. The total mass is the mass density times the volume, so

$$M(r) = \frac{4}{3}\pi r^3 \rho$$

if we assume ρ to be constant.

2. The hydrostatic equation reads

$$\frac{dP}{dr} = -\rho G \frac{M(r)}{r^2} = -\frac{4}{3}\pi G \rho^2 r$$

where the $M(r)$ from 13.3.1 was inserted. We start by fluffing around with differentials:

$$\frac{dP}{dr} = \frac{dP}{dr}\frac{dT}{dT} = \frac{dT}{dr}\frac{dP}{dT}$$

The pressure is given as $P = \rho kT/(\mu m_H)$. Then

$$\frac{dP}{dr} = \frac{dT}{dr}\frac{dP}{dT} = \frac{dT}{dr}\frac{d}{dT}\left(\frac{\rho kT}{\mu m_H}\right) = \frac{dT}{dr}\frac{\rho k}{\mu m_H}$$

Insert this expression into the hydrostatic equation and obtain

$$\frac{dT}{dr} = -\frac{4}{3}\pi G \rho r \frac{\mu m_H}{k} \tag{0.1}$$

3. We now integrate this solution from 0 to r. Letting

$$C = \pi G \frac{\mu m_H}{k}$$

equation 0.1 becomes

$$\frac{dT}{dr} = -\frac{4}{3} C\rho \cdot r$$

integrating with regards to r from 0 to R gives

$$T(R) - T_C = -\frac{4}{3} C\rho \cdot \int_0^R r = -\frac{2}{3} C\rho R^2$$

such that

$$T_C = \frac{2}{3} C\rho R^2 + T(R)$$

4. Assuming the Sun to be spherical with a homogeneous (homogeneous means that $\rho(\vec{x}) \equiv \rho_0$ is constant) density, the total mass is expressed as

$$M = V \cdot \rho = \frac{4}{3}\pi r^3 \cdot \rho$$

solving for ρ

$$\rho = M \frac{3}{4\pi R^3} \approx 1.4 \cdot 10^3 kg/m^3$$

We now use this ρ for estimating the core temperature of the sun:

$$T_C = T(R) + \frac{2}{3} R^2 \pi G \rho \frac{\mu m_H}{k} \approx 11.5\,\text{million}\,\text{K}$$

where $R = 700\,000$km, k the Boltzmann-constant, $\mu = 1$ (assuming only protons populate the sun), $T(R) \approx 0$ as the surface temperature is way lower than the core temperature, m_H is the proton mass and G the gravitational constant. The "real" temperature when accounting for varying density ρ is $\sim$ 15 million K. Pretty hot, that is.

5. The *pp*-chain dominates as the core temperature $T_C < 20$ million K.

6. We already saw that
$$\rho = M\frac{3}{4\pi R^3}$$
inserting into
$$T_C = \frac{2}{3}C\rho R^2 + T(R)$$
we find
$$T_C = \frac{2}{3}CR^2 M\frac{3}{4\pi R^3} \propto \frac{M}{R}$$

7. The temperature in the core T_C is proportional to
$$T_C \propto \frac{M}{R}$$
so if the temperature increases by a factor of 10, then for a constant mass M the radius has to be decreased by a factor 10.

8. This is a nice exercise, as one has to utilize all previous knowledge from this exercise. It is basically just a repetition of things already done, but with a different pressure P. Return to the fact that
$$\frac{dP}{dr} = -\rho\frac{GM}{r^2} = \frac{dP}{dT}\frac{dT}{dr} \qquad (0.2)$$
where now $P = \frac{1}{3}aT^4$ is pure good old relativistic radiation pressure. Then
$$\frac{dP}{dT} = \frac{4}{3}aT^3$$
inserting this back into 0.2 to obtain
$$\frac{dT}{dr} = -\rho\frac{GM}{r^2}\left(\frac{dP}{dT}\right)^{-1} = -\rho\frac{GM}{r^2}\frac{3}{4aT^3}$$
Separating the r and T on each side, we obtain a separable differential equation:
$$T^3 dT = -\rho GM\frac{3}{4a}\frac{1}{r^2} = -\frac{\pi G}{a}r\rho^2 dr$$
where we used that the mass $M = \frac{4}{3}\pi r^3\rho$. Integrating both sides gives
$$\int_{T_C}^{T(R)} T^3 dT = -\rho^2\frac{\pi G}{a}\int_0^R r dr$$
such that
$$\frac{1}{4}\left(T_C^4 - T(R)^4\right) = \rho^2\frac{\pi G}{2a}R^2$$
Solving for T_C alone gives
$$T_C^4 = T(R)^4 + \rho^2\frac{2\pi G}{a}R^2$$
Take the 4th root on both sides, and Voilà! We're done.

Exercise 3D.3

1. We are now given a variable (and much more realistic) mass density of a star which is dependent on r and the radius R:

$$\rho(r) = \frac{\rho_C}{1 + \left(\frac{r}{R}\right)^2}$$

The mass inside a spherical shell of radius r is given as $M = \int \rho \cdot dV$, where the volume element $dV = 4\pi r^2 dr$. Then

$$M(r) = \int_0^r \rho(r) 4\pi r^2 dr = 4\pi \int_0^r \frac{\rho_C r^2}{1 + \left(\frac{r}{R}\right)^2} dr$$

Substituting $x = r/R$ gives $r = xR$ and $dr = Rdx$, such that

$$M(r) = 4\pi \int_0^r \frac{\rho_C x^2 R^2}{1 + x^2} Rdx = 4\pi \rho_C R^3 \int_0^x \frac{x^2}{1 + x^2} dx$$

Using the fact that

$$\int_0^x \frac{x^2}{1 + x^2} dx = x - \arctan x$$

the mass is expressed as

$$M(r) = 4\pi \rho_C R^3 \left(\frac{r}{R} - \arctan \frac{r}{R}\right)$$

2. The hydrostatic equilibrium is expressed as

$$\frac{dP}{dr} = -\rho(r)\frac{GM}{r^2} = -4\pi \frac{\rho_C}{1 + \left(\frac{r}{R}\right)^2} \rho_C R^3 \left(\frac{r}{R} - \arctan \frac{r}{R}\right) \frac{G}{r^2} \tag{0.3}$$

We use the ideal gas law $P = \rho(r)kT(r)/(\mu m_H)$, and take the derivative with respect to r. Then

$$\frac{dP}{dr} = \frac{d}{dr}\left(\frac{T(r)\rho(r)k}{\mu m_h}\right)$$

Inserting this expression into 0.3 and move the constants μ, m_H and k to the right hand side yields

$$\frac{d}{dr}\Big(\rho(r)T(r)\Big) = -\frac{\mu m_H}{k} 4\pi \frac{\rho_C^2}{1 + \left(\frac{r}{R}\right)^2} R^3 \left(\frac{r}{R} - \arctan \frac{r}{R}\right) \frac{G}{r^2}$$

puh.

3. This is again a separable differential equation, so we separate the $r's$ and the $T's$ on each side:

$$\rho(r)T(r) - \rho_C T_C = -\int_0^r \frac{\mu m_H}{k} 4\pi \frac{\rho_C^2}{1+\left(\frac{r}{R}\right)^2} R^3 \left(\frac{r}{R} - \arctan\frac{r}{R}\right)\frac{G}{r^2} dr$$

Ni-ice. Now move the constants outside the integral:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^3 G\right)\int_0^r \frac{1}{1+\left(\frac{r}{R}\right)^2}\left(\frac{r}{R} - \arctan\frac{r}{R}\right)\frac{1}{r^2} dr$$

and use the same substitution as in exercise 13.4.1:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^2 G\right)\int_0^x \frac{1}{1+x^2}\left(x - \arctan x\right)\frac{1}{x^2} dx$$

where one of the R's in the denominator disappeared due to the change of variable. Including the $1/x^2$, we split the integral into two parts:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^2 G\right)\int_0^x \left(\frac{1}{(1+x^2)x} - \frac{\arctan x}{(1+x^2)x^2}\right) dx \qquad (0.4)$$

4. We magically use that

$$\int_0^x \frac{1}{x(x^2+1)} dx = \ln x - \frac{1}{2}\ln(x^2+1)$$

and

$$\int_0^x \frac{\arctan x}{x^2(x^2+1)} dx = -\frac{1}{2}(\arctan x)^2 - \frac{1}{x}\arctan x + \ln x - \frac{1}{2}\ln(x^2+1)$$

Inserting these two fellows into equation 0.4, the logarithmic parts luckily cancel (as $\ln_{x\to 0} x = -\infty$!). Then:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^2 G\right)\left(\frac{1}{2}\left(\arctan(\frac{r}{R})\right)^2 + \frac{R}{r}\arctan\frac{r}{R} - 1\right)$$

The extra -1 has a curious origin: in the limit when $x \to \infty$, then by L'hôpital's rule, $\lim_{x\to 0} \arctan(x)/x = 1$. Rearranging terms and dividing by ρ_C results in

$$T_C = \frac{\rho(r)}{\rho_C}T(r) + \left(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\right)\left(\frac{1}{2}\left(\arctan(\frac{r}{R})\right)^2 + \frac{R}{r}\arctan\frac{r}{R} - 1\right)$$

inserting for $\rho(r)$ gives

$$T_C = \frac{1}{1+\left(\frac{r}{R}\right)^2} T(r) + \Big(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\Big)\Big(\frac{1}{2}\big(\arctan(\frac{r}{R})\big)^2 + \frac{R}{r}\arctan\frac{r}{R} - 1\Big) \tag{0.5}$$

which is the end result.

5. What happens when the arctan's $r \propto x \to \infty$? From basic arithmetic's, we know that $\lim_{x\to\pi/2}\tan(x) = \infty$, so $\lim_{x\to\infty}\arctan x = \pi/2$. Inserting this into equation (0.5) one obtains

$$T_C = \Big(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\Big)\Big(\frac{1}{2}(\frac{\pi}{2})^2 - 1\Big)$$

where the 1st and 3rd terms disappear as $\lim_{x\to\infty} 1/(1+x^2) = 0$.

6. From

$$\rho(r) = \frac{\rho_C}{1+\left(\frac{r}{R}\right)^2}$$

it is easy to see that the density $\rho(r) = \frac{1}{2}\rho_C$ when $r = R$. We now need to decide what this R is. The core stops where $r = 0.2R_{sun}$, and at this point $\rho(r) = \frac{1}{10}\rho_C$. Then

$$\frac{1}{10}\rho_C = \frac{\rho_C}{1+\left(\frac{0.2R_{sun}}{R}\right)^2}$$

where R is the point that the density is halved. Inverting both sides and removing ρ_C yields

$$10 = 1 + \left(\frac{0.2R_{sun}}{R}\right)^2$$

such that

$$\sqrt{9} = \frac{0.2R_{sun}}{R}$$

or

$$R = \frac{0.2R_{sun}}{3} \approx 0.067R_{sun}$$

7. We now use the approximation in the core:

$$T_C = \Big(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\Big)\Big(\frac{1}{2}(\frac{\pi}{2})^2 - 1\Big)$$

where R was given in 13.4.6. Solve for ρ_C:

$$\rho_C = \frac{T_C k}{\mu m_H G R^2 4\pi(\pi^2/8 - 1)} \approx 2.9 \cdot 10^5 kg$$

or 200 times the mean density (assuming $1400kg/m^3$), about a factor two wrong. Not bad!

Exercise 3D.4

1. We use that $M \propto T^2$, that is,

$$M = CT^2$$

where we obtain C by using what we know about the sun: $C = M/T^2 \approx 6 \cdot 10^{22}$. The small star with $M = 0.5M_{sun}$ has temperature

$$T = \sqrt{0.5 * M/C} \approx 4000K$$

while the larger star $M = 5M_{sun}$

$$T = \sqrt{5 * M/C} \approx 13000K$$

and the largest star $M = 40M_{sun}$

$$T = \sqrt{40 * M/C} \approx 36000K$$

2. We use Wien's displacement law $T = 2.9 \cdot 10^6 nmK/\lambda = 14400K$. From this, $M = CT^2 = 6.2M_{sun}$.

Exercise 3D.5

We use that the radiation pressure is given as $P \sim T^4$ such that the hydrostatic equation reads

$$\frac{dP}{dR} = \frac{dP}{dT}\frac{dT}{dR} \propto T^3 \frac{T}{R} \propto g\rho \propto \frac{M^2}{R^5}$$

in other words,

$$T^4 \propto \frac{M^2}{R^4} \tag{0.6}$$

using the fact (equation 2 in the lecture notes) that

$$L \propto \frac{R^4 T^4}{M} = M$$

when equation (0.6) inserted

Exercise 3D.8

1. The hydrostatic equation says that

$$\frac{dP}{dr} \propto g\rho \propto \frac{M^2}{R^5}$$

but assuming that $P \propto r^n$ for any n, we find that

$$\frac{dP}{dr} \propto R^{n-1} = \frac{R^n}{R} \propto \frac{P}{R}$$

such that

$$P \propto \frac{M^2}{R^4}$$

2. For an ideal gas,

$$P \propto \rho T = \frac{M}{R^3} T = \frac{M^2}{R^4}$$

such that

$$T \propto \frac{M}{R}$$

3. The efficiency of the *pp*-chain is

$$\epsilon_{pp} = \epsilon_{0,pp} X_H^2 \rho T_6^2 \approx 0.005$$

while

$$\epsilon_{CNO} = \epsilon_{0,CNO} X_H X_{CNO} \rho T_6^{20} \approx 0.009$$

so the CNO-cycle dominated this star

4. For the triple-α process to occur, we need a helium-abundant core, that is, $X_{he} = 1$. In this case, we find

$$\epsilon_{03\alpha} \rho^2 \Big(\frac{T_6}{100}\Big)^{41} = \epsilon_{0CNO} X_H X_{CNO} \rho 18^{20}$$

where we used that $T_8 = T_6/100$. Solving for T_6 yields

$$T_6 = \Big(\frac{\epsilon_{0CNO} X_H X_{CNO}}{\epsilon_{03\alpha} \rho} \cdot 100^{41} \cdot 18^{20}\Big)^{1/41} \approx 131\,\text{million K} \qquad (0.7)$$

5. We use that

$$R \propto \frac{M}{T} = C \cdot \frac{1}{T}$$

for M constant when the core is contracting and C is a constant. We determine C by using what we know about the star on the main sequence ($R = 0.2R_{sun}, T_{sun} = 18MK$), so $C = RT$. For a core temperature to reach 131 million K, the radius of the core needs to be

$$R = \frac{C}{T} = 0.2R_{sun} 18 \frac{1}{131} = 0.0275 R_{sun}$$

6. A more suitable density can be found as such

$$\rho_{before}(0.2R_{sun})^3 = \rho_{after}(0.0275R_{sun})^3$$

such that

$$\rho_{after} = \rho_{before}(\frac{0.2}{0.0275})^3 = 384\rho_{before}$$

Using equation (0.7) we obtain a new core temperature of $T \approx 114$ million K.