Solution to the analytical part of project 3, AST3220, 2024

Problem 1

a) EdS model $\Rightarrow$ $k=0$, $H = H_0(1+z)^{3/2}$

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3}$$

Definition of the redshift:

$$1+z = \frac{a_0}{a}$$

$$\Rightarrow \quad 1+z = \frac{a_0}{a_0(t/t_0)^{2/3}} = \left(\frac{t_0}{t}\right)^{2/3}$$

$$\Rightarrow \quad (1+z)^{3/2} = \frac{t_0}{t}$$

$$\Rightarrow \quad t(z) = \frac{t_0}{(1+z)^{3/2}}$$

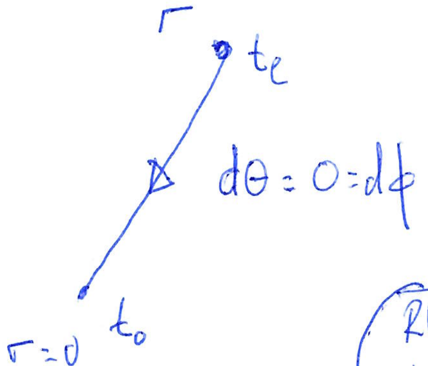
b) With $z = z_1 = 3$,

$$t(z_1) = \frac{t_0}{4^{3/2}} = \underline{\underline{\frac{t_0}{8}}}$$

With $z = z_2 = 8$:

$$t(z_2) = \frac{t_0}{9^{3/2}} = \underline{\underline{\frac{t_0}{27}}}$$

c) Light emitted at $t = t_e$, observed at $t = t_0$



Light: $ds^2 = 0$

RW with $k = 0$, $d\theta = 0 = d\phi$

$$\Rightarrow \quad c^2 dt^2 - a^2(t)\, dr^2 = 0$$

$$\Rightarrow \quad dr = -\frac{c\,dt}{a(t)}$$

Light moving inwards

$$\Rightarrow \quad \int_0^r dr' = r = -\int_{t_0}^{t_e} \frac{c\,dt}{a(t)} = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}$$

For EdS:

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a_0 \left(\frac{t}{t_0}\right)^{2/3}} = \frac{c t_0^{2/3}}{a_0} \int_{t_e}^{t_0} t^{-2/3}\, dt$$

$$= \frac{c t_0^{2/3}}{a_0} \Big|_{t_e}^{t_0} 3 t^{1/3} = \frac{3 c t_0^{2/3}}{a_0} \left(t_0^{1/3} - t_e^{1/3}\right)$$

$$\Rightarrow \quad r = \frac{3ct_0}{a_0}\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right]$$

Also:
$$1+z = \frac{a_0}{a(t_e)} = \frac{a_0}{a_0\left(\frac{t_e}{t_0}\right)^{2/3}} = \left(\frac{t_0}{t_e}\right)^{2/3}$$

$$\Rightarrow \quad \left(\frac{t_e}{t_0}\right)^{1/3} = \frac{1}{\sqrt{1+z}}$$

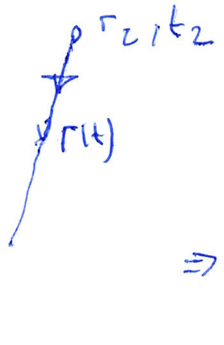
So
$$r = \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

$\underline{z = z_1 = 3}$
$$r_1 = \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{4}}\right) = \underline{\underline{\frac{3ct_0}{2a_0}}}$$

$\underline{z = z_2 = 8}$
$$r_2 = \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{9}}\right) = \underline{\underline{\frac{2ct_0}{a_0}}}$$

d) Light emitted at $t = t_2 = t(z_2) = \frac{t_0}{27} = t_e$

from $r = r_2 = \frac{2ct_0}{a_0}$:



Light: $ds^2 = 0$

$RW(k=0,\ d\theta = d\phi = 0) \Rightarrow c^2 dt^2 - a^2(t) dr^2 = 0$

light moving inwards

$$\Rightarrow \quad dr = -\frac{c\,dt}{a(t)}$$

$$\Rightarrow \quad \int_{r_2}^{r(t)} dr = -\int_{t_e}^{t} \frac{c\,dt'}{a(t')}$$

$$\Rightarrow \quad r(t) - \frac{2ct_0}{a_0} = -\int_{t_e}^{t} \frac{c\,dt'}{a_0\left(\frac{t'}{t_0}\right)^{2/3}} = -\frac{ct_0^{2/3}}{a_0}\int_{t_e}^{t} t'^{-2/3}\,dt'$$

$$= -\frac{ct_0^{2/3}}{a_0} \Big| _{t_e}^{t} 3t'^{1/3}$$

$$= -\frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - t_e^{1/3}\right)$$

$$= -\frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \underbrace{\left(\frac{t_e}{t_0}\right)^{1/3}}_{\left(1/27\right)^{1/3}}\right]$$

$$= -\frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \frac{1}{3}\right]$$

$$= \frac{ct_0}{a_0} - \frac{3ct_0}{a_0}\left(\frac{t}{t_0}\right)^{1/3}$$

$$\Rightarrow \quad r(t) = \frac{3ct_0}{a_0} - \frac{3ct_0}{a_0}\left(\frac{t}{t_0}\right)^{1/3}$$

$$= \frac{3ct_0}{a_0}\left[1 - \left(\frac{t}{t_0}\right)^{1/3}\right]$$

e) The redshift measured by the observer at $r = r_1$ is given by

$$1 + z_{12} = \frac{a(t_{12})}{a(t_e)} = \frac{a_0\left(\frac{t_{12}}{t_0}\right)^{2/3}}{a_0\left(\frac{t_e}{t_0}\right)^{2/3}} = \left(\frac{t_{12}}{t_e}\right)^{2/3}$$

where $t_e = t_2 = \frac{t_0}{27}$

We find t_{12} from the condition

$$r(t_{12}) = r_1$$

from c)

$$\Rightarrow \quad \frac{3ct_0}{a_0}\left[1 - \left(\frac{t_{12}}{t_0}\right)^{1/3}\right] = \frac{3ct_0}{2a_0}$$

$$\Rightarrow \quad \left(\frac{t_{12}}{t_0}\right)^{1/3} = \frac{1}{2}$$

$$\Rightarrow \quad t_{12} = \frac{t_0}{8}$$

So

$$1 + z_{12} = \left(\frac{t_0/8}{t_0/27}\right)^{2/3} = \left(\frac{27}{8}\right)^{2/3} = \frac{9}{4}$$

$$\Rightarrow \quad \underline{\underline{z_{12} = \frac{5}{4}}}$$

Problem 2

a) The Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \qquad (FI)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \qquad (FII)$$

We are given that $\rho + \frac{3p}{c^2} > 0$, and from FII we see that we therefore have

$$\underline{\ddot{a} < 0}$$

$H_0 > 0$ means that the Universe is expanding today, at $t = t_0$

From FI: because expanding today

$$\frac{\dot{a}}{a} = \underset{\uparrow}{+}\sqrt{\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}}$$

If $k=0$ or $k=-1$, the expression under the square root is always positive, so the universe was expanding in the past, and will continue to expand in the future.

If $k=+1$, there could be a point where the expression vanishes, and there we would have a switch from expansion to contraction. But we know that

$$(*) \quad \frac{8\pi G}{3}\rho > \frac{kc^2}{a^2} \quad \text{at } t=t_0,$$

and we are given that ρ decreases faster with a than $1/a^2$. Since a was smaller in the past, this means that $(*)$ was also satisfied at all times in the past. We can therefore conclude that the universe has been strictly expanding up until (at least) $t=t_0$.

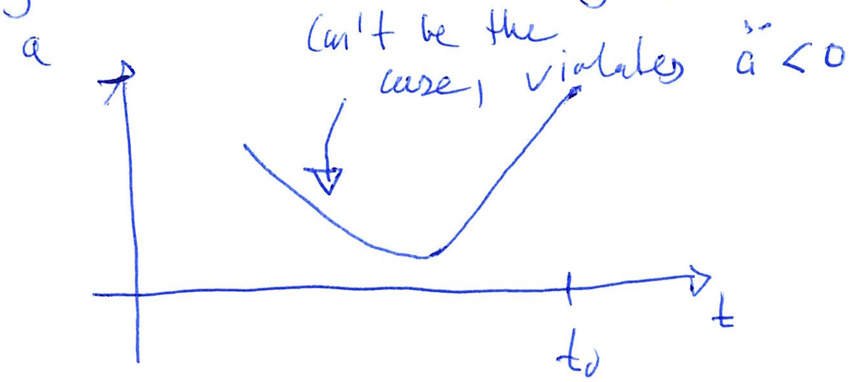
So now we know that

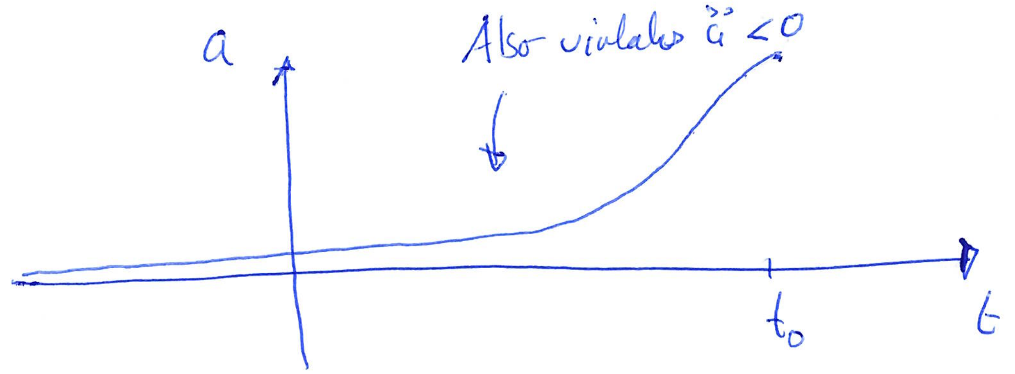
$$\dot{a} > 0$$

<u>and</u> $\quad \ddot{a} < 0$

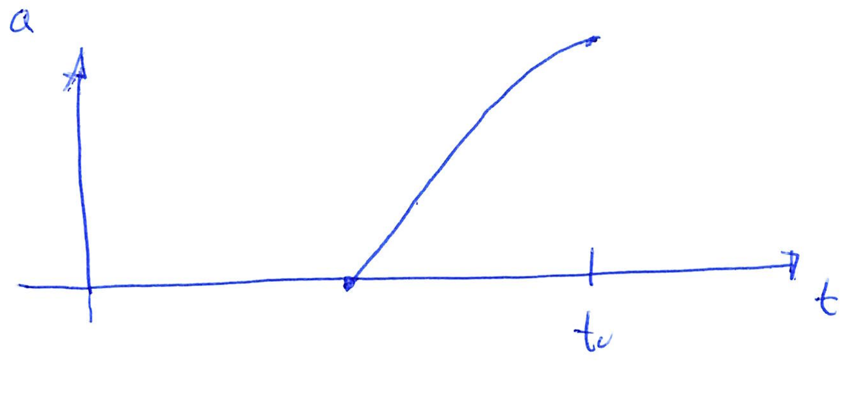
for all $t \leq t_0$.

What does this mean? Well, we can try to draw the graph of a vs t:





Only possibility:

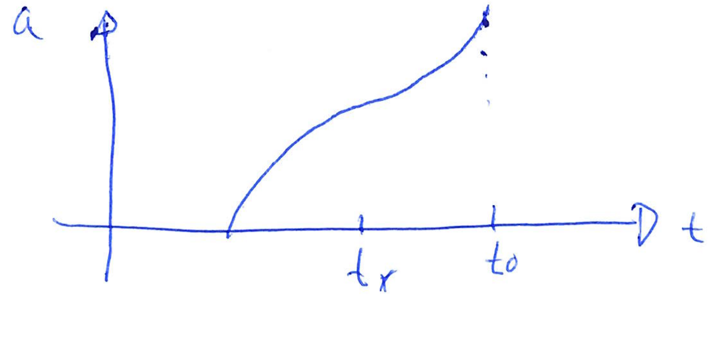


$\Rightarrow$ Must be a time in the past when $a = 0$. $\square$

b) In this model, $\ddot{a} > 0$ today.
However, since $\rho_m \propto a^{-3}$ and
ρ_Λ = constant, as we go into
the past and a becomes smaller,
$\ddot{a}$ will change sign, so $\ddot{a} < 0$
before some time t^*.
Furthermore, since $\Omega_{m0} + \Omega_{\Lambda 0} + \Omega_{k0} = 1$,
and $\Omega_{m0} = 0{,}3$, $\Omega_{\Lambda 0} = 0{,}7$,
we have $\Omega_{k0} = 0 \Rightarrow k = 0$,
so we can be sure that the universe
was always expanding in the past.
Before t^* we therefore have both

$$\dot{a} > 0 \quad \text{and}$$
$$\ddot{a} < 0$$

for all $t < t_*$, and then
the argument from a) goes through.
So also in this model, $a = 0$ at
some time in the past



Problem 3

a) We change variables in the integral, first from t to a using

$$\dot{a} = \frac{da}{dt} \Rightarrow dt = \frac{da}{\dot{a}} = \frac{da}{aH} \quad \left(H = \frac{\dot{a}}{a}\right):$$

$$d_{P,PH} = a(t)\int_0^t \frac{c\,dt'}{a(t')} = a(t)\int_{\underset{0}{\overset{\shortparallel}{a(t=0)}}}^{a(t)} \frac{c\,da'}{a'\cdot a'H(a')}$$

$$= a(t)\int_0^{a(t)} \frac{c\,da'}{a'^2 H(a')}$$

Next we change from a to z using

$$a' = \frac{a_0}{1+z'} \Rightarrow da' = -\frac{a_0\,dz'}{(1+z')^2}$$

$$1+z' = \frac{a_0}{a'} \;; \quad a' = 0 \Rightarrow z' \to \infty$$

$$a' = a(t) \Rightarrow 1+z' = \frac{a_0}{a(t)} \equiv 1+z$$

So

$$d_{P,PH}(z) = a(t)\int_\infty^z c\cdot\left(\frac{1+z'}{a_0}\right)^2 \cdot \frac{1}{H(z')}\left(-\frac{a_0\,dz'}{(1+z')^2}\right)$$

$$= \frac{c\,a(t)}{a_0}\int_z^\infty \frac{dz'}{H(z')} = \underline{\underline{\frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')}}}$$

b) Matter-dominated universe:

$$H(z) = H_0 \sqrt{\Omega_{m0}} \, (1+z)^{3/2}$$

$$\rightarrow \quad d_{p,PH}(z) = \frac{c}{1+z} \int_z^\infty \frac{dz'}{H_0 \sqrt{\Omega_{m0}} (1+z')^{3/2}}$$

$$= \frac{c}{H_0 (1+z) \sqrt{\Omega_{m0}}} \int_z^\infty \frac{dz'}{(1+z')^{3/2}}$$

$$= \frac{c}{H_0 \sqrt{\Omega_{m0}} (1+z)} \Big|_z^\infty -2 (1+z')^{-\frac{1}{2}}$$

$$= \frac{c}{H_0 \sqrt{\Omega_{m0}} (1+z)} \cdot \frac{2}{(1+z)^{1/2}}$$

$$= \frac{2c}{H_0 \sqrt{\Omega_{m0}} (1+z)^{3/2}} = \frac{2c}{H(z)} \sim \frac{c}{H(z)}$$

Radiation-dominated universe:

$$H(z) = H_0 \sqrt{\Omega_{r0}} \, (1+z)^2$$

$$\rightarrow \quad d_{p,PH}(z) = \frac{c}{1+z} \int_z^\infty \frac{dz'}{H_0 \sqrt{\Omega_{r0}} (1+z')^2}$$

$$= \frac{c}{H_0 \sqrt{\Omega_{r0}} (1+z)} \int_z^\infty \frac{dz'}{(1+z')^2}$$

$$= \frac{c}{H_0 \sqrt{\Omega_{r0}} (1+z)} \Big|_z^\infty - \frac{1}{1+z'} = \frac{c}{H_0 \sqrt{\Omega_{r0}} (1+z)^2} = \frac{c}{H(z)}$$

c) The particle horizon depends on the behaviour of a at times between 0 and t, so we need to know when the universe went from being dominated by radiation to being dominated by matter. We have

$$\rho_m = \rho_{m0}\left(\frac{a}{a_0}\right)^{-3} = \Omega_{m0}\rho_{c0}(1+z)^3$$

and

$$\rho_r = \rho_{r0}\left(\frac{a}{a_0}\right)^{-4} = \Omega_{r0}\rho_{c0}(1+z)^4 .$$

The transition happened at

$$\Omega_{m0}\rho_{c0}(1+z_{eq})^3 = \Omega_{r0}\rho_{c0}(1+z_{eq})^4$$

$$\Rightarrow \quad 1+z_{eq} = \frac{\Omega_{m0}}{\Omega_{r0}} = \frac{0{,}3}{10^{-4}} = 3\cdot 10^3$$

For $z > z_{eq}$ the universe was dominated by radiation, for $z < z_{eq}$ by matter.

We want to solve

$$d_{P,PH}(z) = 10 \text{ km} = 10^4 \text{ m}$$

But which expression should we use for $d_{P,PH}(z)$?

Let's try the matter-dominated one first:

$$d_{p,PH}(z) = \frac{2c}{H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}} = 10^4 m \qquad \frac{c}{H_0} = \frac{2997.9}{h} Mpc$$

$$\Rightarrow \quad (1+z)^{3/2} = \frac{\frac{2c}{H_0\sqrt{\Omega_{m0}}}}{10^4 m} = \frac{\frac{2 \cdot 2997.9 \cdot 10^6 \cdot 3{,}086 \cdot 10^{16}}{0{,}7 \cdot \sqrt{0{,}3}} m}{10^4 m}$$

$$= \frac{4{,}83 \cdot 10^{26} m}{10^4 m} = 4{,}83 \cdot 10^{22}$$

$$\Rightarrow \quad 1+z = 6{,}15 \cdot 10^{17}$$

This redshift is deep within the radiation-dominated era, so this result can't be accurate. We should use the expression for a radiation dominated universe instead. So we must solve

$$\frac{c}{H_0\sqrt{\Omega_{r0}}(1+z)^2} = 10^4 m$$

$$\Rightarrow \quad (1+z)^2 = \frac{\frac{c}{H_0\sqrt{\Omega_{r0}}}}{10^4 m} = \frac{1{,}32 \cdot 10^{28} m}{10^4 m} = 1{,}32 \cdot 10^{24}$$

$$\Rightarrow \quad \underline{\underline{1+z = 1{,}15 \cdot 10^{12}}}$$

The critical density is

$$\rho_{c0} = 1{,}878 \cdot 10^{-29}\, h^2\, g\, cm^{-3}$$

$$= 1{,}878 \cdot 10^{-26}\, h^2\, kg\, m^{-3}$$

$$\underset{h=0{,}7}{=} 9{,}202 \cdot 10^{-27}\, kg\, m^{-3}$$

The matter density at this redshift was therefore

$$\rho_m = \Omega_{m0}\, \rho_{c0}\, (1+z)^3 = 0{,}3 \cdot 9{,}202 \cdot 10^{-27}\, kg\, m^{-3} \cdot (1{,}15 \cdot 10^{12})^3$$

$$= \underline{\underline{4{,}2 \cdot 10^{9}\, kg\, m^{-3}}}$$

while the radiation density was

$$\rho_r = \Omega_{r0}\, \rho_{c0}\, (1+z)^4 = 10^{-4} \cdot 9{,}202 \cdot 10^{-27}\, kg\, m^{-3}\, (1{,}15 \cdot 10^{12})^4$$

$$= \underline{\underline{1{,}6 \cdot 10^{18}\, kg\, m^{-3}}}$$

(bear in mind that this is really <u>kinetic energy</u>, since radiation has zero rest mass)

The average density of a typical neutron star:

$$\rho_{NS} = \frac{3M}{4\pi R^3} \sim \frac{M}{4R^3} = \frac{1{,}5 \cdot 2 \cdot 10^{30}\, kg}{4 \cdot 10^{12}\, m^3}$$

$$= \frac{3}{4} \cdot 10^{18}\, kg\, m^{-3} \sim 8 \cdot 10^{17}\, kg\, m^{-3}$$

So the matter density at that time was much smaller than that of a neutron star, while the radiation (energy) density was comparable to it.

d) The relationship between redshift and the CMB temperature gives

$$T = T_0 (1+z) = 2{,}725\,K \cdot 1{,}15 \cdot 10^{12}$$
$$= \underline{\underline{3{,}1 \cdot 10^{12}\,K}}$$

e) The trick to finding an expression for the age of the Universe is to start from

$$\dot{a} = \frac{da}{dt}$$

$$\Rightarrow \quad dt = \frac{da}{\dot{a}}$$

We integrate, assuming $a(t=0) = 0$ (otherwise it would not be clear what we mean by the age of the universe)

$$t(z) = \int_0^{t(z)} dt = \int_0^a \frac{da'}{\dot{a}'} = \int_0^a \frac{da'}{a' H(a')}$$

Next we change integration variable from a' to z':

$$1+z' = \frac{a_0}{a'} \; ; \quad a' = 0 \Rightarrow z' \to \infty \; , \quad a' = a \Rightarrow z' = z$$

$$a' = \frac{a_0}{1+z'} \; , \quad da' = -\frac{a_0 \, dz'}{(1+z')^2}$$

$$\Rightarrow \quad t(z) = -\int_{\infty}^{z} \left(\frac{1+z'}{a_0}\right) \frac{1}{H(z')} \frac{a_0 \, dz'}{(1+z')^2}$$

$$= \underline{\underline{\int_{z}^{\infty} \frac{dz'}{(1+z')H(z')}}}$$

For a radiation-dominated universe:

$$t(z) = \int_{z}^{\infty} \frac{dz'}{(1+z') H_0 \sqrt{\Omega_{r0}} (1+z')^2}$$

$$= \frac{1}{H_0 \sqrt{\Omega_{r0}}} \int_{z}^{\infty} \frac{dz'}{(1+z')^3} = \frac{1}{H_0 \sqrt{\Omega_{r0}}} \Big[-\frac{1}{2(1+z')^2} \Big]_{z}^{\infty}$$

$$= \frac{1}{H_0 \sqrt{\Omega_{r0}}} \cdot \frac{1}{2(1+z)^2} = \underline{\underline{\frac{1}{2H(z)}}}$$

With $h = 0{,}7$, $\frac{1}{H_0} = 1{,}4 \cdot 10^{10}$ yrs,

so

$$t(z = 1{,}15 \cdot 10^{12}) = 1{,}4 \cdot 10^{10} \text{ yrs} \cdot \frac{1}{2 \cdot 10^{-2} \cdot (1{,}15 \cdot 10^{12})^2}$$

$$= 5{,}3 \cdot 10^{-13} \text{ yrs} = 5{,}3 \cdot 10^{-13} \cdot 3{,}15 \cdot 10^{7} \text{ s}$$

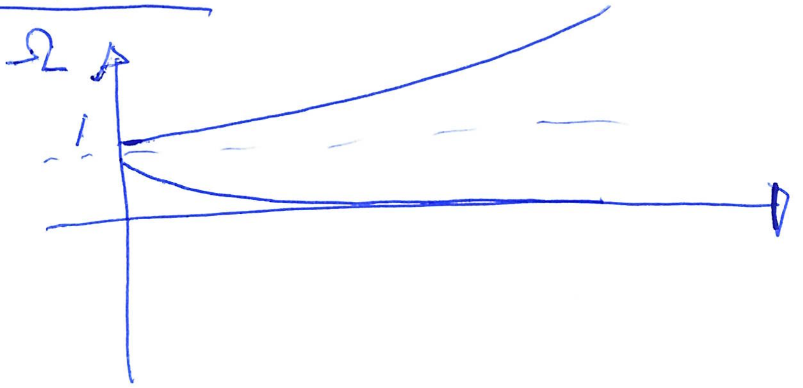
$$= \underline{\underline{1{,}7 \cdot 10^{-5} \text{ s} = 17 \, \mu\text{s}}}$$

Problem 4.

Very briefly

Causal processes could, in the Big Bang model without inflation, only have made the CMB temperature even in patches of size ~ 1° on the sky. So why is T = 2,725 K to high precision across the whole sky?

Flatness problem



Without inflation, if Ω starts out ≠ 1, it is driven farther and farther away from 1. So why is Ω ≈ 1 today?

Problem 5

$$V(\phi) = \lambda \phi^p ; \quad \lambda > 0$$

The slow-roll parameters:

$$\epsilon = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{E_p^2}{16\pi}\left(\frac{p\lambda\phi^{p-1}}{\lambda\phi^p}\right)^2$$

$$= \frac{p^2 E_p^2}{16\pi\phi^2}$$

$$\eta = \frac{E_p^2}{8\pi}\frac{V''}{V} = \frac{E_p^2}{8\pi}\frac{p(p-1)\lambda\phi^{p-2}}{\lambda\phi^p}$$

$$= \frac{p(p-1)E_p^2}{8\pi\phi^2}$$

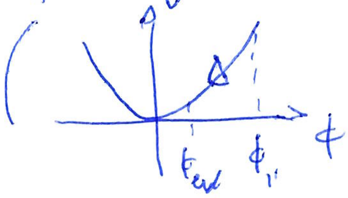
Slow-roll conditions: $\epsilon << 1, \ |\eta| << 1$

End of inflation: $\epsilon = 1 \Rightarrow \phi = \phi_{end}$ given by

$$\frac{p^2 E_p^2}{16\pi\phi_{end}^2} = 1$$

$$\Rightarrow \quad \phi_{end} = \frac{p E_p}{4\sqrt{\pi}}$$

Total number of e-folds during inflation:



$$N_{tot} = \frac{8\pi}{E_p^2} \int_{\phi_{end}}^{\phi_i} \frac{V}{V'} \, d\phi$$

$$= \frac{8\pi}{E_p^2} \int_{\phi_{end}}^{\phi_i} \frac{\lambda \phi^p}{p \lambda \phi^{p-1}} \, d\phi$$

$$= \frac{8\pi}{p E_p^2} \int_{\phi_{end}}^{\phi_i} \phi \, d\phi$$

$$= \frac{4\pi}{p E_p^2} \left(\phi_i^2 - \phi_{end}^2 \right)$$

$$= \frac{4\pi \phi_i^2}{p E_p^2} - \frac{4\pi}{p E_p^2} \frac{p^2 E_p^2}{16\pi}$$

$$= \frac{p}{4} \frac{16\pi \phi_i^2}{p^2 E_p^2} - \frac{p}{4} = \frac{p}{4} \left(\frac{1}{\epsilon_i} - 1 \right)$$

Obviously we must choose ϕ_i so that the slow-roll conditions are fulfilled at the beginning of inflation, so that

$$\epsilon_i \ll 1 \Rightarrow \frac{1}{\epsilon_i} \gg 1,$$

and therefore $\underline{\underline{N_{tot} \gg 1}}$ $\square$

Problem 6

a) The scalar field has energy density

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

and pressure

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

If $V(\phi) = 0$, then

$\rho_\phi = p_\phi$, and

$$w_\phi = \frac{p_\phi}{\rho_\phi} = 1 > 0,$$

and we cannot have $\ddot{a} > 0$,

so we <u>will not have inflation</u>.

b) We have in this case

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} = \frac{\frac{1}{2}\cdot 2V(\phi) - V(\phi)}{\frac{1}{2}\cdot 2V(\phi) + V(\phi)} = 0,$$

so in this case the scalar field will look like non-relativistic matter, and, as in a, we will have

$\ddot{a} < 0 \Longrightarrow$ <u>no inflation</u>.

Problem 7

$$V(\phi) = V_0 e^{-\lambda\phi} \ ; \ V_0 > 0 \, , \lambda > 0$$

$$M_p = \frac{1}{\sqrt{8\pi G}} \rightarrow 8\pi G = \frac{1}{M_p^2}$$

a) SRA: $$3H\dot{\phi} = -V'(\phi)$$
$$H^2 = \frac{8\pi G}{3} V(\phi)$$

$$V'(\phi) = -\lambda V_0 e^{-\lambda\phi} \ , \ \text{so}$$

$$3H\dot{\phi} = \lambda V_0 e^{-\lambda\phi} \quad 1)$$

$$H^2 = \frac{V_0}{3M_p^2} e^{-\lambda\phi} \quad 2)$$

b) Start by taking the square root of 2):

$$H = \frac{\sqrt{V_0}}{\sqrt{3} M_p} e^{-\frac{1}{2}\lambda\phi}$$

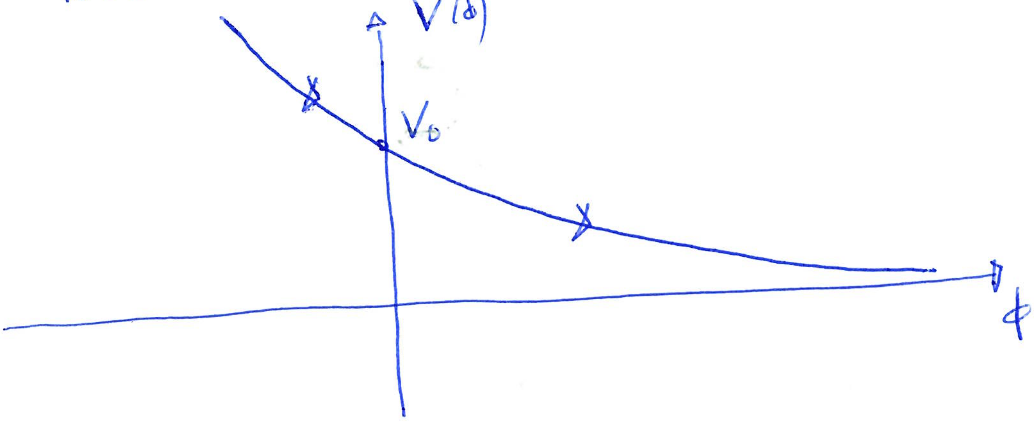
Insert this in 1):

$$3 \cdot \frac{\sqrt{V_0}}{\sqrt{3} M_p} e^{-\frac{1}{2}\lambda\phi} \cdot \dot{\phi} = \lambda V_0 e^{-\lambda\phi}$$

$$\Rightarrow \quad e^{\frac{1}{2}\lambda\phi} \frac{d\phi}{dt} = \lambda V_0 \cdot \frac{M_p}{\sqrt{3V_0}} = \lambda M_p \sqrt{\frac{V_0}{3}}$$

$$\Rightarrow \quad e^{\frac{1}{2}\lambda\phi}\, d\phi = \lambda M_p \sqrt{\frac{V_0}{3}}\, dt = \sqrt{\frac{\lambda^2 M_p^2 V_0}{3}}\, dt$$

This potential is a bit different from the power-law potentials we usually deal with. It looks like this



The field has to start out at large negative values and will then roll towards the minimum at $V = 0$ for $\phi \to +\infty$

For simplicity, I let inflation start at $t = 0$, and I take $\phi \to -\infty$ as $t \to 0$. Then $\longrightarrow$

$$\int_{-\infty}^{\phi(t)} e^{\frac{1}{2}\lambda\phi}\, d\phi = \sqrt{\frac{\lambda^2 M_p^2 V_0}{3}} \int_0^t dt$$

$$\Rightarrow \left. \frac{2}{\lambda} e^{\frac{1}{2}\lambda\phi} \right|_{-\infty}^{\phi(t)} = \sqrt{\frac{\lambda^2 M_p^2 V_0}{3}}\, t$$

$$\Rightarrow \frac{2}{\lambda} e^{\frac{1}{2}\lambda\phi(t)} = \sqrt{\frac{\lambda^2 M_p^2 V_0}{3}}\, t \qquad (**)$$

$$\Rightarrow \phi(t) = \frac{2}{\lambda} \ln\left[\frac{\lambda}{2}\sqrt{\frac{\lambda^2 M_p^2 V_0}{3}}\, t\right]$$

$$= \underline{\underline{\frac{2}{\lambda} \ln\left[\frac{\lambda^2 M_p}{2}\sqrt{\frac{V_0}{3}}\, t\right]}}$$

We can now find $a(t)$ from

$$\frac{1}{a}\frac{da}{dt} = H = \frac{\sqrt{V_0}}{\sqrt{3}\, M_p} e^{-\frac{1}{2}\lambda\phi}$$

From $(**)$ we see that

$$e^{\frac{1}{2}\lambda\phi} = \frac{\lambda}{2}\sqrt{\frac{\lambda^2 M_p^2 V_0}{3}}\, t = \frac{\lambda^2 M_p}{2\sqrt{3}}\, t \cdot \sqrt{V_0}$$

$$\Rightarrow e^{-\frac{1}{2}\lambda\phi} = \frac{1}{e^{\frac{1}{2}\lambda\phi}} = \frac{2\sqrt{3}}{\lambda^2 M_p \sqrt{V_0}}\frac{1}{t}$$

~~Demand~~ Hence ;

$$\frac{1}{a}\frac{da}{dt} = \frac{\cancel{\sqrt{V_0}}}{\cancel{\sqrt{3}}\,M_p}\,\frac{2\cancel{\sqrt{3}}}{\lambda^2 M_p \cancel{\sqrt{V_0}}}\,\frac{1}{t} = \frac{2}{\lambda^2 M_p^2}\,\frac{1}{t}$$

$$\Rightarrow \quad \frac{da}{a} = \frac{2}{\lambda^2 M_p^2}\,\frac{dt}{t}$$

$$\Rightarrow \quad \int \frac{da}{a} = \frac{2}{\lambda^2 M_p^2}\int \frac{dt}{t}$$

$$\Rightarrow \quad \ln a = \frac{2}{\lambda^2 M_p^2}\ln t + \overbrace{\ln a_i}^{\text{constant of integration}}$$

$$\Rightarrow \quad \underline{\underline{a(t) = a_i\, t^{\frac{2}{\lambda^2 M_p^2}}}}$$

where a_i is determined by a boundary condition of some sort that need not worry us.

c) Without the SRA:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$H^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

From what we found in b), it seems reasonable to guess that

$$a(t) = C t^{\alpha}$$

$$\phi(t) = \frac{2}{\lambda} \ln (Bt)$$

This gives $\quad H = \frac{\dot{a}}{a} = \frac{\alpha C t^{\alpha-1}}{C t^{\alpha}} = \frac{\alpha}{t}$

$$V(\phi) = V_0 e^{-\lambda \phi} = V_0 e^{-\lambda \cdot \frac{2}{\lambda} \ln (Bt)}$$

$$= V_0 e^{-2 \ln (Bt)} = V_0 \left(e^{-\ln (Bt)}\right)^2$$

$$= V_0 \frac{1}{(Bt)^2} = \frac{V_0}{B^2 t^2}$$

$$V'(\phi) = -\lambda V_0 e^{-\lambda \phi} = -\lambda V = -\frac{\lambda V_0}{B^2 t^2}$$

Furthermore

$$\dot{\phi} = \frac{2}{\lambda} \frac{1}{Bt} \cdot B = \frac{2}{\lambda t}$$

$$\ddot{\phi} = -\frac{2}{\lambda t^2}$$

We insert these results in the

full equations of motion :

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$\Rightarrow \quad -\frac{2}{\lambda t^2} + 3 \cdot \frac{\alpha}{t} \cdot \frac{2}{\lambda t} - \frac{\lambda V_0}{B^2 t^2} = 0 \qquad i)$$

$$H^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$\Rightarrow \quad \frac{\alpha^2}{t^2} = \frac{1}{3M_p^2}\left[\frac{1}{2} \cdot \frac{4}{\lambda^2 t^2} + \frac{V_0}{B^2 t^2}\right] \qquad ii)$$

We see that the time t cancels out in both $i)$ and $ii)$, leaving us with two equations for the two constants α and B:

$$\text{I)} \quad -\frac{2}{\lambda} + \frac{6\alpha}{\lambda} - \frac{\lambda V_0}{B^2} = 0$$

$$\text{II)} \quad \alpha^2 = \frac{2}{3\lambda^2 M_p^2} + \frac{V_0}{3M_p^2 B^2}$$

$$\text{I)} \Rightarrow \quad \frac{6\alpha}{\lambda} = \frac{2}{\lambda} + \frac{\lambda V_0}{B^2}$$

I multiply this equation by $\frac{1}{3\lambda M_p^2}$:

$$\frac{6\alpha}{\lambda} \cdot \frac{1}{3\lambda M_p^2} = \frac{2}{\lambda}\frac{1}{3\lambda M_p^2} + \frac{\lambda V_0}{B^2}\frac{1}{3\lambda M_p^2}$$

$$\Rightarrow \quad \frac{2\alpha}{\lambda^2 M_p^2} = \frac{2}{3\lambda^2 M_p^2} + \frac{V_0}{3M_p^2 B^2} \qquad \mathrm{I}')$$

Next I subtract I' from II:

$$\alpha^2 - \frac{2\alpha}{\lambda^2 M_p^2} = \frac{2}{3\lambda^2 M_p^2} + \frac{V_0}{3M_p^2 B^2} - \frac{2}{3\lambda^2 M_p^2} - \frac{V_0}{3M_p^2 B^2} = 0$$

$$\Rightarrow \quad \alpha\left(\alpha - \frac{2}{\lambda^2 M_p^2}\right) = 0$$

$\alpha = 0$ is a solution, but it gives $a \propto t^0 =$ constant, which is not interesting. The relevant solution is therefore

$$\alpha = \frac{2}{\lambda^2 M_p^2}.$$

which is the same result as in the SRA.
I insert this in $\mathrm{I})$:

$$\frac{6\alpha}{\lambda} \cdot \frac{2}{\lambda^2 M_p^2} = \frac{2}{\lambda} + \frac{\lambda V_0}{B^2}$$

$$\Rightarrow \quad \frac{\lambda V_0}{B^2} = \frac{2}{\lambda}\left(\frac{6}{\lambda^2 M_p^2} - 1\right) = \frac{2}{\lambda}\,\frac{6 - \lambda^2 M_p^2}{\lambda^2 M_p^2}$$

$$\Rightarrow \quad \frac{B^2}{\lambda V_0} = \frac{\lambda}{2} \, \frac{\lambda^2 M_p^2}{6 - \lambda^2 M_p^2}$$

$$\Rightarrow \quad B^2 = \frac{\lambda^2 V_0}{2} \, \frac{\lambda^2 M_p^2}{6 - \lambda^2 M_p^2} = \frac{\lambda^4 M_p^2}{4} \, \frac{2 V_0}{6 - \lambda^2 M_p^2}$$

The SRA result was

$$\phi(t) = \frac{2}{\lambda} \ln \left[\frac{\lambda^2 M_p}{2} \sqrt{\frac{V_0}{3}} \, t \right],$$

and we have now found the exact solution

$$\phi(t) = \frac{2}{\lambda} \ln \left[\frac{\lambda^2 M_p}{2} \sqrt{\frac{2 V_0}{6 - \lambda^2 M_p^2}} \right]$$

To relate these expressions, it is useful to calculate the SR parameters for this potential.

$$\epsilon = \frac{E_p^2}{16\pi} \left(\frac{V'}{V} \right)^2 = \frac{E_p^2}{16\pi} \left(\frac{-\lambda V_0 e^{-\lambda\phi}}{V_0 e^{-\lambda\phi}} \right)^2$$

$$= \frac{\lambda^2 E_p^2}{16\pi}$$

In units where $\hbar = c = 1$,

$$E_p^2 = \frac{1}{G} = 8\pi M_p^2,$$

so

$$\epsilon = \frac{\lambda^2 M_p^2}{2}$$

Also :

$$\eta = \frac{E_p^2}{8\pi} \frac{V''}{V} = M_p^2 \frac{\lambda^2 V_0 e^{-\lambda\phi}}{V_1 e^{-\lambda\phi}} = \lambda^2 M_p^2$$

We now see that we can write the ~~SRA~~ exact solution for $\phi(t)$ as

$$\phi(t) = \frac{2}{\lambda} \ln \left[\frac{\lambda^2 M_p}{2} \sqrt{\frac{2V_0}{6-2\epsilon}}\, t \right]$$

$$= \frac{2}{\lambda} \ln \left[\frac{\lambda^2 M_p}{2} \sqrt{\frac{2V_0}{6-\eta}}\, t \right]$$

If the SR conditions

$$\epsilon \ll 1$$
$$|\eta| \ll 1$$

are satisfied, we see that

$$\phi(t) \simeq \frac{2}{\lambda} \ln \left[\frac{\lambda^2 M_p}{2} \sqrt{\frac{2V_0}{6}}\, t \right]$$

$$= \frac{2}{\lambda} \ln \left[\frac{\lambda^2 M_p}{2} \sqrt{\frac{V_0}{3}}\, t \right]$$

which matches the SR solution.

The scale factor is the same as in the SRA, but we note that we can write it as

$$a(t) = C t^{1/\epsilon},$$

so the exponent in the power-law will be large if $\epsilon \ll 1$.

d) The main problem with this model is that

$$\epsilon = \frac{\lambda^2 M_p^2}{2} = \frac{\eta}{2} = \text{constant}$$

This means that if we have slow-roll ($\epsilon \ll 1$, $|\eta| \ll 1$), or even just inflation ($\epsilon < 1$), it will never end.

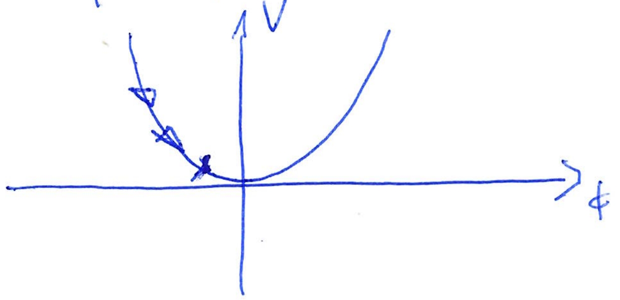
A realistic model for inflation must allow inflation to end and the Universe to enter a radiation-dominated phase.

This model doesn't, so it is, sadly, unrealistic.

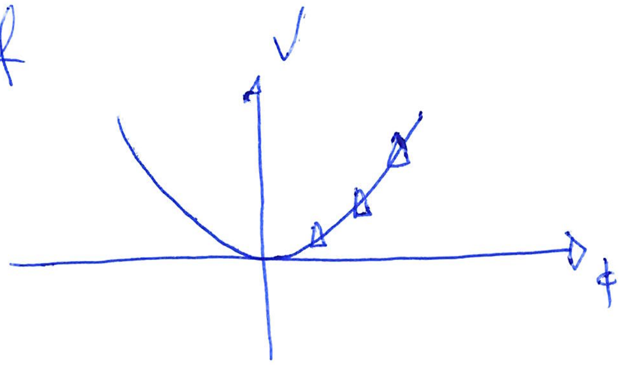
Problem 8

Assume : $\dot{\phi} > 0$

⌈ Means that in the case of a symmetric potential, ϕ rolls like this:



instead of



⌋

a) The equations of motion:

(1) $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$

(2) $H^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$

Want to prove:

$$\dot{\phi} = -2M_p^2 H'(\phi)$$

where $\qquad H'(\phi) = \dfrac{dH}{d\phi}$

This suggests that we start with taking the derivative of (2) with respect to ϕ:

$$2H\,H'(\phi) = \frac{1}{3M_p^2}\left[\dot{\phi}\,\frac{d\dot{\phi}}{d\phi} + V'(\phi)\right]$$

$$= \frac{1}{3M_p^2}\left[\dot{\phi}\,\frac{d\dot{\phi}}{dt}\,\overbrace{\frac{dt}{d\phi}}^{\frac{1}{\dot{\phi}}} + V'(\phi)\right]$$

$$= \frac{1}{3M_p^2}\left[\ddot{\phi} + V'(\phi)\right]$$

But from (1) we have

$$\ddot{\phi} + V'(\phi) = -3H\dot{\phi}$$

so

$$2HH'(\phi) = \frac{1}{3M_p^2}(-3H\dot{\phi}) = -\frac{H}{M_p^2}\dot{\phi}$$

$$\Rightarrow \qquad \underline{\underline{\dot{\phi} = -2M_p^2 H'(\phi)}} \quad \square$$

b) We proceed by simply inserting the result from a) in (2): (which is FI)

$$H^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\left(-2M_p^2 H'(\phi)\right)^2 + V(\phi)\right]$$

$$\Rightarrow \quad 3M_p^2 H^2 = 2M_p^4 [H'(\phi)]^2 + V(\phi)$$

$$\Rightarrow \quad 2M_p^4 [H'(\phi)]^2 - 3M_p^2 H^2(\phi) = -V(\phi)$$

$$\Rightarrow \quad (3) \quad \underline{\underline{[H'(\phi)]^2 - \frac{3}{2M_p^2} H^2(\phi) = -\frac{1}{2M_p^4} V(\phi)}} \quad \square$$

c) We have

$$H(\phi) = H_0(\phi) + \delta H(\phi)$$

where

$$[H_0'(\phi)]^2 - \frac{3}{2M_p^2} H_0^2(\phi) = -\frac{1}{2M_p^4} V(\phi)$$

We demand that $H(\phi)$ also is a solution of (3):

$$[H'(\phi)]^2 - \frac{3}{2M_p^2} H^2(\phi) = -\frac{1}{2M_p^4} V(\phi)$$

$$\Rightarrow \quad [H_0'(\phi) + \delta H'(\phi)]^2 - \frac{3}{2M_p^2}[H_0(\phi) + \delta H(\phi)]^2 = -\frac{1}{2M_p^4} V(\phi)$$

Since we only work to first order in δH, we get

$$[H_0'(\phi)]^2 + 2H_0'(\phi)\,\delta H'(\phi) - \frac{3}{2M_p^2} H_0^2(\phi) - \frac{3}{2M_p^2}\cdot 2H_0(\phi)\,\delta H(\phi) = -\frac{1}{2M_p^4}V(\phi)$$

$$\text{''} \quad -\frac{1}{2M_p^4}V'(\phi)$$

$$\Rightarrow \quad 2H_0'(\phi)\,\delta H'(\phi) - \frac{3}{M_p^2}H_0(\phi)\,\delta H(\phi) = 0$$

$$\Rightarrow \quad \underline{\underline{H_0'(\phi)\,\delta H'(\phi) = \frac{3}{2M_p^2}H_0(\phi)\,\delta H(\phi)}} \quad \square$$

d) This is not too hard to solve if we change notation:

$$\frac{dH_0}{d\phi}\frac{d\delta H}{d\phi} = \frac{3}{2M_p^2}H_0\,\delta H$$

$$\Rightarrow \quad \frac{d\delta H}{\delta H} = \frac{3}{2M_p^2}\frac{H_0}{dH_0/d\phi}\,d\phi$$

$$\Rightarrow \quad \int_{\delta H_i = \delta H(\phi_i)}^{\delta H(\phi)} \frac{d\delta H}{\delta H} = \frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}\,d\varphi$$

$$\Rightarrow \quad \ln\left(\frac{\delta H(\phi)}{\delta H(\phi_i)}\right) = \frac{3}{2M_p^2}\int_{\phi_i}^{\phi} \frac{H_0(\varphi)}{H_0'(\varphi)}\, d\varphi$$

$$\Rightarrow \quad \underline{\underline{\delta H(\phi) = \delta H(\phi_i)\exp\left[\frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}\, d\varphi\right]}} \quad \square$$

From a) we have

$$H_0'(\varphi) = -\frac{\dot{\varphi}}{2M_p^2}$$

so that

$$\frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0}{H_0'}\, d\varphi = -3\int_{\phi_i}^{\phi} H_0(\varphi)\frac{d\varphi}{\dot{\varphi}}$$

By definition $\dot{\varphi} = \frac{d\varphi}{dt}$, so $\frac{d\varphi}{\dot{\varphi}} = dt$

and the integral is therefore equal to

$$-3\int_{t_i}^{t} H_0(t)\, dt = -3\int_{t_i}^{t}\frac{\dot{a}\, dt}{a}$$

$$= -3\int_{a(t_i)}^{a(t)}\frac{da}{a} = -3\ln\left[\frac{a(t)}{a(t_i)}\right] = -3N(t),$$

where $N(t)$ is the number of e-folds

at time t. So

$$\delta H(t) = \delta H(t_i)\, e^{-3N(t)}$$

and we see that the exponential factor will very soon kill off the perturbation. $\square$