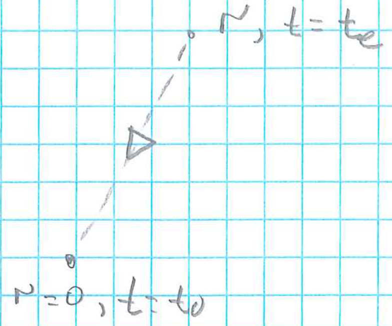
Problem 1

a)



Light : $ds^2 = 0$

$d\theta = d\phi = 0$,

so

$$c^2 dt^2 - a^2(t)\frac{dr^2}{1-kr^2} = 0$$

$$\Rightarrow \frac{dr}{\sqrt{1-kr^2}} = -\frac{c\,dt}{a(t)}$$

because $dt>0 \Rightarrow dr<0$ in this case

$$\Rightarrow \int_r^0 \frac{dr'}{\sqrt{1-kr'^2}} = -\int_{t_e}^{t_0}\frac{c\,dt}{a(t)}$$

$$\Rightarrow \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_{t_e}^{t_0}\frac{c\,dt}{a(t)}$$

b) $|k| \leq 1$, so $r \ll 1$ and $0 \leq r' \leq r$

$$\Rightarrow \frac{1}{\sqrt{1-kr'^2}} \simeq 1$$

$\frac{t_0 - t_e}{t_0} \ll 1 \Rightarrow$ integration over short range in t, can consider $a(t)$ to be constant, and equal to $a(t_e)$ over the range of integration.

So :

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} \simeq \int_0^r dr' = r$$

$$\int_{t_e}^{t_0}\frac{c\,dt}{a(t)} \simeq \frac{c}{a(t_e)}\int_{t_e}^{t_0} dt = \frac{c(t_0-t_e)}{a(t_e)},$$

and therefore

$$r \simeq \frac{c(t_0-t_e)}{a(t_e)}$$

c) Taylor expansion of $a(t)$ around $t=t_0$:

$$a(t) \simeq a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0} \cdot (t-t_0)$$

$$= a(t_0) - a(t_0) \underbrace{\frac{\dot{a}(t_0)}{a(t_0)}}_{\substack{= H(t_0) \\ = H_0}} (t_0 - t)$$

$$= a(t_0)\left[1 - H_0(t_0 - t)\right]$$

For $t = t_e$ (note typo in problem):

$$a(t_e) \simeq a(t_0)\left[1 - H_0(t_0 - t_e)\right]$$

d) The redshift is given by

$$1+z = \frac{a_0}{a(t_e)} = \frac{\cancel{a(t_0)}}{\cancel{a(t_0)}\left[1 - H_0(t_0 - t_e)\right]}$$

$$\simeq 1 + H_0(t_0 - t_e) \qquad \left(\frac{1}{1-x} \simeq 1+x \text{ for } x \ll 1\right)$$

$\left(H_0 \sim \frac{1}{t_0}, \text{ so } H_0(t_0 - t_e) \sim \frac{t_0 - t_e}{t_0} \ll 1\right)$

$$\Rightarrow \underline{\underline{z \simeq H_0(t_0 - t_e)}}$$

e) From the definition:

$$d_p(t_0) = a(t_0)\, S_k^{-1}(r)$$

But $S_k^{-1}(r) = \left\{\begin{array}{ll} \sin^{-1} r, & k=+1 \\ r, & k=0 \\ \sinh^{-1} r, & k=-1 \end{array}\right\} \simeq r$ (indep. of k)

for $r \ll 1$, so

$$d_p(t_0) \simeq a(t_0)\, r$$

We found in b):

$$r \approx \frac{c(t_0 - t_e)}{a(t_e)},$$

so

$$d_p(t_0) = \underbrace{\frac{a(t_0)}{a(t_e)}}_{1+z} c(t_0 - t_e)$$

From d): $t_0 - t_e \approx \frac{z}{H_0}$,

so

$$d_p(t_0) = (1+z)\, c\, \frac{z}{H_0} \simeq \frac{cz}{H_0}$$, to first order in z.

Or:

$$\underline{\underline{cz \simeq H_0\, d_p(t_0)}}$$

f) By definition,

$$d_L = a_0 r (1+z) \simeq a_0 r \text{ for } z \ll 1,$$

so

$$d_L \simeq d_p(t_0),$$

and the H-L law can be written as

$$cz \simeq H_0 d_L$$

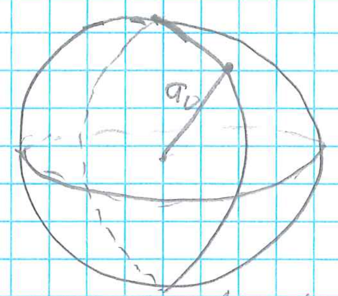
Both z and d_L are defined in a way that makes them observable, so on this form, the law is testable.

Problem 2

We found in the lectures that the scale factor in the Einstein universe is given by

$$a_0 = \frac{c}{\sqrt{4\pi G \rho_0}}$$

If we think of the 2d analogy, this universe is like the surface of a sphere with radius a_0.



A round-trip corresponds to covering the circumference of a circle of radius a_0,

$$\ell = 2\pi a_0 = \frac{2\pi c}{\sqrt{4\pi G \rho_0}} = c\sqrt{\frac{\pi}{G\rho_0}},$$

and the time it takes for light to do this is

$$\tau = \frac{\ell}{c} = \sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi}{6{,}674 \cdot 10^{-11}\, m^3 kg^{-1} s^{-2} \cdot 10^{-27}\, kg\, m^{-3}}}$$

$$\simeq 6{,}86 \cdot 10^{18}\, s \simeq \underline{\underline{15{,}4 \cdot 10^{9}\ yrs}}$$

Problem 3

FI with just Λ:

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho_\Lambda a^2 = \frac{8\pi G}{3}\frac{\Lambda}{8\pi G}a^2 = \frac{\Lambda}{3}a^2$$

If $\Lambda < 0$, the RHS is negative

But if $k=0$ or $k=+1$, the LHS is always non-negative. Therefore there can be no solution in these cases.

For $k=-1$:

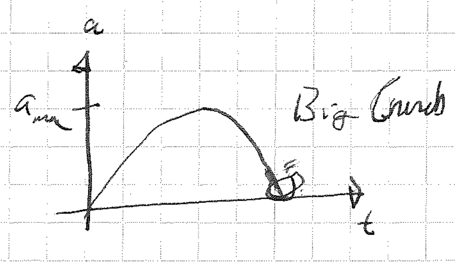
$$\dot{a}^2 - c^2 = \frac{\Lambda}{3}a^2 = -\frac{|\Lambda|}{3}a^2$$

$$\Rightarrow \dot{a}^2 = c^2 - \frac{|\Lambda|}{3}a^2$$

Since $\dot{a}^2$ is non-negative, there must be a maximum value of a, the value for which $\dot{a} = 0$:

$$c^2 - \frac{|\Lambda|}{3}a_{max}^2 = 0$$

$$\Rightarrow a_{max} = \sqrt{\frac{3c^2}{|\Lambda|}}$$



Solving FI:

$$\left(\frac{da}{dt}\right)^2 = c^2 - \frac{|\Lambda|}{3}a^2 = c^2\left[1 - \left(\frac{a}{a_{max}}\right)^2\right]$$

$$\Rightarrow \frac{da}{\sqrt{1-\left(\frac{a}{a_{max}}\right)^2}} = c\,dt$$

$$\Rightarrow \int_0^a \frac{da'}{\sqrt{1-\left(\frac{a'}{a_{max}}\right)^2}} = \int_0^t c\,dt = ct$$

$$\frac{a'}{a_{max}} = \sin\theta \qquad \sqrt{1-\left(\frac{a'}{a_{max}}\right)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$da' = a_{max}\cos\theta\, d\theta \qquad \begin{aligned} a' = 0 &\to \theta = 0 \\ a' = a &\to \theta = \arcsin\left(\frac{a}{a_{max}}\right)\end{aligned}$$

$$\Rightarrow \int_0^{\arcsin\left(\frac{a}{a_{max}}\right)} \frac{a_{max}\cos\theta\, d\theta}{\cos\theta} = ct$$

$$\Rightarrow a_{max} \arcsin\left(\frac{a}{a_{max}}\right) = ct$$

$$\Rightarrow \underline{\underline{a(t) = a_{max} \sin\left(\frac{ct}{a_{max}}\right)}} \; ; \; a_{max} = \sqrt{\frac{3c^2}{|\Lambda|}}$$

Crunch (collapse) when

$$\frac{ct}{a_{max}} = \pi$$

$$\Rightarrow t = t_{crunch} = \frac{\pi a_{max}}{c} = \frac{\pi}{c}\sqrt{\frac{3c^2}{|\Lambda|}} = \pi\sqrt{\frac{3}{|\Lambda|}}$$

Based on our experience, the Universe must prevail at least 10^{10} yrs for intelligent life to appear. Since $\frac{1}{H_0} \sim 10^{10}$ yrs, we can take

$$t_{crunch} \gtrsim \frac{1}{H_0}$$

as our criterion, and then

$$\pi\sqrt{\frac{3}{|\Lambda|}} \gtrsim \frac{1}{H_0} \qquad (H_0 \simeq 2 \cdot 10^{-18}\, s^{-1})$$

$$\sqrt{\frac{|\Lambda|}{3}} \lesssim \pi H_0$$

$$\Rightarrow |\Lambda| \lesssim 3\pi^2 H_0^2 \sim 3 \cdot 10 \cdot 4 \cdot 10^{-36}\, s^{-2}$$

$$\sim \underline{\underline{10^{-34}\, s^{-2}}}$$

In units of ρ_{c0}:

$$|\Omega_\Lambda| = \frac{8\pi G}{3H_0^2}|\rho_\Lambda| = \frac{8\pi G}{3H_0^2}\frac{|\Lambda|}{8\pi G} = \frac{|\Lambda|}{3H_0^2}$$

$$\lesssim \frac{3\pi^2 H_0^2}{3H_0^2} = \pi^2 \underline{\underline{\simeq 10}}$$

Problem 4

Phantom energy : $p = w\rho c^2$, with $w < -1$

Here : $w = -2$

$k = 0$, phantom energy dominates.

a) We found $\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$

With $w = -2$:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3\cdot(1-2)} = \underline{\underline{\rho_0 \left(\frac{a}{a_0}\right)^{3}}}.$$

Energy density <u>increases</u> as the Universe expands!

b) Just for some variation, start from FI on its original form:

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho a^2$$

Here : $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = H_0^2 \underbrace{\frac{8\pi G}{3H_0^2}}_{\rho_{c0}^{-1}} \rho_0 \left(\frac{a}{a_0}\right)^3$$

$$= H_0^2 \frac{\rho_0}{\rho_{c0}} \left(\frac{a}{a_0}\right)^3 = H_0^2 \left(\frac{a}{a_0}\right)^3$$

because $\rho_0 = \rho_{c0}$ when $k = 0$.

Then $$\frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = H_0 \left(\frac{a}{a_0}\right)^{3/2}$$

$$\Rightarrow \quad \frac{da}{a\left(\frac{a}{a_0}\right)^{3/2}} = H_0\, dt$$

Change variable: $x = \frac{a}{a_0} \Rightarrow \frac{da}{a} = \frac{dx}{x}$

$$\frac{dx}{x \cdot x^{3/2}} = H_0\, dt$$

Integrate from $t = t_0$ up to general $t > t_0$.
When $t = t_0$, $x = \frac{a}{a_0} = \frac{a_0}{a_0} = 1$, so

$$H_0 \int_{t_0}^{t} dt' = \int_1^x \frac{dx'}{(x')^{5/2}}$$

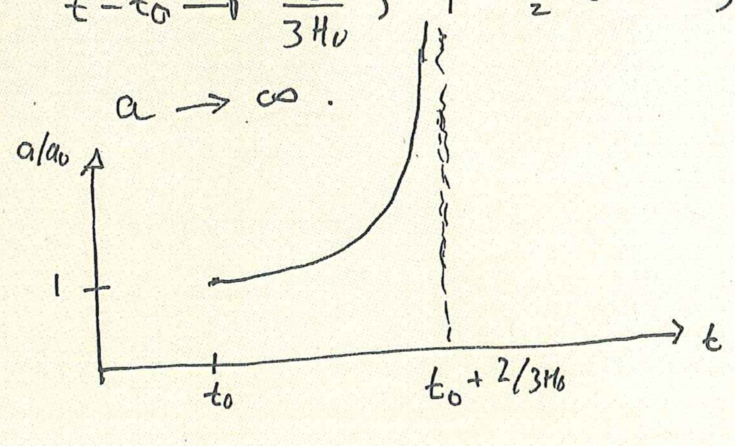
$$H_0(t - t_0) = \left| -\frac{2}{3}\frac{1}{(x')^{3/2}} \right|_1^x = -\frac{2}{3}\frac{1}{x^{3/2}} + \frac{2}{3}$$

$$\Rightarrow \frac{1}{x^{3/2}} = 1 - \frac{3}{2} H_0(t - t_0)$$

$$\Rightarrow x = \frac{a}{a_0} = \frac{1}{\left[1 - \frac{3}{2} H_0(t - t_0)\right]^{2/3}}$$

c) As $t - t_0 \rightarrow \frac{2}{3H_0}$, $1 - \frac{3}{2} H_0(t - t_0) \rightarrow 0$,

so $a \rightarrow \infty$.



All distances stretched to ∞ in a finite time, as if all of space is ripped apart

$\longrightarrow$ "Big Rip"

Problem 5

a) $$\left(\frac{\dot a}{a}\right)^2 + \frac{1}{a^2} = \frac{\Lambda}{3}$$

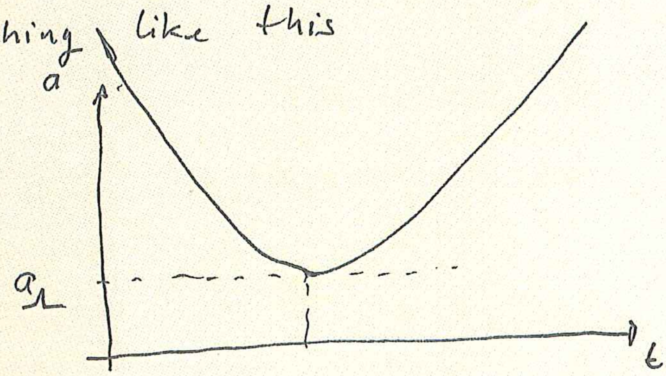
$$\Rightarrow \quad \left(\frac{\dot a}{a}\right)^2 = \frac{\Lambda}{3} - \frac{1}{a^2} \geq 0$$

$$\Rightarrow \quad \frac{\Lambda}{3} \geq \frac{1}{a^2}$$

$$\Rightarrow \quad a^2 \geq \frac{3}{\Lambda} \equiv a_\Lambda^2$$

b) FII : $$\frac{\ddot a}{a} = \frac{\Lambda}{3} > 0$$

So the solution for a(t) must look something like this



No problem letting $t \to -\infty$, so the Universe could be infinitely old.

c) FI again:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\Lambda}{3} - \frac{1}{a^2} = \frac{1}{a_\Lambda^2} - \frac{1}{a^2}$$

$$\Rightarrow \quad \left(\frac{da}{dt}\right)^2 = \left(\frac{a}{a_\Lambda}\right)^2 - 1$$

$$\Rightarrow \quad \frac{da}{\sqrt{\left(\frac{a}{a_\Lambda}\right)^2 - 1}} = dt$$

Choose $a = a_\Lambda$ at $t = 0$, integrate:

$$\int_0^t dt' = \int_{a_\Lambda}^{a} \frac{da'}{\sqrt{\left(\frac{a'}{a_\Lambda}\right)^2 - 1}} \underset{\substack{x = \frac{a'}{a_\Lambda} \\ da' = a_\Lambda dx}}{=} a_\Lambda \int_1^{a/a_\Lambda} \frac{dx}{\sqrt{x^2-1}}$$

$$= a_\Lambda \Big|_1^{a/a_\Lambda} \cosh^{-1} x = a_\Lambda \cosh^{-1}\left(\frac{a}{a_\Lambda}\right)$$

$$\Rightarrow \quad \underline{\underline{a = a_\Lambda \cosh\left(\frac{t}{a_\Lambda}\right)}}$$

d) $$V(a) = \frac{9\pi^2 a_\Lambda^2}{4G^2}\left[\left(\frac{a}{a_\Lambda}\right)^2 - \left(\frac{a}{a_\Lambda}\right)^4\right] \equiv K\,(x^2 - x^4)$$

$x \geq 0$ since $a \geq 0$

Look at $f(x) = x^2 - x^4$

$$f'(x) = 2x - 4x^3 = 2x(1-2x^2) = 2x(1+\sqrt{2}x)(1-\sqrt{2}x)$$

$$f''(x) = 2 - 8x^2 = 2(1-4x^2) = 2(1+2x)(1-2x)$$

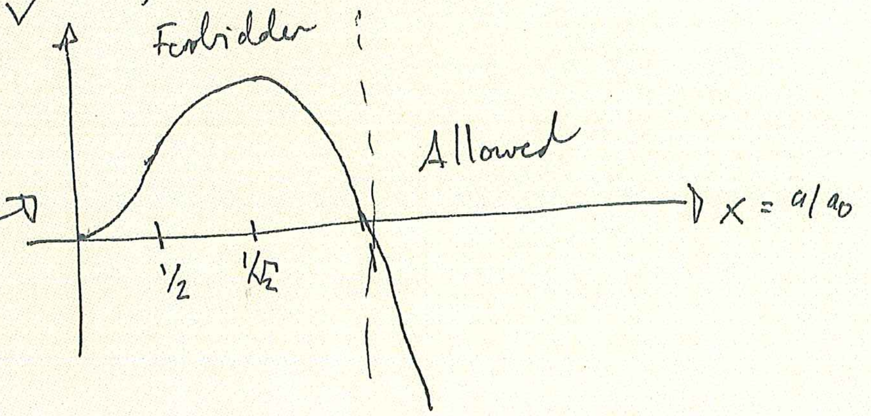
$f = 0$ for $x = 0$ and $x = 1$, $f < 0$ for $x > 1$

$f'(x) = 0$ for $x = 0$ and $x = 1/\sqrt{2} \simeq 0{,}707$

$f'(x) > 0$ for $0 < x < 1/\sqrt{2}$, negative for $x > \frac{1}{\sqrt{2}}$

$f''(x) = 0$ for $x = \frac{1}{2}$, > 0 for $x < \frac{1}{2}$, < 0 for $x > \frac{1}{2}$

So V looks something like this



Forbidden regions : V must be $\leq E$, since the "particle" must have ~~pos~~ non-negative "kinetic energy". But $E=0$, so only regions with $V \leq 0$ are allowed.
This means ~~that~~ that only $x \geq 1$ $(a \geq a_1)$ is allowed classically, which fits nicely with what we found in a).

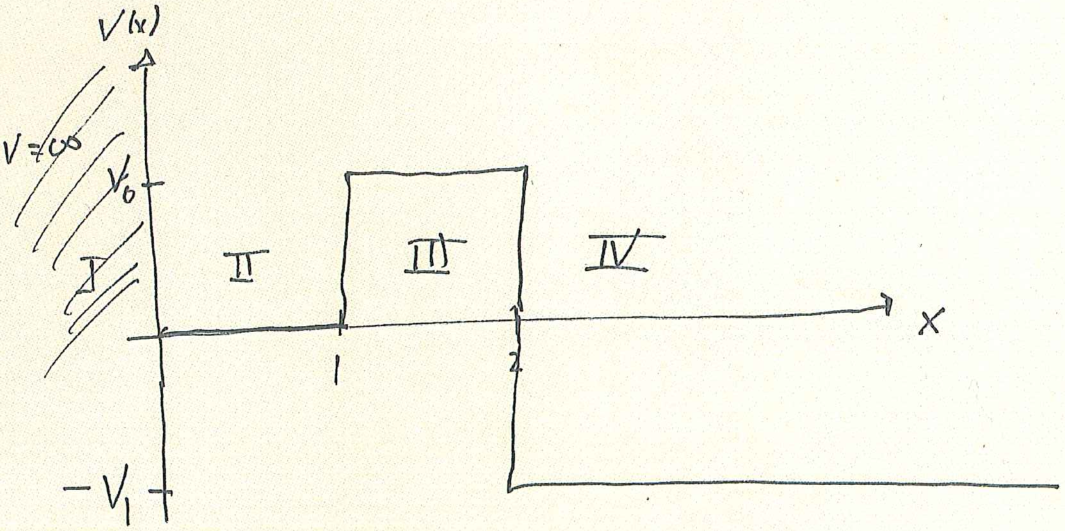
e) $$-\frac{d^2\psi}{da^2} + \frac{9\pi^2 a_1^2}{4G^2}\left[\left(\frac{a}{a_1}\right)^2 - \left(\frac{a}{a_1}\right)^4\right]\psi = 0$$

$$x = \frac{a}{a_1}, \quad \frac{d}{da} = \frac{dx}{da}\frac{d}{dx} = \frac{1}{a_1}\frac{d}{dx}, \quad \frac{d^2}{da^2} = \frac{1}{a_1^2}\frac{d^2}{dx^2}$$

$$\Rightarrow \quad -\frac{1}{a_1^2}\frac{d^2\psi}{dx^2} + \frac{9\pi^2 a_1^2}{4G^2}(x^2 - x^4)\psi = 0$$

$$-\frac{d^2\psi}{dx^2} + \frac{9\pi^2 a_1^4}{4G^2}(x^2 - x^4)\psi = 0$$

e)



In region I, $V=\infty$, so $\underline{\underline{\psi_I = 0}}$

In region II: $-\dfrac{d^2\psi}{dx^2} + \underbrace{V(x)}_{=0}\psi = 0$

$$\frac{d^2\psi}{dx^2} = 0$$

$$\Rightarrow \underline{\underline{\psi = A + Bx \equiv \psi_{II}(x)}}$$

In region III:

$$-\frac{d^2\psi}{dx^2} + V_0\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = \underbrace{V_0}_{>0}\psi$$

$$\Rightarrow \underline{\underline{\psi = Ce^{\sqrt{V_0}\,x} + De^{-\sqrt{V_0}\,x} \equiv \psi_{III}(x)}}$$

In region IV:

$$-\frac{d^2\psi}{dx^2} - V_1\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = \underbrace{-V_1}_{<0}\psi$$

$$\Rightarrow \underline{\underline{\psi = Ee^{i\sqrt{V_1}\,x} + Fe^{-i\sqrt{V_1}\,x} \equiv \psi_{IV}(x)}}$$

f) Cont. of ψ at $x = 0$:

$$\psi_I(0) = \psi_{II}(0)$$

$$\Rightarrow \quad \underline{0 = A}$$

Cont. of ψ at $x = 1$:

$$\psi_{II}(1) = \psi_{III}(1)$$

$$\Rightarrow \quad \underline{\underline{B = Ce^{\sqrt{V_0}} + De^{-\sqrt{V_0}}}}$$

Cont. of ψ' at $x = 1$:

$$\psi_{II}'(1) = \psi_{III}'(1)$$

$$\underline{\underline{B = C\sqrt{V_0}\,e^{\sqrt{V_0}} - D\sqrt{V_0}\,e^{-\sqrt{V_0}}}}$$

Cont. of ψ at $x = 2$:

$$\psi_{III}(2) = \psi_{IV}(2)$$

$$\underline{\underline{Ce^{2\sqrt{V_0}} + De^{-2\sqrt{V_0}} = Ee^{2i\sqrt{V_1}} + Fe^{-2i\sqrt{V_1}}}}$$

Cont. of ψ' at $x = 2$:

$$\psi_{III}'(2) = \psi_{IV}'(2)$$

$$\underline{\underline{C\sqrt{V_0}\,e^{2\sqrt{V_0}} - D\sqrt{V_0}\,e^{-2\sqrt{V_0}} = iE\sqrt{V_1}\,e^{2i\sqrt{V_1}} - iF\sqrt{V_1}\,e^{-2i\sqrt{V_1}}}}$$

g) The transmission probability amplitude is the ratio of the amplitude of the outgoing wave from the barrier in region IV, E, to the amplitude of the wave coming in towards the barrier in region II, B. Rewrite the equations from f):

$$1 = \frac{C}{B} e^{\sqrt{V_0}} + \frac{D}{B} e^{-\sqrt{V_0}} \qquad 1)$$

$$\frac{1}{\sqrt{V_0}} = \frac{C}{B} e^{\sqrt{V_0}} - \frac{D}{B} e^{-\sqrt{V_0}} \qquad 2)$$

$$\frac{C}{B} e^{2\sqrt{V_0}} + \frac{D}{B} e^{-2\sqrt{V_0}} = \frac{E}{B} e^{2i\sqrt{V_1}} + \frac{F}{B} e^{-2i\sqrt{V_1}} \qquad 3)$$

$$-i\sqrt{\frac{V_0}{V_1}} \frac{C}{B} e^{2\sqrt{V_0}} + i\sqrt{\frac{V_0}{V_1}} \frac{D}{B} e^{-2\sqrt{V_0}} = \frac{E}{B} e^{2i\sqrt{V_1}} - \frac{F}{B} e^{-2i\sqrt{V_1}} \qquad 4)$$

1) + 2) gives
$$2 \frac{C}{B} e^{\sqrt{V_0}} = 1 + \frac{1}{\sqrt{V_0}}$$
$$\Rightarrow \quad \frac{C}{B} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{V_0}}\right) e^{-\sqrt{V_0}}$$

1) − 2) gives
$$2 \frac{D}{B} e^{-\sqrt{V_0}} = 1 - \frac{1}{\sqrt{V_0}}$$
$$\Rightarrow \quad \frac{D}{B} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{V_0}}\right) e^{\sqrt{V_0}}$$

3) + 4) gives

$$2\,\frac{E}{B}\,e^{2i\sqrt{V_1}} = \frac{C}{B}\,e^{2\sqrt{V_0}}\left(1 - i\sqrt{\frac{V_0}{V_1}}\right) + \frac{D}{B}\,e^{-2\sqrt{V_0}}\left(1 + i\sqrt{\frac{V_0}{V_1}}\right)$$

These are real numbers

$$\Rightarrow \quad \frac{E}{B} = \frac{1}{2}\,e^{-2i\sqrt{V_1}}\left[\left(\frac{C}{B}e^{2\sqrt{V_0}} + \frac{D}{B}e^{-2\sqrt{V_0}}\right) - i\sqrt{\frac{V_0}{V_1}}\left(\frac{C}{B}e^{2\sqrt{V_0}} - \frac{D}{B}e^{-2\sqrt{V_0}}\right)\right]$$

and we have expressed C/B and D/B in terms of V_0. (don't bother to substitute for them.)

h)
$$\left|\frac{E}{B}\right|^2 = \left(\frac{1}{2}\right)^2\left|e^{-2i\sqrt{V_1}}\right|^2\left[\left(\frac{C}{B}e^{2\sqrt{V_0}} + \frac{D}{B}e^{-2\sqrt{V_0}}\right)^2 + \frac{V_0}{V_1}\left(\frac{C}{B}e^{2\sqrt{V_0}} - \frac{D}{B}e^{-2\sqrt{V_0}}\right)^2\right]$$

$$= \frac{1}{4}\left[\left(\frac{C}{B}e^{2\sqrt{V_0}} + \frac{D}{B}e^{-2\sqrt{V_0}}\right)^2 + \frac{V_0}{V_1}\left(\frac{C}{B}e^{2\sqrt{V_0}} - \frac{D}{B}e^{-2\sqrt{V_0}}\right)^2\right]$$

As $V_1 \to \infty$, the second term is negligible in comparison with the first, and the main contribution to $|E/B|^2$ will be $\frac{1}{4}\left(\frac{C}{B}\right)^2 e^{4\sqrt{V_0}}$,

but $\frac{C}{B} \sim e^{-\sqrt{V_0}}$, so $\left|\frac{E}{B}\right|^2 \sim e^{2\sqrt{V_0}}$

(not $e^{4\sqrt{V_0}}$ as the problem text says. Sorry!)

i) $|E/B|^2$ gives the probability of a making the transition from $a=0$ to $a>0$ spontaneously, $a=0$ can be interpreted as "no universe", so in this sense $|E/B|^2$ gives the probability of the Universe tunneling into existence from nothing.