Problem 1

a) We just need to find $\ddot{a}$:

$$a(t) = Ct^2$$
$$\dot{a} = 2Ct$$
$$\ddot{a} = 2C > 0, \text{ since } C > 0$$

But

$$H = \frac{\dot{a}}{a} = \frac{2Ct}{Ct^2} = \frac{2}{t}$$

So the expansion is accelerating, but H <u>decreases</u> with time.

b) Same thing here:

$$a = a_0 e^{H_0(t-t_0)}$$
$$\dot{a} = a_0 H_0 e^{H_0(t-t_0)} = H_0 a$$
$$\ddot{a} = H_0^2 a > 0 \rightarrow \text{accelerating}$$

But $\frac{\dot{a}}{a} = H_0 = \text{constant}$,

so H <u>does not</u> increase with time.

c) $H = \frac{\dot{a}}{a}$, so

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a - \dot{a}\cdot\dot{a}}{a^2} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2$$

FI with $k = 0$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

FII:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

So

$$\frac{dH}{dt} = -\frac{4\pi G}{3}\rho - 4\pi G\frac{P}{c^2} - \frac{8\pi G}{3}\rho$$

$$= -4\pi G\left(\rho + \frac{P}{c^2}\right) < 0 \quad \text{if } \rho + \frac{P}{c^2} > 0$$

QED

Problem 2

a) This is the Milne model (see lectures)

$$\rho = p = 0 \,, \quad k = -1$$

Then FI gives

$$\dot{a}^2 - c^2 = 0$$

$$\Rightarrow \quad \dot{a} = c$$

$$\Rightarrow \quad a = ct$$

Since $a_0 = ct_0$ (t_0 = today)

we can also write this as

$$\underline{a = a_0 \left(\frac{t}{t_0}\right)}$$

b) $k=0$, but universe with $p = -\frac{1}{3}\rho c^2$

$$\Rightarrow \quad \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1-\frac{1}{3})} = \rho_0 \left(\frac{a}{a_0}\right)^{-2}$$

FI:

$$\left(\frac{\dot{a}}{a}\right)^2 + 0 = \frac{8\pi G}{3}\rho_0 \left(\frac{a}{a_0}\right)^{-2}$$

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \underbrace{\frac{8\pi G}{3H_0^2}\rho_0}_{\substack{=\rho_0/\rho_{c0} \\ =1 \text{ for } k=0}} \left(\frac{a}{a_0}\right)^{-2}$$

$$x = a/a_0 \,, \quad \tau = H_0 t$$

$$\Rightarrow \quad \frac{1}{x^2}\left(\frac{dx}{d\tau}\right)^2 = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dx}{d\tau} = 1$$

$$\Rightarrow \quad x = \tau \quad (\text{choose } x(0) = 0)$$

$$\Rightarrow \quad \frac{a}{a_0} = H_0 t$$

$$t = t_0 \rightarrow \quad 1 = H_0 t_0 \Rightarrow H_0 = \frac{1}{t_0}$$

$$\Rightarrow \quad \underline{a = a_0\left(\frac{t}{t_0}\right)}$$

So there two models, which are physically completely different, give the same solution for $a = a(t)$.

c) Can we distinguish between them using $d_L(z)$?

Found in previous problem set, and in summary of week 4 that for Milne

$$d_L(z) = \frac{c}{H_0}\left(\frac{z^2+2z}{2}\right)$$

For $k=0$, $w=-\frac{1}{3}$ model:

$$d_L(z) = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{1\cdot(1+z')^2}}$$

$h(z) = H(z)/H_0$ for this model

$$= \frac{c}{H_0}(1+z)\int_0^z \frac{dz'}{1+z'}$$

$$= \frac{c}{H_0}(1+z)\ln(1+z)$$

So $d_L(z)$ behaves differently from that of the Milne model

At $z=1$: $d_L = \frac{c}{H_0}\frac{1+2}{2} = \frac{3}{2}\frac{c}{H_0}$ (Milne)

$$d_L = \frac{c}{H_0}2\ln 2 \simeq 1{,}39\frac{c}{H_0} \quad (k=0, w=-\tfrac{1}{3}),$$

a relative difference of 7%.

At $z=2$: $d_L = \frac{c}{H_0}\frac{4+4}{2} = 4\frac{c}{H_0}$ (Milne)

$$d_L = \frac{c}{H_0}3\ln 3 \simeq 3{,}30\frac{c}{H_0} \quad (k=0, w=-\tfrac{1}{3})$$

a relative difference of 17,5%. This is within the possible for observations to distinguish between

Problem 3

a) Have shown in the lectures that

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{x\,h(x)}$$

where $x = a/a_0$, and

$$h(x) = \frac{H(x)}{H_0} = \frac{H(a/a_0)}{H_0}$$

Rewrite as an integral over redshift:

$$1+z = \frac{a_0}{a} = \frac{1}{x}$$

$$dz = -\frac{1}{x^2}\,dx = -(1+z)^2\,dx$$

$$dx = -\frac{dz}{(1+z)^2}$$

$$x = 1 \rightarrow 1+z = 1 \Rightarrow z = 0$$

$$x = 0 \rightarrow 1+z = \lim_{x\to 0}\frac{1}{x} = \infty$$

So

$$t_0 = \frac{1}{H_0}\int_\infty^0 (1+z)\frac{1}{h(z)}\left(-\frac{dz}{(1+z)^2}\right)$$

$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\,h(z)}$$

For a model with NR matter and spatial curvature,

$$h^2 = \frac{H^2}{H_0^2} = \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2$$

and $\Omega_{m0} + \Omega_{k0} = 1$, so

$$h(z) = \sqrt{\Omega_{m0}(1+z)^3 + (1-\Omega_{m0})(1+z)^2}$$

and the result follows.

b) Consider t_0 as a function of Ω_{m0}:

$$t_0(\Omega_{m0}) = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m0}(1+z)^3+(1-\Omega_{m0})(1+z)^2}}$$

Take the derivative

$$\frac{d}{d\Omega_{m0}} t_0(\Omega_{m0}) = \frac{1}{H_0}\int_0^\infty \frac{dz}{1+z}\,\frac{d}{d\Omega_{m0}}\left[\Omega_{m0}(1+z)^3+(1-\Omega_{m0})(1+z)^2\right]^{-1/2}$$

$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{1+z}\left\{-\frac{1}{2}\left[\Omega_{m0}(1+z)^3+(1-\Omega_{m0})(1+z)^2\right]^{-3/2}\right.$$

$$\left.\cdot\left[(1+z)^3-(1+z)^2\right]\right\}$$

$$= -\frac{1}{2H_0}\int_0^\infty \frac{dz}{1+z}\,\frac{(1+z)^2\cdot z}{\left[\Omega_{m0}(1+z)^3+(1-\Omega_{m0})(1+z)^2\right]^{3/2}}$$

$$= -\frac{1}{2H_0}\int_0^\infty dz\,\frac{z(1+z)}{\left[\Omega_{m0}(1+z)^3+(1-\Omega_{m0})(1+z)^2\right]^{3/2}}$$

The integral is clearly positive, so we see that

$$\frac{d}{d\Omega_{m0}} t_0(\Omega_{m0}) < 0,$$

and t_0 is therefore a strictly decreasing function of Ω_{m0}, bounded upwards by the value at $\Omega_{m0}=0$!

$$t_0(\Omega_{m0}=0) = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{0+(1+z)^2}}$$

$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)^2} = \frac{1}{H_0}\Big|_0^\infty -\frac{1}{1+z} = \frac{1}{H_0}$$

So, for $\Omega_{m0} > 0$,

$$t_0 < \frac{1}{H_0}, \quad \text{q.e.d.}$$

Problem 4

a) With M, Λ & k:

$$FI \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\left[\rho_{m0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{\Lambda 0}\right]$$

$$\Rightarrow \frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 + \underbrace{\frac{kc^2}{a_0^2 H_0^2}}_{= -\Omega_{k0}}\frac{a_0^2}{a^2} = \underbrace{\frac{8\pi G}{3H_0^2}}_{= 1/\rho_{c0}}\left[\rho_{m0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{\Lambda 0}\right]$$

$$\Rightarrow \frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \Omega_{m0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a}{a_0}\right)^{-2}$$

$t = t_0$: $1 = \Omega_{m0} + \Omega_{\Lambda 0} + \Omega_{k0} \Rightarrow \Omega_{k0} = 1 - \Omega_{m0} - \Omega_{\Lambda 0}$

New variables: $x = a/a_0$, $\tau = H_0 t$

$$\frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = \frac{1}{a_0 x}\left(\frac{d}{d\tau}a_0 x\right)\frac{d\tau}{dt}$$

$$= \frac{H_0}{x}\frac{dx}{d\tau}$$

$$\Rightarrow \cancel{\frac{1}{H_0^2}}\frac{\cancel{H_0^2}}{x^2}\left(\frac{dx}{d\tau}\right)^2 = \Omega_{m0}x^{-3} + \Omega_{\Lambda 0} + \Omega_{k0}x^{-2}$$

$$\Rightarrow \left(\frac{dx}{d\tau}\right)^2 + \underbrace{\left[-\frac{\Omega_{m0}}{x} - \Omega_{\Lambda 0}x^2\right]}_{= U(x)} = \Omega_{k0} = 1 - \Omega_{m0} - \Omega_{\Lambda 0}$$

b) $\left(\frac{dx}{d\tau}\right)^2 \geq 0$, so

$$U(x) = -\left(\frac{\Omega_{m0}}{x} + \Omega_{\Lambda 0}x^2\right) = \Omega_{k0} - \left(\frac{dx}{d\tau}\right)^2 \leq \Omega_{k0}$$

$$U(x) = -\frac{\Omega_{m0}}{x} - \Omega_{\Lambda 0}\, x^2$$

$$U'(x) = 0 \Rightarrow \frac{\Omega_{m0}}{x^2} - 2\Omega_{\Lambda 0}\, x = 0$$

$$x^3 = \frac{\Omega_{m0}}{2\Omega_{\Lambda 0}}$$

$$x = \left(\frac{\Omega_{m0}}{2\Omega_{\Lambda 0}}\right)^{1/3}$$

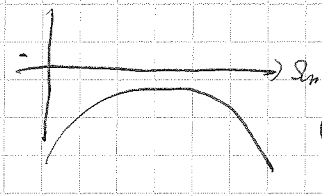
$$U_{max} = \frac{1}{x}\left(-\Omega_{m0} - \Omega_{\Lambda 0}\, x^3\right)$$

$$= \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3}\left(-\Omega_{m0} - \cancel{\Omega_{\Lambda 0}} \cdot \frac{\Omega_{m0}}{2\cancel{\Omega_{\Lambda 0}}}\right)$$

$$= -\frac{3}{2}\,\Omega_{m0}\left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3}$$

1) $\Omega_{m0} = 0{,}3,\ \Omega_{\Lambda 0} = 0{,}7 \rightarrow \Omega_{k0} = 0$

$$U_{max} = -0{,}75 < \Omega_{k0}$$

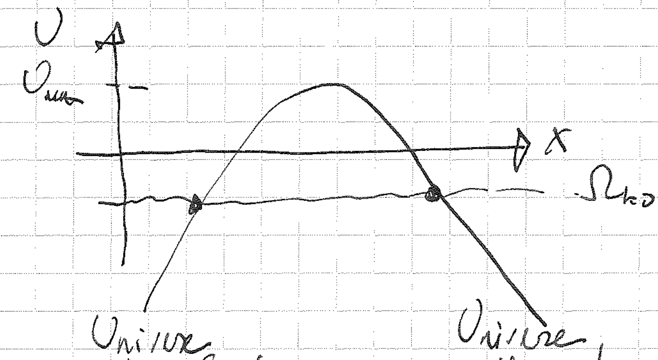


Universe expanding from $a=0$ to $a=\infty$

2) $\Omega_{m0} = 2{,}0,\ \Omega_{\Lambda 0} = -0{,}5$

$$\rightarrow U_{max} = 2{,}38$$

$$\Omega_{k0} = 1 - 2 + 0{,}5 = -0{,}5 < U_{max}$$



Universe with a Crunch

Universe with a bounce

3. $\Omega_{m0} = 0{,}5$, $\Omega_{\Lambda 0} = 3{,}0$

$\rightarrow \quad U_{max} \approx -1{,}72$

$\Omega_{k0} = 1 - 0{,}5 - 3 = -2{,}5 < U_{max}$

Same situation as in 2.

d) DGP:

$$\frac{H^2(z)}{H_0^2} = \left(\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{r_c}} + \sqrt{\Omega_{r_c}}\right)^2 + \Omega_{k0}(1+z)^2$$

$\underline{z=0}$ $\quad H^2(z=0) = H_0^2$, so

$$1 = \left(\sqrt{\Omega_{m0} + \Omega_{r_c}} + \sqrt{\Omega_{r_c}}\right)^2 + \Omega_{k0}$$

$$\Rightarrow \quad 1 = \Omega_{m0} + \Omega_{r_c} + 2\sqrt{\Omega_{r_c}}\sqrt{\Omega_{m0} + \Omega_{r_c}} + \Omega_{r_c} + \Omega_{k0}$$

$$\Rightarrow \quad \boxed{\Omega_{m0} + \Omega_{k0} + 2\,\Omega_{r_c} + 2\sqrt{\Omega_{r_c}}\sqrt{\Omega_{m0} + \Omega_{r_c}} = 1}$$

e) As usual, $1+z = \frac{a_0}{a} = \frac{1}{x} = x^{-1}$

and

$$\frac{H^2}{H_0^2} = \frac{1}{H_0^2}\left(\frac{1}{a_0 x}\frac{d}{d\tau}(a_0 x)\frac{d\tau}{dt}\right)^2$$

$$= \frac{1}{H_0^2}\left(\frac{1}{a_0 x}\cdot a_0 \frac{dx}{d\tau}\cdot H_0\right)^2$$

$$= \frac{1}{H_0^2} H_0^2 \frac{1}{x^2}\left(\frac{dx}{d\tau}\right)^2 = \frac{1}{x^2}\left(\frac{dx}{d\tau}\right)^2$$

FI for DGP becomes

$$\frac{1}{x^2}\left(\frac{dx}{d\tau}\right)^2 = \left(\sqrt{\Omega_{m0}x^{-3} + \Omega_{r_c}} + \sqrt{\Omega_{r_c}}\right)^2 + \Omega_{k0}x^{-2}$$

$$\Rightarrow \quad \left(\frac{dx}{d\tau}\right)^2 + \left[-x^2\left(\sqrt{\Omega_{m0}x^{-3}+\Omega_{rc}} + \sqrt{\Omega_{rc}}\right)\right] = \Omega_{k0}$$

$$\Omega_{k0} = 1 - \Omega_{m0} - 2\Omega_{rc} - 2\sqrt{\Omega_{rc}}\sqrt{\Omega_{m0}+\Omega_{rc}}$$
$$= \text{constant}$$

f) Here, $U(x) = -x^2\left(\sqrt{\Omega_{m0}x^{-3}+\Omega_{rc}} + \sqrt{\Omega_{rc}}\right)$

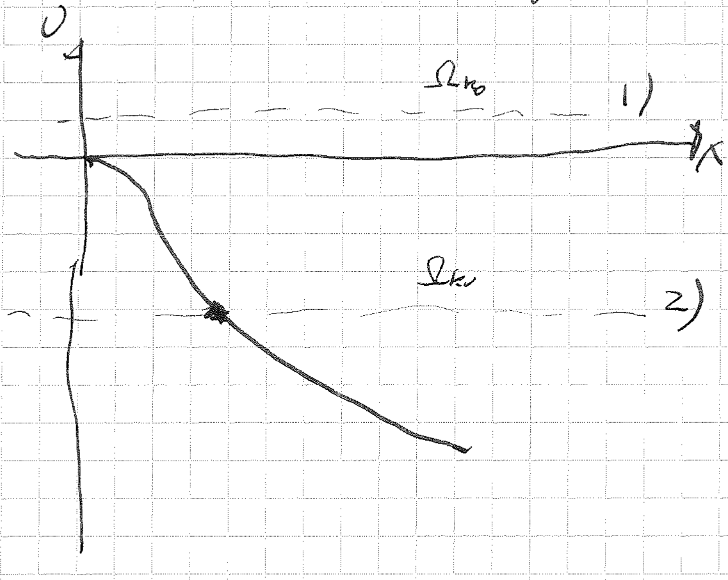
$$= -\sqrt{\Omega_{m0}x + \Omega_{rc}x^4} - \sqrt{\Omega_{rc}}\,x^2$$

We see that $U(0) = 0$,

and $U(x) \simeq -\sqrt{\Omega_{m0}x}$ for small x,

$U(x) \simeq -2\sqrt{\Omega_{rc}}\,x^2$ for large x,

so U will look something like this



Only two types of situations can occur then: 1), where $\Omega_{k0} \geq 0$ and the Universe expands from 0 to ∞

or 2) where the Universe has a minimum size, and we have a Big Bounce (dep on initial conditions)