Proof of Mattig's formula

For universe models with non-relativistic matter only, we have

$$d_L = \frac{2c}{H_0 \Omega_{m0}^2} \left[\Omega_{m0} z + (\Omega_{m0} - 2)\left(\sqrt{1+\Omega_{m0} z} - 1\right) \right] \quad (*)$$

Proof The general expression for the luminosity distance is

$$d_L = \frac{c(1+z)}{H_0 \sqrt{|\Omega_{k0}|}} S_k \left[\sqrt{|\Omega_{k0}|} \int_0^z \frac{dz'}{H(z')/H_0} \right]$$

where
$$S_k(x) = \begin{cases} \sin x, & k = +1 \\ x, & k = 0 \\ \sinh x, & k = -1 \end{cases}$$

In this type of model, $k = 0$ corresponds to $\Omega_{m0} = 1$, which is the EdS universe for which we have already found

$$d_L = \frac{2c}{H_0} \left(1 + z - \sqrt{1+z} \right)$$

We check that $(*)$ reproduces this result:

$$d_L = \frac{2c}{H_0 \cdot 1} \left[z + (1-2)\left(\sqrt{1+1\cdot z} - 1\right) \right]$$

$$= \frac{2c}{H_0} \left(z - \sqrt{1+z} + 1 \right),$$

So this special case checks out. 2.

Next, let's look at $\Omega_{m0} < 1$

$\Rightarrow \Omega_{k0} = 1 - \Omega_{m0} > 0 \Rightarrow k = -1$

We don't need to drag H_0 and c around, so let's write

$$\frac{H_0 d_L}{c} = \frac{1+z}{\sqrt{1-\Omega_{m0}}} \sinh\left[\sqrt{1-\Omega_{m0}} \int_0^z \frac{dz'}{H(z')/H_0}\right]$$

FI gives

$$\frac{H}{H_0} = \sqrt{\Omega_{m0}(1+z)^3 + (1-\Omega_{m0})(1+z)^2}$$

$$= (1+z)\sqrt{\Omega_{m0}(1+z) + 1 - \Omega_{m0}}$$

$$= (1+z)\sqrt{1+\Omega_{m0} z}$$

So the expression inside $\sinh(\ )$ becomes

$$I = \sqrt{1-\Omega_{m0}} \int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_{m0} z'}}$$

One way of handling the integral is to substitute

$$u = \sqrt{1+\Omega_{m0} z'}$$

$$z = \frac{u^2-1}{\Omega_{m0}}$$

$$dz = \frac{2u\,du}{\Omega_{m0}}$$

Then

$$I = \sqrt{1-\Omega_{m0}} \int_1^{\sqrt{1+\Omega_{m0} z}} \frac{1}{1+\frac{u^2-1}{\Omega_{m0}}} \frac{1}{u} \frac{2u}{\Omega_{m0}}\,du$$

$$= \sqrt{1-\Omega_{m0}} \int_1^{\sqrt{1+\Omega_{m0} z}} \frac{2}{u^2-1+\Omega_{m0}}\,du$$

By defining $a^2 = 1-\Omega_{m0}$, we can write this as

$$I = a \int_1^{\sqrt{1+\Omega_{m0} z}} \frac{2}{u^2-a^2}\,du$$

$$= a \int_1^{\sqrt{1+\Omega_{m0} z}} \frac{2}{(u-a)(u+a)}\,du$$

Since

$$\frac{1}{u-a} - \frac{1}{u+a} = \frac{u+a-(u-a)}{(u-a)(u+a)} = \frac{2a}{(u-a)(u+a)},$$

we get

$$I = \int_1^{\sqrt{1+\Omega_{m0} z}} \left(\frac{1}{u-a} - \frac{1}{u+a} \right) du$$

$$= \Big|_1^{\sqrt{1+\Omega_{m0} z}} \ln\left(\frac{u-a}{u+a}\right)$$

$$\rightarrow \quad I = \ln\left(\frac{\sqrt{1+\Omega_{m0}z} - \sqrt{1-\Omega_{m0}}}{\sqrt{1+\Omega_{m0}z} + \sqrt{1-\Omega_{m0}}}\right) - \ln\left(\frac{1-\sqrt{1-\Omega_{m0}}}{1+\sqrt{1-\Omega_{m0}}}\right)$$

$$\equiv \ln A - \ln B$$

So

$$\frac{H_0 d_L}{c} = \frac{1+z}{\sqrt{1-\Omega_{m0}}}\sinh I$$

where

$$\sinh I = \frac{1}{2}\left(e^{I} - e^{-I}\right)$$

$$= \frac{1}{2}\left(e^{\ln A - \ln B} - e^{-\ln A + \ln B}\right)$$

$$= \frac{1}{2}\left(\frac{A}{B} - \frac{B}{A}\right) = \frac{1}{2}\,\frac{A^2 - B^2}{AB}$$

$$= \frac{(A-B)(A+B)}{2AB}$$

Now comes the tedious part:

$$A - B = \frac{\sqrt{1+\Omega_{m0}z} - \sqrt{1-\Omega_{m0}}}{\sqrt{1+\Omega_{m0}z} + \sqrt{1-\Omega_{m0}}} - \frac{1-\sqrt{1-\Omega_{m0}}}{1+\sqrt{1-\Omega_{m0}}}$$

$$= \frac{\left(\sqrt{1+\Omega_{m0}z} - \sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right) - \left(1-\sqrt{1-\Omega_{m0}}\right)\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

$$= \frac{\cancel{\sqrt{1+\Omega_{m0}z}} + \sqrt{1+\Omega_{m0}z}\sqrt{1-\Omega_{m0}} - \sqrt{1-\Omega_{m0}} - \cancel{1+\Omega_{m0}} - \cancel{\sqrt{1+\Omega_{m0}z}} - \sqrt{1-\Omega_{m0}} + \sqrt{\cdots}\sqrt{\cdots} \cancel{-1-\Omega_{m0}}}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

$$= \frac{2\left(\sqrt{1+\Omega_{m0}z}\sqrt{1-\Omega_{m0}} - \sqrt{1-\Omega_{m0}}\right)}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

$$= \frac{2\sqrt{1-\Omega_{m0}}\left(\sqrt{1+\Omega_{m0}z} - 1\right)}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

and

$$A+B = \frac{\sqrt{1+\Omega_{m0}z} - \sqrt{1-\Omega_{m0}}}{\sqrt{1+\Omega_{m0}z} + \sqrt{1-\Omega_{m0}}} + \frac{1-\sqrt{1-\Omega_{m0}}}{1+\sqrt{1-\Omega_{m0}}}$$

$$= \frac{\left(\sqrt{1+\Omega_{m0}z} - \sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right) + \left(1-\sqrt{1-\Omega_{m0}}\right)\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

$$= \frac{\sqrt{1+\Omega_{m0}z} + \cancel{\sqrt{1+\Omega_{m0}z}\sqrt{1-\Omega_{m0}}} \cancel{-\sqrt{1-\Omega_{m0}}} - 1+\Omega_{m0} + \sqrt{1+\Omega_{m0}z} + \cancel{\sqrt{1-\Omega_{m0}}} - \cancel{\sqrt{1-\Omega_{m0}}\sqrt{1+\Omega_{m0}z}} - 1+\Omega_{m0}}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

$$= \frac{2\left[\sqrt{1+\Omega_{m0}z} - (1-\Omega_{m0})\right]}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}$$

so

$$(A-B)(A+B) = \frac{4\sqrt{1-\Omega_{m0}}\left(\sqrt{1+\Omega_{m0}z}-1\right)\left[\sqrt{1+\Omega_{m0}z}-(1-\Omega_{m0})\right]}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)^2\left(1+\sqrt{1-\Omega_{m0}}\right)^2}$$

Next

$$\frac{1}{2AB} = \frac{1}{2}\,\frac{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)}{\left(\sqrt{1+\Omega_{m0}z}-\sqrt{1-\Omega_{m0}}\right)}\cdot\frac{\left(1+\sqrt{1-\Omega_{m0}}\right)}{\left(1-\sqrt{1-\Omega_{m0}}\right)}$$

Finally

$$\frac{(A-B)(A+B)}{2AB} = \frac{4\sqrt{1-\Omega_{m0}}\left(\sqrt{1+\Omega_{m0}z}-1\right)\left[\sqrt{1+\Omega_{m0}z}-(1-\Omega_{m0})\right]}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)^2\left(1+\sqrt{1-\Omega_{m0}}\right)^2}$$

$$\cdot\frac{1}{2}\,\frac{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)}{\left(\sqrt{1+\Omega_{m0}z}-\sqrt{1-\Omega_{m0}}\right)\left(1-\sqrt{1-\Omega_{m0}}\right)}$$

$$= 2\sqrt{1-\Omega_{m0}}\,\frac{\left(\sqrt{1+\Omega_{m0}z}-1\right)\left[\sqrt{1+\Omega_{m0}z}-(1-\Omega_{m0})\right]}{\left(\sqrt{1+\Omega_{m0}z}+\sqrt{1-\Omega_{m0}}\right)\left(\sqrt{1+\Omega_{m0}z}-\sqrt{1-\Omega_{m0}}\right)\left(1+\sqrt{1-\Omega_{m0}}\right)\left(1-\sqrt{1-\Omega_{m0}}\right)}$$

$$= 2\sqrt{1-\Omega_{m0}}\,\frac{\left(\sqrt{1+\Omega_{m0}z}-1\right)\left[\sqrt{1+\Omega_{m0}z}-(1-\Omega_{m0})\right]}{\left(1+\Omega_{m0}z-1+\Omega_{m0}\right)\left(1-1+\Omega_{m0}\right)}$$

$$= \frac{2\sqrt{1-\Omega_{m0}}}{\Omega_{m0}^2(1+z)}\left(\sqrt{1+\Omega_{m0}z}-1\right)\left[\sqrt{1+\Omega_{m0}z}-(1-\Omega_{m0})\right]$$

So the luminosity distance becomes

$$\frac{H_0 d_L}{c} = \frac{(1+z)}{\sqrt{1-\Omega_{m0}}} \frac{2\sqrt{1-\Omega_{m0}}}{\Omega_{m0}^2 (1+z)} \left(\sqrt{1+\Omega_{m0} z} - 1\right)\left[\sqrt{1+\Omega_{m0} z} - (1-\Omega_{m0})\right]$$

$$= \frac{2}{\Omega_{m0}^2}\left[1+\Omega_{m0} z - \sqrt{1+\Omega_{m0} z} + \Omega_{m0}\sqrt{1+\Omega_{m0} z} - \sqrt{1+\Omega_{m0} z} + 1 - \Omega_{m0}\right]$$

$$= \frac{2}{\Omega_{m0}^2}\left[2 - \Omega_{m0} - 2\sqrt{1+\Omega_{m0} z} + \Omega_{m0}\sqrt{1+\Omega_{m0} z} + \Omega_{m0} z\right]$$

$$= \frac{2}{\Omega_{m0}^2}\left[\Omega_{m0} z + (\Omega_{m0} - 2)\left(\sqrt{1+\Omega_{m0} z} - 1\right)\right],$$

as we wanted to show.

Note that there is nothing in this result that suggest that anything dramatic happens when Ω_{m0} crosses from <1 to >1.

For $\Omega_{m0} > 1$, we have

$$\frac{H_0 d_L}{c} = \frac{1+z}{\sqrt{\Omega_{m0}-1}} \sin\left[\underbrace{\sqrt{\Omega_{m0}-1}\int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_{m0} z'}}}_{= I_{\Omega>1}}\right]$$

Apart from the factor $\sqrt{\Omega_{m0}-1}$ which was $\sqrt{1-\Omega_{m0}}$ for $\Omega_{m0} < 1$, this is the same expression as that we calculated for $\Omega_{m0} < 1$.

There is nothing about the integral that indicates that $\Omega_{m0} > 1$ is problematic, so we would expect the same result as for $\Omega_{m0} < 1$, or, in other words

$$I_{\Omega>1} = \sqrt{\Omega_{m0}-1} \int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_{m0} z'}}$$

$$= i\sqrt{1-\Omega_{m0}} \int_0^z \frac{dz'}{(1+z')\sqrt{1+\Omega_{m0} z'}}$$

$$= i\, I_{\Omega<1}$$

Therefore:

$$\frac{H_0 d_L}{c} = \frac{1+z}{\sqrt{\Omega_{m0}-1}} \sin I_{\Omega>1}$$

$$= \frac{1+z}{i\sqrt{1-\Omega_{m0}}} \frac{1}{2i}\left(e^{iI_{\Omega>1}} - e^{-iI_{\Omega>1}}\right)$$

$$= \frac{1+z}{\sqrt{1-\Omega_{m0}}} \left(-\frac{1}{2}\right)\left(e^{-I_{\Omega<1}} - e^{I_{\Omega<1}}\right)$$

$$= \frac{1+z}{\sqrt{1-\Omega_{m0}}} \frac{1}{2}\left(e^{I_{\Omega<1}} - e^{-I_{\Omega<1}}\right)$$

$$= \frac{1+z}{\sqrt{1-\Omega_{m0}}} \sinh I_{\Omega<1}$$

$$= \frac{2}{\Omega_{m0}^2}\left[\Omega_{m0} z + (\Omega_{m0}-2)\left(\sqrt{1+\Omega_{m0} z} - 1\right)\right],$$

showing that Mattig's formula is valid for all values of Ω_{m0}.

b) $\Omega_{m0} \to 0$ in Mattig's formula:

Then $\sqrt{1+\Omega_{m0} z} \simeq 1 + \frac{1}{2}\Omega_{m0} z - \frac{1}{8}\Omega_{m0}^2 z^2$

(go to order Ω_{m0}^2 because of prefactor of Ω_{m0}^{-2})

So

$$d_L \simeq \frac{2c}{H_0 \Omega_{m0}^2}\left[\Omega_{m0} z + (\Omega_{m0}-2)\left(\cancel{1} + \frac{1}{2}\Omega_{m0} z - \frac{1}{8}\Omega_{m0}^2 z^2 - \cancel{1}\right)\right]$$

$$\simeq \frac{2c}{H_0 \Omega_{m0}^2}\left[\cancel{\Omega_{m0} z} + \frac{1}{2}\Omega_{m0}^2 z - \underbrace{\frac{1}{8}\Omega_{m0}^3 z^2}_{\mathcal{O}(\Omega_{m0}^3)\simeq 0} - \cancel{\Omega_{m0} z} + \frac{1}{4}\Omega_{m0}^2 z^2\right]$$

$$= \frac{2c}{H_0 \Omega_{m0}^2}\left(\frac{1}{2}\Omega_{m0}^2 z + \frac{1}{4}\Omega_{m0}^2 z^2\right)$$

$$= \frac{c}{H_0}\left(\frac{1}{2}z^2 + z\right),$$ which

agrees with our earlier result for the Milne model.

Problem 1

a) The general expression for d_L is

$$d_L = \frac{c(1+z)}{H_0\sqrt{|\Omega_{k0}|}} S_k^{\,1}\left[\sqrt{|\Omega_{k0}|}\int_0^z \frac{dz'}{H(z')/H_0}\right]$$

where $S_k^{\,1}(x) = \begin{cases} \sin x, & k=+1 \\ x, & k=0 \\ \sinh x, & k=-1 \end{cases}$

Here we have a spatially flat universe, so $k=0$

and

$$d_L = \frac{c(1+z)}{H_0\sqrt{|\Omega_{k0}|}}\sqrt{|\Omega_{k0}|}\int_0^z \frac{dz'}{H(z')/H_0} = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{H(z')/H_0}$$

For a model with dust and Λ:

$$H^2 = H_0^2\left[\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}\right]$$

But $\Omega_{m0} + \Omega_{\Lambda 0} = 1$, so

$$\frac{H}{H_0} = \sqrt{\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}}$$

Therefore

$$d_L = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{m0}(1+z')^3 + 1 - \Omega_{m0}}}$$

b) i) $\Omega_{m0} = 1$ gives

$$d_L = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{(1+z')^{3/2}} = \frac{c(1+z)}{H_0} \Big|_0^z -2(1+z')^{-1/2}$$

$$= \frac{2c(1+z)}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right) = \underline{\underline{\frac{2c}{H_0}\left(1+z-\sqrt{1+z}\right)}}$$

ii) $\Omega_{m0} = 0$ gives

$$d_L = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{1} = \underline{\underline{\frac{c}{H_0}(z+z^2)}}$$

c) We have

$$\Omega_{m0}(1+z')^3 + 1 - \Omega_{m0} \approx \Omega_{m0}(1+3z') + 1 - \Omega_{m0}$$

$$= \Omega_{m0} + 3\Omega_{m0} z' + 1 - \Omega_{m0} = 1 + 3\Omega_{m0} z'$$

for $z' \ll 1$ ($z' \leq z$ in the integration, so $z \ll 1 \Rightarrow z' \ll 1$),

and $$\frac{1}{\sqrt{\Omega_{m0}(1+z')^3 + 1 - \Omega_{m0}}} \simeq \frac{1}{[1+3\Omega_{m0} z']^{1/2}} = (1+3\Omega_{m0} z')^{-1/2}$$

$$\simeq 1 - \frac{3}{2}\Omega_{m0} z'$$

So $$d_L \simeq \frac{c(1+z)}{H_0} \int_0^z \left(1 - \frac{3}{2}\Omega_{m0} z'\right) dz' = \frac{c(1+z)}{H_0}\left(z - \frac{3}{4}\Omega_{m0} z^2\right)$$

$$= \frac{c}{H_0}\left(z - \frac{3}{4}\Omega_{m0} z^2 + z^2 - \frac{3}{4}\Omega_{m0} z^3\right) \approx \underline{\underline{\frac{cz}{H_0}}} \text{ for } z \ll 1$$

d) $$d_L = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{m0}(1+z')^3+1-\Omega_{m0}}}$$

$$\Rightarrow \frac{H_0 d_L}{c} = (1+z)\int_0^z \frac{dz'}{\sqrt{\Omega_{m0}(1+z')^3+1-\Omega_{m0}}}$$

Introduce $s \equiv \left(\frac{1-\Omega_{m0}}{\Omega_{m0}}\right)^{1/3}$

$$\Rightarrow \frac{H_0 d_L}{c} = \frac{(1+z)}{\sqrt{\Omega_{m0}}}\int_0^z \frac{dz'}{\sqrt{(1+z')^3+\left(\frac{1-\Omega_{m0}}{\Omega_{m0}}\right)}}$$

$$= \frac{(1+z)}{\sqrt{\Omega_{m0}}}\int_0^z \frac{dz'}{\sqrt{(1+z')^3+s^3}}$$

Now we substitute

$$u = \frac{s}{1+z'}$$

$$\Rightarrow 1+z' = \frac{s}{u}$$

$$\Rightarrow dz' = -\frac{s}{u^2}du$$

$$z' = 0 \Rightarrow u = s$$

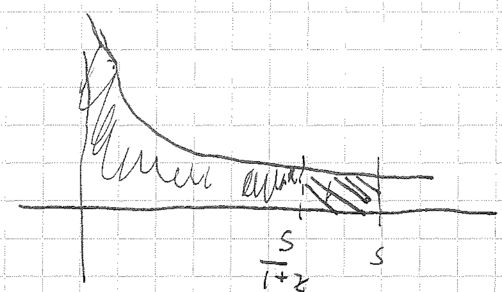
$$z' = z \Rightarrow u = \frac{s}{1+z}$$

So

$$\frac{H_0 d_L}{c} = \frac{(1+z)}{\sqrt{\Omega_{m0}}}\int_s^{s/(1+z)} \left(-\frac{s}{u^2}du\right)\frac{1}{\sqrt{\frac{s^3}{u^3}+s^3}}$$

$$= \frac{(1+z)}{\sqrt{\Omega_{m0}}}\int_{s/(1+z)}^{s} \frac{s}{s^{3/2}}\frac{du}{u^2\sqrt{u^{-3}+1}}$$

$$= \frac{(1+z)}{\sqrt{s\,\Omega_{m0}}}\int_{s/(1+z)}^{s} \frac{du}{\sqrt{u^4+u}}$$



Graphically it is clear that

$$\int_{s/(1+z)}^{s} \cdots = \int_0^s \cdots - \int_0^{s/(1+z)} \cdots ,$$

So

$$\frac{H_0 d_L}{c} = \frac{(1+z)}{\sqrt{s\Omega_{m0}}} \left[\int_0^s \frac{du}{\sqrt{u^4+u}} - \int_0^{s/(1+z)} \frac{du}{\sqrt{u^4+u}} \right]$$

$$= \frac{(1+z)}{\sqrt{s\Omega_{m0}}} \left[T(s) - T\left(\frac{s}{1+z}\right) \right]$$

where $$T(s) = \int_0^s \frac{du}{\sqrt{u^4+u}}$$

e) $m - M = 5\log\left(\frac{d_L}{10\,pc}\right)$

Sn 1997ap : $z = 0{,}83$, $m = 24{,}32$

1) $24{,}32 - M = 5\log\left(\frac{d_L(z=0{,}83)}{10\,pc}\right)$

Sn 1992P : $z = 0{,}026$, $m = 16{,}08$

2) $16{,}08 - M = 5\log\left(\frac{d_L(z=0{,}026)}{10\,pc}\right)$

1) − 2) $\Rightarrow$ $24{,}32 - 16{,}08 = 5\log\left(\frac{d_L(z=0{,}83)}{d_L(z=0{,}026)}\right)$

$\Rightarrow d_L(z=0{,}83) = d_L(z=0{,}026)\cdot 10^{\frac{24{,}32-16{,}08}{5}}$

$\simeq 44{,}46\, d_L(z=0{,}026)$

0,026 << 1, so we can use

$d_L(z) \simeq \frac{cz}{H_0}$ for $z \ll 1$,

and then

$d_L(z=0{,}83) = 44{,}46\cdot 0{,}026\,\frac{c}{H_0}$

$\simeq \underline{\underline{1{,}16\,\frac{c}{H_0}}}$

f) For EdS :

$d_L(z=0{,}83) = \frac{2c}{H_0}\left(1 + 0{,}83 - \sqrt{1+0{,}83}\right) \simeq \underline{\underline{0{,}95\,\frac{c}{H_0}}}$

For dS :

$d_L(z=0{,}83) = 0{,}83\cdot(1+0{,}83)\frac{c}{H_0} \simeq \underline{\underline{1{,}52\,\frac{c}{H_0}}}$

⊃: Pure matter → d_L too low
Pure Λ → d_L too high

g) See plot (full line)
Numerical experimentation gives $d_L(z=0{,}83) = 1{,}16\frac{c}{H_0}$
for $\Omega_{m0} = 0{,}41$, $\Omega_{\Lambda 0} = 1 - \Omega_{m0} = 0{,}59$

h) See plot (dashed line)
I find that $\Omega_{m0} = 0{,}047$ gives
$d_L(z=0{,}83) \simeq 1{,}16\frac{c}{H_0}$.

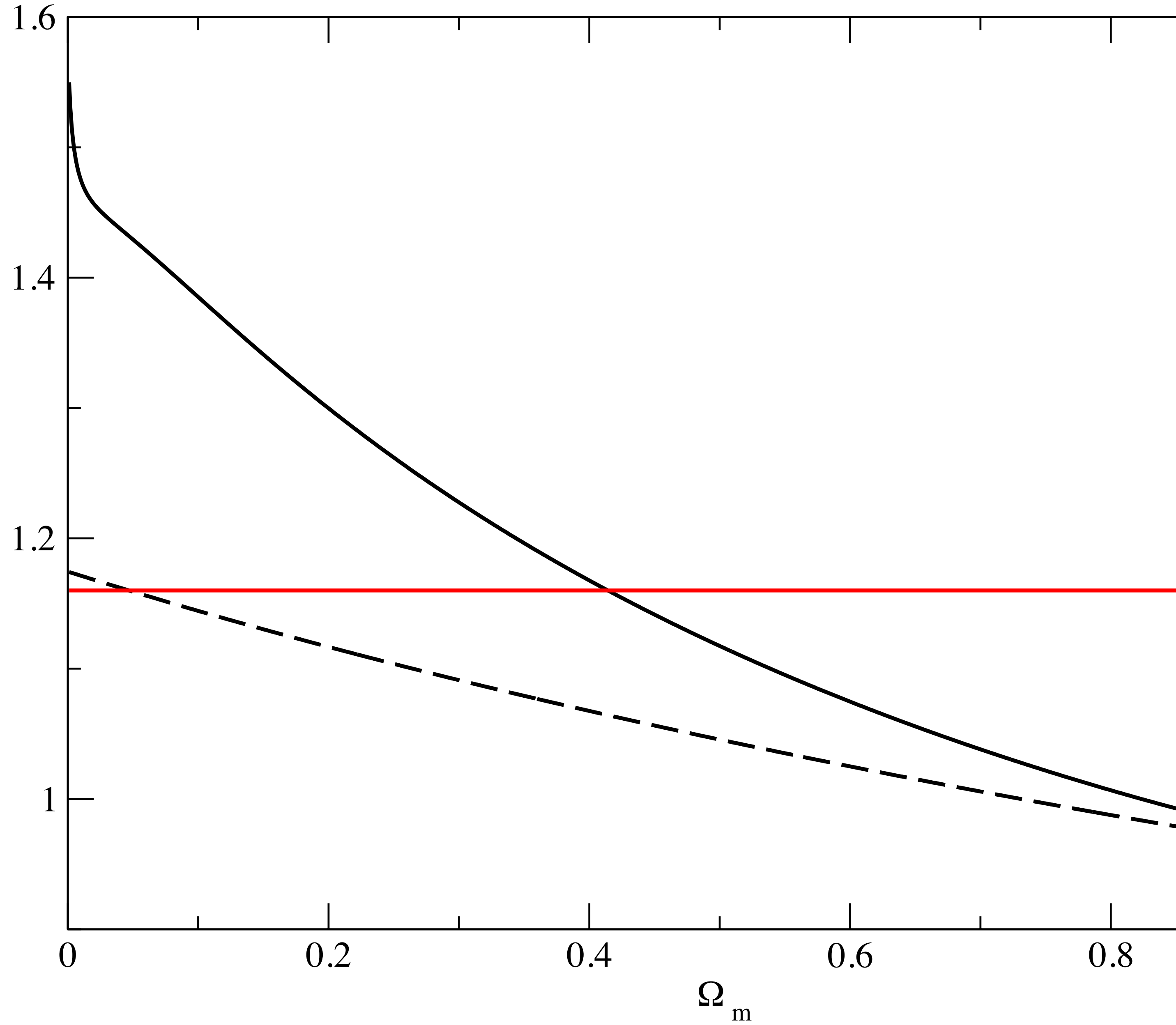
hh) Milne:

$$d_L(z=0{,}83) = \frac{c}{H_0}\left(z + \frac{1}{2}z^2\right)$$

$$= \frac{c}{H_0}\left(0{,}83 + \frac{1}{2}\cdot 0{,}83^2\right) \simeq \underline{\underline{1{,}17\frac{c}{H_0}}}$$

Pretty close!

i) This measurements, on its own, does not give us good reasons to prefer any model above the others. $\Omega_{m0} = 0{,}047$ is low, but we need other data to refute this value. And, ~~in some sense~~ the Milne model very nearly matches the measurement <u>without any adjustable parameter</u>, and is this sense is to be preferred on the grounds of simplicity!



1.6
1.4
1.2
1
0
0.2
0.4
0.6
0.8
Ω_m

Problem 3

FII : $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \rho_\Lambda + \frac{3p_m}{c^2} + \frac{3p_\Lambda}{c^2}\right)$

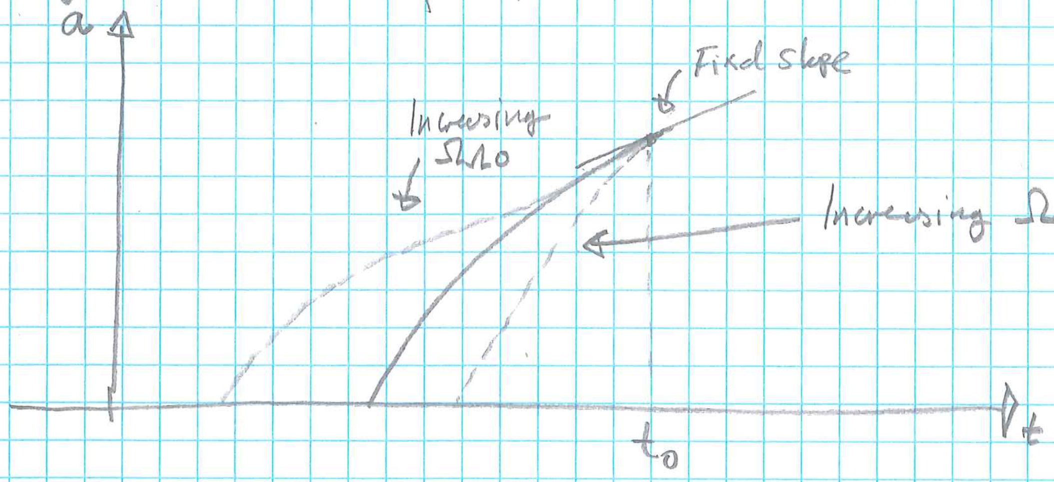
$= \dots = -\frac{1}{2}H_0^2\left[\Omega_{m0}\left(\frac{a_0}{a}\right)^3 - 2\Omega_{\Lambda 0}\right]$

Ω_{m_0} : Tries to make $\ddot{a} < 0$, $a(t)$ bend like ∩

$\Omega_{\Lambda 0}$: —— " —— $\ddot{a} > 0$, —" —— ∪

H_0 fixed $\Rightarrow$ slope of $a(t)$ fixed at $t = t_0$.

Age : Time from $a(t = t_0)$ to $a = 0$



(Remember : We choose $t = 0$ to be where $a(t) = 0$ in the past)

Making $\ddot{a}$ more negative, makes $a(t)$ hit 0 later.

Making $\ddot{a}$ less negative, makes $a(t)$ hit 0 later

Problem 4

$$ds^2 = dt^2 - a^2(t)\, d\vec{x} \cdot d\vec{x}$$

a) Proper time is defined so that $d\tau^2 = ds^2$

For an observer who follows the expansion,

$d\vec{x} = 0$, so

$$d\tau^2 = ds^2 = dt^2 - 0$$

$\therefore$ $\underline{\underline{\tau = t}}$

b) $g_{\mu\nu} \neq 0$ only if $\mu = \nu$, so

$$g_{\mu\nu} dx^\mu dx^\nu = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

$$= dt^2 - a^2(t)\, dx^2 - a^2(t)\, dy^2 - a^2(t)\, dz^2$$

$$= dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) = \underline{\underline{ds^2}}$$

c) $L = L\left(x, \frac{dx}{dt}\right) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x)$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \; ; \quad \frac{\partial L}{\partial\left(\frac{dx}{dt}\right)} = m \frac{dx}{dt}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx}{dt}\right)}\right) = 0$$

$$\Rightarrow \quad -\frac{\partial V}{\partial x} - \frac{d}{dt}\left(m \frac{dx}{dt}\right) = 0$$

$$\Rightarrow \quad \underline{\underline{m \frac{d^2 x}{dt^2} = -\frac{\partial V}{\partial x}}}$$

d)

$$L = T = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$= \frac{1}{2} m \left\{ g_{00} \left(\frac{dx^0}{d\tau}\right)^2 + g_{11}\left(\frac{dx^1}{d\tau}\right)^2 + g_{22}\left(\frac{dx^2}{d\tau}\right)^2 + g_{33}\left(\frac{dx^3}{d\tau}\right)^2 \right\}$$

$$= \frac{1}{2} m \left\{ \left(\frac{dt}{d\tau}\right)^2 - a^2(t) \left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \right] \right\}$$

e) For $\mu = 0$:

$$\frac{\partial L}{\partial x^0} = \frac{\partial L}{\partial t} = \frac{1}{2} m \frac{\partial}{\partial t} \left\{ \left(\frac{dt}{d\tau}\right)^2 - a^2(t) \frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} \right\}$$

$$= -m a \dot{a} \frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau}$$

$$\frac{\partial L}{\partial \left(\frac{dx^0}{d\tau}\right)} = \frac{\partial L}{\partial \left(\frac{dt}{d\tau}\right)} = m \frac{dt}{d\tau}$$

$$\frac{\partial L}{\partial x^0} - \frac{d}{d\tau} \left(\frac{\partial L}{\partial \left(\frac{dx^0}{d\tau}\right)} \right) = 0$$

$$\Rightarrow \quad - m a \dot{a} \frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} - m \frac{d^2 t}{d\tau^2} = 0$$

$$\Rightarrow \quad \frac{d^2 t}{d\tau^2} + a \dot{a} \frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} = 0$$

For $\mu = 1$:

$$\frac{\partial L}{\partial x^1} = \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \left(\frac{dx^1}{d\tau}\right)} = \frac{\partial L}{\partial \left(\frac{dx}{d\tau}\right)} = - m a^2(t) \frac{dx}{d\tau}$$

$$\frac{\partial L}{\partial x^1} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(\frac{dx^1}{d\tau}\right)}\right) = 0$$

$$\Rightarrow \quad 0 - \frac{d}{d\tau}\left(-m\,a^2(t)\frac{dx}{d\tau}\right) = 0$$

$$\Rightarrow \quad \frac{d}{d\tau}\left(a^2(t)\right)\frac{dx}{d\tau} + a^2(t)\frac{d^2x}{d\tau^2} = 0$$

Remember: $t = t(\tau)$

$$\Rightarrow \quad a^2(t)\frac{d^2x}{d\tau^2} + 2a\frac{da}{dt}\frac{dt}{d\tau}\frac{dx}{d\tau} = 0$$

$$\Rightarrow \quad \frac{d^2x}{d\tau^2} + 2H\frac{dt}{d\tau}\frac{dx}{d\tau} = 0$$

The equations for $x^2 = y$ and $x^3 = z$ will have the same form as the equation for $x^1 = x$, because L is symmetric in x, y, z.

f) We found

$$\frac{d^2t}{d\tau^2} + a\dot{a}\,\frac{d\vec{x}}{d\tau}\cdot\frac{d\vec{x}}{d\tau} = 0$$

With the given definitions,

$$\frac{d\vec{x}}{d\tau} = \frac{1}{m}\vec{P}$$

and

$$\frac{d^2t}{d\tau^2} = \frac{d}{d\tau}\frac{dt}{d\tau} = \frac{E}{m}\frac{d}{dt}\frac{E}{m} = \frac{E}{m^2}\frac{dE}{dt},$$

so

$$\frac{E}{m^2}\frac{dE}{dt} + a\dot{a}\,\frac{1}{m^2}\,\vec{P}\cdot\vec{P} = 0$$

$$\Rightarrow \quad E\frac{dE}{dt} + a\dot{a}\,\vec{P}\cdot\vec{P} = 0$$

g)
$$g_{\mu\nu} P^\mu P^\nu = m^2$$

$$\Rightarrow \quad (P^0)^2 - a^2\left[(P^1)^2 + (P^2)^2 + (P^3)^2\right] = m^2$$

$$\Rightarrow \quad E^2 - a^2 \vec{P}\cdot\vec{P} = m^2$$

$$\Rightarrow \quad \underline{\underline{a^2 \vec{P}\cdot\vec{P} = E^2 - m^2 = p^2 + m^2 - m^2 = p^2}}$$

h)
$$E\frac{dE}{dt} + a\dot{a}\,\vec{P}\cdot\vec{P} = 0$$

$$\Rightarrow \quad E\frac{dE}{dt} + \frac{\dot{a}}{a}\underbrace{a^2 \vec{P}\cdot\vec{P}}_{= E^2 - m^2} = 0$$

$$\Rightarrow \quad E\frac{dE}{dt} + \frac{1}{a}\frac{da}{dt}(E^2 - m^2) = 0$$

Separable equation :

$$\frac{E}{E^2 - m^2}\,dE = -\frac{1}{a}\,da$$

$$\Rightarrow \quad \int \frac{E}{E^2-m^2}\,dE = -\int \frac{1}{a}\,da = -\ln a + C, \quad C = \text{constant}$$

We have
$$\int \frac{E}{E^2-m^2}\,dE \underset{\substack{u = E^2 - m^2 \\ du = 2E\,dE}}{=} \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(E^2 - m^2)$$

$$= \frac{1}{2}\ln p^2 = \ln p,$$

so
$$\ln p = -\ln a + C$$

$$\Rightarrow \quad p = e^{-\ln a + C} = \frac{e^C}{a} \propto \underline{\underline{\frac{1}{a}}}$$

i) $$p = \frac{a(t_f)}{a(t)} p_f = \frac{a_f}{a} p_f$$

$$d\tau = \frac{m}{E} dt$$

$$\Rightarrow \quad H(\tau)\, d\tau = \frac{m}{E} \frac{\dot{a}}{a} dt = \frac{m}{E} \frac{da}{a} \qquad \left(\dot{a} = \frac{da}{dt} \Rightarrow \dot{a}\, dt = da \right)$$

$$\Rightarrow \quad \int_{\tau_i}^{\tau_f} H(\tau)\, d\tau = \int_{a(\tau_i)}^{a(\tau_f)} \frac{m\, da}{a E}$$

$$= \int_{a_i}^{a_f} \frac{m\, da}{a\sqrt{p^2 + m^2}} = \int_{a_i}^{a_f} \frac{m\, da}{a\sqrt{\frac{a_f^2}{a^2} p_f^2 + m^2}}$$

$$= \int_{a_i}^{a_f} \frac{m\, da}{\sqrt{a_f^2 p_f^2 + m^2 a^2}} = \int_{a_i}^{a_f} \frac{da}{\sqrt{a^2 + \frac{p_f^2 a_f^2}{m^2}}}$$

j) We have

$$\int_{a_i}^{a_f} \frac{da}{\sqrt{a^2 + \frac{p_f^2 a_f^2}{m^2}}} \underset{\uparrow}{=} \int_{\sinh^{-1}\left(\frac{m a_i}{a_f p_f}\right)}^{\sinh^{-1}\left(\frac{m a_f}{a_f p_f}\right)} \frac{\frac{a_f p_f}{m} \cosh x\, dx}{\sqrt{\frac{a_f^2 p_f^2}{m^2} \underbrace{(1 + \sinh^2 x)}_{= \cosh^2 x}}}$$

$$\boxed{\begin{aligned} a &= \frac{a_f p_f}{m} \sinh x \\ da &= \frac{a_f p_f}{m} \cosh x\, dx \\ x &= \sinh^{-1}\left(\frac{m a}{a_f p_f}\right) \end{aligned}}$$

$$= \sinh^{-1}\left(\frac{m \cancel{a_f}}{\cancel{a_f} p_f}\right) - \sinh^{-1}\left(\frac{m a_i}{a_f p_f}\right)$$

$$= \ln\left[\frac{m}{p_f} + \sqrt{\frac{m^2}{p_f^2} + 1}\right] - \ln\left(\frac{m a_i}{a_f p_f} + \sqrt{\frac{m^2 a_i^2}{a_f^2 p_f^2} + 1}\right)$$

$$= \ln\left(\frac{\frac{m}{p_f} + \sqrt{\frac{m^2}{p_f^2}+1}}{\frac{m a_i}{a_f p_f} + \sqrt{\frac{m^2 a_i^2}{a_f^2 p_f^2}+1}}\right)$$

So

$$\int_{\tau_i}^{\tau_f} H(\tau)\,d\tau = \ln\left(\frac{\frac{m}{p_f} + \sqrt{\frac{m^2}{p_f^2}+1}}{\frac{m a_i}{a_f p_f} + \sqrt{\frac{m^2 a_i^2}{a_f^2 p_f^2}+1}}\right)$$

$\ln x$ is strictly increasing with x, so maximal if the argument has a maximal value.
In this case it has, because the denominator has a minimum for $a_i = 0$, so

$$\int_{\tau_i}^{\tau_f} H(\tau)\,d\tau \leq \ln\left(\frac{\frac{m}{p_f} + \sqrt{\frac{m^2}{p_f^2}+1}}{0 + \sqrt{0+1}}\right)$$

$$= \ln\left(\frac{m}{p_f} + \frac{1}{p_f}\sqrt{m^2+p_f^2}\right) = \underline{\ln\left(\frac{E_f+m}{p_f}\right)}$$

k)

$$H_{ave} \equiv \frac{1}{\tau_f - \tau_i}\int_{\tau_i}^{\tau_f} H(\tau)\,d\tau$$

$$\Rightarrow \quad H_{ave} \leq \frac{1}{\tau_f-\tau_i}\ln\left(\frac{E_f+m}{p_f}\right)$$

$E_f + m = \sqrt{p_f^2+m^2} + m > p_f$, so

$\ln\left(\frac{E_f+m}{p_f}\right) > 0$

If $H_{ave} > 0$, we would get the inequality

$$0 < H_{ave} \leq 0$$

if $\tau_f - \tau_i$ could $\to \infty$

This is a contradiction (Have can't be a positive number and at most 0 at the same time),

so we must conclude that $\tau_f - \tau_i$ must be finite.

Proper time has to start (and end) at some point.

The dS universe can only be infinitely old for observers who follow the expansion.