Problem 1

a) The proper distance for the particle horizon (PH) is given by

$$d_{PH}(z) = \frac{c}{(1+z)H_0} \int_z^\infty \frac{dz'}{H(z')/H_0}$$

The Einstein-de Sitter model has $\Omega_{m0} = 1$, all other $\Omega_i = 0$, so

$$\frac{H}{H_0} = \sqrt{(1+z)^3} = (1+z)^{3/2}$$

This gives

$$d_{PH}(z) = \frac{c}{(1+z)H_0} \int_z^\infty \frac{dz'}{(1+z')^{3/2}}$$

$$= \frac{c}{(1+z)H_0} \Big|_z^\infty -2(1+z')^{-\frac{1}{2}}$$

$$= \frac{2c}{H_0(1+z)^{3/2}}$$

b) The density at redshift z is

$$\rho_M(z) = \rho_{m0}(1+z)^3 = \rho_{c0}(1+z)^3$$

$$= \frac{3H_0^2}{8\pi G}(1+z)^3$$

The mass within the PH is therefore

$$M_{PH}(z) = \rho_m(z) \cdot \frac{4\pi}{3} d_{PH}^3(z)$$

$$= \frac{\cancel{3}H_0^2}{\cancel{8}\cancel{\pi}G}(1+z)^3 \frac{4\cancel{\pi}}{\cancel{3}} \cdot \frac{\cancel{8}c^3}{H_0^3(1+z)^{9/2}}$$

$$= \underline{\frac{4c^3}{GH_0} \frac{1}{(1+z)^{3/2}}}$$

The numerical prefactor is

$$\frac{4\cdot(3\cdot10^8\,m\,s^{-1})^3}{7\cdot10^{-11}\,m^3\,kg^{-1}\,s^{-2}\cdot 70\cdot\frac{10^3\,m}{3\cdot10^6\cdot10^{16}\,m}\,s^{-1}}$$

$$\simeq \frac{4\cdot30\cdot10^{24}\,\cancel{m^3}\,s^{-3}}{2\cdot10^2\cdot10^{-11}\cdot10^{-19}\,\cancel{m^3}\,kg^{-1}\,\cancel{s^{-3}}}$$

$$\simeq 6\cdot10^{53}\,kg = 3\cdot10^{23}\,M_\odot$$

where $1\,M_\odot = 2\cdot10^{30}\,kg$

So

$$M_{PH}(z) = \frac{3\cdot10^{23}\,M_\odot}{(1+z)^{3/2}}$$

c) For comoving observers, distances stretch as $d \propto \frac{a}{a_0} \propto \frac{1}{1+z}$, and the mass within a comoving volume is conserved:

$$M \propto \rho d^3 \propto (1+z)^3\frac{1}{(1+z)^3} = \text{constant}$$

The particle horizon is $\propto \frac{1}{(1+z)^{3/2}}$, so we should not expect the mass contained inside it to be constant. As time goes by, more and more distant regions of the Universe comes into view, and therefore the mass inside the PH will increase.

d) Let z_* = redshift where $M_{PH}(z) = 10^{15} M_\odot$

Then
$$\frac{3\cdot 10^{23} M_\odot}{(1+z_*)^{3/2}} = 10^{15} M_\odot$$

$$\Rightarrow \quad \underline{(1+z_*)^{3/2}} = \underline{3\cdot 10^{8}}$$

$$\Rightarrow \quad 1+z_* = 3^{2/3}\cdot 10^{16/3}$$
$$\simeq 2\cdot 10^{1/3}\cdot 10^{5} \simeq \underline{\underline{4\cdot 10^{5}}}$$

Size of the PH at z_*:

$$d_{PH}(z_*) = \frac{2c}{H_0(1+z_*)^{3/2}} \approx \frac{\cancel{2}\cdot \cancel{3\cdot 10^{8}}\, m/s}{\cancel{2}\cdot 10^{-18} s^{-1}} \frac{1}{\cancel{3\cdot 10^{8}}}$$

$$= 10^{18} m = 10^{2}\cdot \underbrace{10^{16} m}_{\simeq 1 ly}$$

$$\simeq 100\, ly \simeq \underline{\underline{30\, pc}}$$

For EdS:
$$\frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{2/3}$$

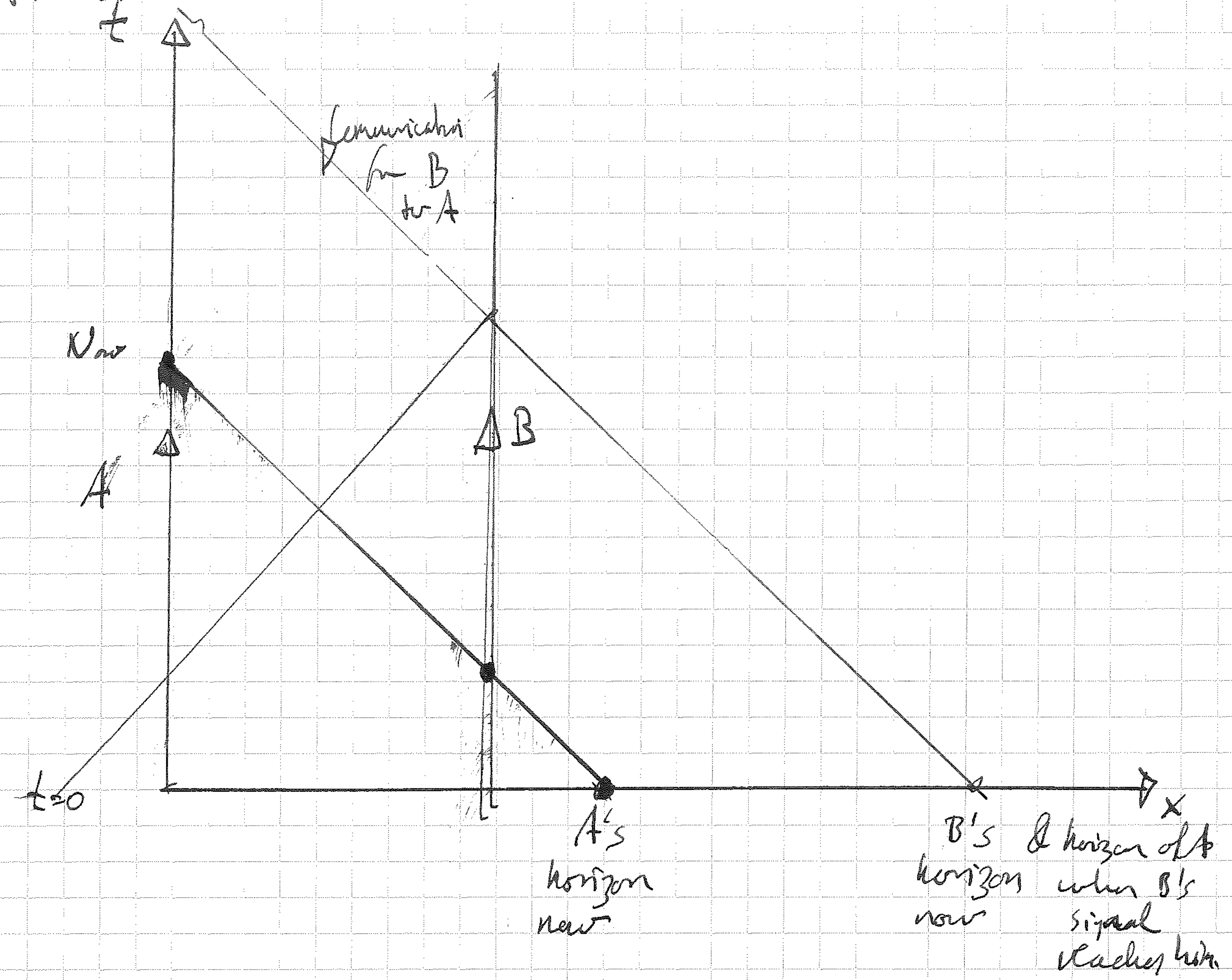
$$\Rightarrow \quad \frac{1}{1+z} = \left(\frac{t}{t_0}\right)^{2/3}$$

$$\Rightarrow \quad t(z) = \frac{t_0}{(1+z)^{3/2}}$$

with
$$t_0 = \frac{2}{3H_0} \approx \frac{\cancel{2}}{3}\frac{1}{\cancel{2}\cdot 10^{-18} s^{-1}} = \frac{1}{3} 10^{18} s$$
$$\approx 3\cdot 10^{17} s \simeq 10^{10} yrs$$

So
$$t(z_*) = \frac{10^{10}\, yrs}{3\cdot 10^{8}} \simeq \underline{\underline{30\, yrs}}$$

Problem 2



B cannot communicate anything to A faster than at the speed of light. If B wants to send information about the region outside of A's present horizon, it will not reach A before a time has past such that that region now is *inside* A's horizon. Any signal from B that reaches A *now* must have been sent from the region that's already inside A's horizon now. See the figure

Problem 3

Density of air:

From Wikipedia, the density of air at 1 atm pressure and 15°C is 1,225 kg m^{-3}

Baryon density:

$$\rho_b(z) = \rho_{b0}(1+z)^3 = \Omega_{b0}\underset{3H_0^2/8\pi G}{\rho_{c0}}(1+z)^3$$

$$\approx 5\cdot 10^{-2} \cdot \frac{\cancel{3}\cdot(2\cdot 10^{-18}\,s^{-1})^2}{8\cdot\cancel{3}\cdot 7\cdot 10^{-11}\,m^3\,kg^{-1}\,s^{-2}}(1+z)^3$$

$$\approx \overbrace{5\cdot 10^{-2}\cdot \frac{4\cdot 10^{-36}\,s^{-2}}{8\cdot 7}}^{=\frac{5}{14}\approx\frac{1}{3}} \cdot 10^{-11}\,m^3\,kg^{-1}\,s^{-2}}(1+z)^3$$

$$\approx 3\cdot 10^{-1}\cdot 10^{-2}\cdot 10^{-25}\,kg\,m^{-3}(1+z)^3$$

$$= 3\cdot 10^{-28}(1+z)^3\,kg\,m^{-3}$$

Let z_{air} be the redshift when ρ_b = density of air. Then

$$3\cdot 10^{-28}(1+z_{air})^3 = 1,225$$

$$(1+z_{air})^3 = \frac{1,225}{3}\cdot 10^{28}$$

$$\approx 3\cdot 10^{27}$$

$$\Rightarrow \quad 1+z_{air} \approx 1,4\cdot 10^9$$

This is in the radiation-dominant era, since $1+z_{air} > 1+z_{eq} \sim 3\cdot 10^3$

But most of the time since then, the Universe has been matter-dominated, and we are extrapolating backwards in time. We are only interested in orders of magnitude, so let's just use the EdS model.

Then, as in problem 1,

$$t(z) \simeq \frac{10^{10}\,\mathrm{yr}}{(1+z)^{3/2}}$$

so

$$t(z_{ai}) \simeq \frac{10^{10} \cdot 3 \cdot 10^{7}\,\mathrm{s}}{\left(3 \cdot 10^{27}\right)^{1/2}}$$

$$\simeq \frac{3 \cdot 10^{17}\,\mathrm{s}}{3^{1/2} \cdot 3 \cdot 10^{13}\,\mathrm{s}} \sim \frac{1}{2} \cdot 10^{4}\,\mathrm{s}$$

$$\sim \underline{\underline{5 \cdot 10^{3}\,\mathrm{s}}}$$

$\Rightarrow$ less than two hours after the Big Bang!

Problem 4

For adiabatic expansion we have

$$T\,dS = 0 = dE + p\,dV \quad \text{(1st law of thermodynamics)}$$

and for an ideal, non-relativistic gas

$$p = \frac{N k_B T}{V}$$

$$E = \frac{3}{2} N k_B T$$

where N is the number of particles and V is the volume.

Using these results in the 1st law:

$$\frac{3}{2} N k_B\, dT + N k_B T \frac{dV}{V} = 0$$

$$\Rightarrow \quad \frac{dT}{T} = -\frac{2}{3} \frac{dV}{V}$$

and integrating from an initial state (T_i, V_i) to a final state (T_f, V_f), we get

$$\int_{T_i}^{T_f} \frac{dT}{T} = -\frac{2}{3} \int_{V_i}^{V_f} \frac{dV}{V}$$

$$\Rightarrow \quad \ln\left(\frac{T_f}{T_i}\right) = -\frac{2}{3} \ln\left(\frac{V_f}{V_i}\right) = \ln\left(\frac{V_f}{V_i}\right)^{-2/3}$$

$$\Rightarrow \quad \frac{T_f}{T_i} = \left(\frac{V_f}{V_i}\right)^{-2/3}$$

Since $V \propto a^3$

$$\frac{T_f}{T_i} = \left(\frac{a_f}{a_i}\right)^{-2}, \quad \text{so } T \propto a^{-2}, \text{ q.e.d.}$$