Problem set 8

Problem 1

a) A quark confined to a region of linear dimension R has by Heisenberg's uncertainty principle a momentum

$$p \sim \frac{\hbar}{R}$$

For a free, relativistic quark the energy is then

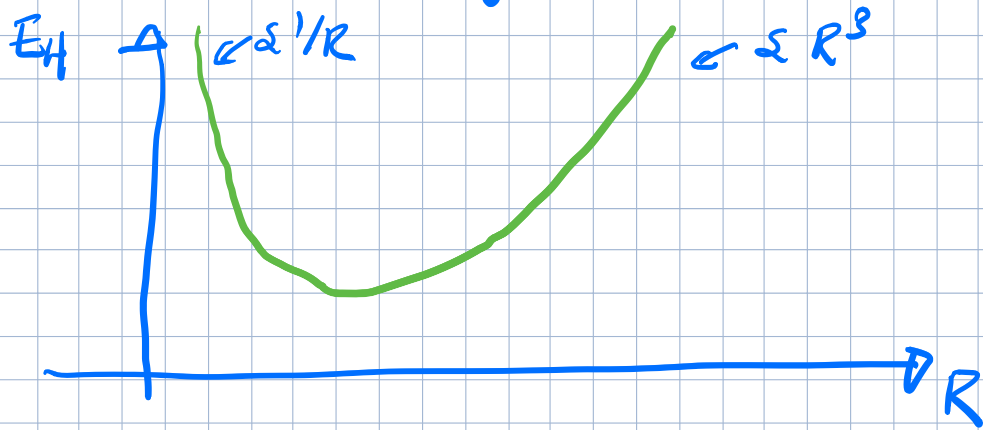
$$E = pc \sim \frac{\hbar c}{R} \propto \underline{\underline{\frac{1}{R}}}$$

b)

$$E_H = \frac{4\pi}{3} B R^3 + \frac{C}{R}$$

For small R, the last term dominates, and $E_H \to \infty$ as $R \to 0$ (assuming C is positive, which is reasonable, since this term represents kinetic energy.) For large R, the first term dominates, and $E_H \to \infty$ as $R \to \infty$.

The graph of E_H should therefore look something like this



E_H should therefore have a minimum, and at this minimum we have

$$\frac{dE_H}{dR} = 0$$

$$\Rightarrow \quad 4\pi B R^2 - \frac{C}{R^2} = 0$$

$$\Rightarrow \quad \underline{\underline{C = 4\pi B R^4}}$$

Inserting this in E_H gives

$$E_{H,min} = \frac{4\pi}{3} B R^3 + 4\pi B R^3$$

$$= \underline{\underline{\frac{16\pi}{3} B R^3}}$$

c) With $E_{H,min} = 10^3$ MeV ($\simeq m_n c^2, m_p c^2$)

and $R = 10^{-15}$ m $= 1$ fm

$$B = \frac{3}{16\pi}\frac{E_{H,min}}{R^3} = \frac{3}{16\pi}\frac{10^3\,\text{MeV}}{1\,\text{fm}^3}$$

$$= \underline{\underline{59{,}7\ \text{MeV fm}^{-3}}}$$

d) We want $B = 200^4\ \text{MeV}^4$ in natural units. Since $\hbar c$ has units MeV · fm, we can translate this ~~to~~ "normal" units by dividing by $(\hbar c)^3$:

$$B = \frac{200^4\ \text{MeV}^4}{(197{,}327)^3\ \text{MeV}^3\,\text{fm}^3}$$

$$\simeq \underline{\underline{208{,}24\ \text{MeV fm}^{-3}}}$$

Using the result from c, this gives the size of the hadron as

$$R = \left(\frac{3}{16\pi}\frac{E_{H,min}}{B}\right)^{1/3}$$

$$= \left(\frac{3}{16\pi}\frac{10^3\ \text{MeV}}{209{,}24\ \text{MeV}\cdot\text{fm}^{-3}}\right)^{1/3}$$

$\simeq \underline{\underline{0,66 \text{ fm}}}$

e) The bag pressure can be found by noting that it acts like a cosmological constant: Its contribution to the energy density is $BV/V = B =$ constant, independent of the volume.

Therefore $P_B = -\rho_B c^2 = \underline{\underline{-B}}$

Alternatively, from the 1st law of thermodynamics, with $dS = 0$

$$T dS = 0 = dE + P dV$$

$$\Rightarrow \quad P = -\frac{dE}{dV},$$

i.e.,

$$P = -\left(\frac{\partial E}{\partial V}\right)_S = -\left(\frac{\partial (BV)}{\partial V}\right)_S = -B.$$

In the following, note that there was a mistake in the original version of the problem: The bag pressure should be included in the quark phase, <u>**not**</u> the hadronic phase.

Early on, in the quark-gluon plasma phase, we have 8 gluons with spin 1, so they are bosons. They are also massless, so each has 2 internal degrees of freedom. The u and d quarks are fermions, each with 2 internal degrees of freedom (spin-$\frac{1}{2}$), 3 degrees of freedom from colour (r, g, b), and each with their own antiparticle. Assuming equilibrium with the photons at temperature T, we therefore get

$$g_{*,QGP} = 8\cdot 2 + \frac{7}{8}\cdot \underbrace{2}_{u,d}\cdot \underbrace{2}_{\bar{u},\bar{d}}\cdot \underbrace{2}_{\text{spin}}\cdot \underbrace{3}_{\text{colour}}$$

$$= 37$$

For a relativistic gas, $P = \frac{1}{3}\rho c^2$, and

$$\rho c^2 = \frac{\pi^2}{30} g_* \frac{(k_B T)^4}{(\hbar c)^3}$$

Adding the (negative) bag pressure then gives

$$P_{QGP} = \frac{1}{3}\frac{\pi^2}{30}\cdot 37 \frac{(k_B T)^4}{(\hbar c)^3} - B$$

$$= \frac{37\pi^2}{90}\frac{(k_B T)^4}{(\hbar c)^3} - B$$

For the hadrons, the bookkeeping is easier. We have 3 of them, and they all have spin 0 (bosons), giving just one internal degree of freedom. Furthermore, π^+ is the antiparticle of π^-, and π^0 is its own antiparticle, so there is no additional contribution from antiparticles. Therefore

$$P_H = \frac{1}{3}\frac{\pi^2}{30}\cdot 3 \cdot \frac{(k_B T)^4}{(\hbar c)^3}$$

$$= \underline{\underline{\frac{\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3}}}$$

f) At the phase transition we have

$$P_H(T_c) = P_{QGP}(T_c)$$

$$\Rightarrow \quad \frac{\pi^2}{30}\frac{(k_B T_c)^4}{(\hbar c)^3} = \frac{37\pi^2}{90}\frac{(k_B T_c)^4}{(\hbar c)^3} - B$$

$$\Rightarrow \quad \frac{\pi^2}{90}(37-3)\frac{(k_B T_c)^4}{(\hbar c)^3} = B$$

$$\Rightarrow \quad k_B T_c = \left(\frac{90}{34\pi^2}\right)^{1/4}\left[(\hbar c)^3 B\right]^{1/4}$$

$$\simeq \underline{\underline{140 \text{ MeV}}} \checkmark \quad (144 \text{ MeV})$$

g) We use the relation between time and temperature derived in the lectures:

$$t = 2{,}423\, g_*^{-1/2} \left(\frac{1\ \mathrm{MeV}}{k_B T}\right)^2 \mathrm{s}$$

in the radiation-dominated era.
As we have seen, before the QCD transition, quarks and gluons contributed 37 relativistic degrees of freedom. In addition, at temperatures $k_B T \geq 140$ MeV, photons are relativistic, and barely of the fermions, electrons, muons, and neutrinos are relativistic (and in equilibrium) up to this point.
This gives

$$g_* = 37 + \underset{\text{photons}}{2} + \frac{7}{8}\left(\underset{e^\pm}{2}\cdot\underset{\text{spin}}{2} + \underset{\mu^\pm}{2}\cdot\underset{\text{spin}}{2} + \overset{\nu_e,\nu_\mu,\nu_\tau}{3}\cdot\underset{\bar{\nu}}{2}\cdot\overset{\text{spin}}{1}\right)$$

$$= 39 + \frac{7}{8}\cdot 14 = 39 + \frac{49}{4} = \underline{\frac{205}{4}}$$

so

$$t = 2{,}423\cdot\left(\frac{205}{4}\right)^{-1/2}\left(\frac{1\ \mathrm{MeV}}{140\ \mathrm{MeV}}\right)^2 \mathrm{s}$$

$$\simeq \underline{\underline{1{,}7\cdot 10^{-5}\ \mathrm{s}}}$$

Problem 2

a) By definition

$$\eta = -\int_t^\infty \frac{c\,dt'}{a(t')}$$

$c/a(t')$ is always positive,

so $\int_{t_1}^{t_2} \frac{c\,dt'}{a(t')} > 0$ for any t_1, t_2

with $t_2 > t_1$. Therefore

$$\int_t^\infty \frac{c\,dt'}{a(t')} > 0 ,$$

so $\eta \leq 0$ and approaches 0

only in the limit $t \to \infty$.

We find $\frac{d\eta}{dt}$ from the fundamental theorem of calculus:

$$\frac{d\eta}{dt} = \frac{d}{dt}\left[-\int_t^\infty \frac{c\,dt'}{a(t')}\right] = \frac{d}{dt}\left[\int_\infty^t \frac{c\,dt'}{a(t')}\right]$$

$$= \frac{c}{a(t)} > 0$$

For $k=0$, the radial coordinate of the event horizon is

$$r_{EH}(t) = \int_t^\infty \frac{c\,dt'}{a(t')}$$

and the proper distance to it at time t is

$$d_{EH}(t) = a(t)\,r_{EH}(t) = a(t)\int_t^\infty \frac{c\,dt'}{a(t')}$$

$$= -a(t)\,\eta \quad . \quad \text{q.e.d.}$$

b) The area of the EH is

$$A_{EH} = 4\pi d_{EH}^2 ,$$

so

$$\frac{dA_{EH}}{dt} = \underbrace{8\pi d_{EH}}_{>0} \frac{d(d_{EH})}{dt} ,$$

and therefore $\frac{dA_{EH}}{dt} \geq 0 \Leftrightarrow \frac{d(d_{EH})}{dt} \geq 0$

Furthermore,

$$\frac{d(d_{EH})}{dt} = \frac{d(d_{EH})}{d\eta} \frac{d\eta}{dt}$$

and from a) we know that $\frac{d\eta}{dt} > 0$,

so $\frac{d(d_{EH})}{dt} \geq 0$ if and only if

$$\frac{d(d_{EH})}{d\eta} \geq 0 , \text{ q.e.d.}$$

c) FI with $k=0$:

$$H^2 = \frac{8\pi G}{3} \rho$$

Take the derivative with respect to t:

$$2H\dot{H} = \frac{8\pi G}{3} \dot{\rho}$$

The continuity equation gives

$$\dot{\rho} = -3H\left(\rho + \frac{P}{c^2}\right) ,$$

so

$$H\dot{H} = \frac{4\pi G}{3}\left[-3H\left(\rho + \frac{P}{c^2}\right)\right]$$

$$= -4\pi G H\left(\rho + \frac{P}{c^2}\right) ,$$

and since $H > 0$,

$$\underline{\dot{H} = -4\pi G\left(\rho + \frac{P}{c^2}\right)} , \text{ q.e.d.}$$

d) $K \equiv \frac{1}{a}\frac{da}{d\eta} \equiv \frac{a'}{a}$

Then

$$K' = \frac{a''a - a' \cdot a'}{a^2} = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2$$

Also,

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right)\underbrace{\frac{d\eta}{dt}}_{=c/a}$$

$$= \frac{c}{a}\frac{d}{d\eta}\left(\frac{1}{a}\frac{da}{d\eta}\underbrace{\frac{d\eta}{dt}}_{=c/a}\right)$$

$$= \frac{c}{a}\frac{d}{d\eta}\left(\frac{c}{a^2}\frac{da}{d\eta}\right)$$

$$= \frac{c}{a}\left(-\frac{2c}{a^3}\frac{da}{d\eta}\right)\frac{da}{d\eta} + \frac{c^2}{a^3}\frac{d^2a}{d\eta^2}$$

$$= -\frac{2c^2}{a^2}\left(\frac{a'}{a}\right)^2 + \frac{c^2}{a^2}\frac{a''}{a}$$

$$= \frac{c^2}{a^2}\left[\underbrace{-\left(\frac{a'}{a}\right)^2}_{=-K^2} + \underbrace{\frac{a''}{a} - \left(\frac{a'}{a}\right)^2}_{=K'}\right]$$

$$= \frac{c^2}{a^2}\left(K' - K^2\right)$$

From c):

$$\frac{c^2}{a^2}\left(K' - K^2\right) = -4\pi G\left(\rho + \frac{P}{c^2}\right)$$

$$\Rightarrow \quad K' - K^2 = -4\pi G\frac{a^2}{c^2}\left(\rho + \frac{P}{c^2}\right)$$

(Mistake in the problem text, but it doesn't matter)

By assumption $\rho + \frac{P}{c^2} \geq 0$, and since $4\pi G\frac{a^2}{c^2} > 0$, this results implies $\underline{K' - K^2 \leq 0}$, q.e.d.

e) Since

$$d_{EM} = -a\eta$$

$$\frac{d}{d\eta}(d_{EM}) = -\frac{da}{d\eta}\eta - a\cdot\frac{d\eta}{d\eta}$$

$$= -a'\eta - a$$

So $\frac{d}{d\eta}(d_{EM}) \geq 0$

$\Leftrightarrow \quad -a'\eta - a \geq 0$

$\Leftrightarrow \quad \underline{a'\eta + a \leq 0}$

So $a'\eta \leq -a$

$\Leftrightarrow \quad \frac{a'}{a}\eta \leq -1$,

and since $\eta \leq 0$,

$$\underline{\underline{K = \frac{a'}{a} \geq -\frac{1}{\eta}}}$$, q.e.d

f) $K' - K^2 \leq 0$

$\Rightarrow \quad \frac{dK}{d\eta} \leq K^2$

$\Rightarrow \quad \frac{dK}{K^2} \leq d\eta$

$\Rightarrow \quad \int_K^{K_0} \frac{dK}{K^2} \leq \int_\eta^0 d\eta = -\eta$

$\Rightarrow \quad \Big|_K^{K_0} -\frac{1}{K} \leq -\eta$

$\Rightarrow \quad -\frac{1}{K_0} + \frac{1}{K} \leq -\eta$

$\Rightarrow \quad \underline{\frac{1}{K_0} - \frac{1}{K} \geq \eta}$, q.e.d.

9) $\underbrace{K \geq -\frac{1}{\eta} \text{ follows from f) if we can}}_{\text{Missing text in the problem}}$

show that $K > 0$ and $K_0 = \infty$

Since $K = \frac{a'}{a} > 0$ follows from the fact that $\dot{a} > 0$ and $\frac{d\eta}{dt} > 0$, we only have to show that $K_0 = \infty$

Recall that

$$\eta = -\int_t^\infty \frac{c\,dt'}{a(t')}$$

and look at $\underset{\text{as } t \to \infty}{a \to \infty}$, $dt' = \frac{da}{\dot{a}}$

$$\int_t^\infty \frac{c\,dt'}{a(t')} = \int_a^\infty \frac{c}{a}\frac{da}{\dot{a}} = \int_a^\infty \frac{c}{a}\frac{da}{\frac{da}{d\eta}\frac{d\eta}{dt}}$$

$$= \int_a^\infty \frac{c}{a}\frac{da}{\frac{c}{a}a'} = \int_a^\infty \frac{da}{a'}$$

$$= \int_a^\infty \frac{da}{aA} < \infty,$$

where the last inequality follows from the assumption that the event horizon exists.

Assume $\frac{1}{|K|}$ is bounded from below, so that

$\frac{1}{|H|} > \varepsilon > 0$ for $a \to \infty$, $t \to \infty$ $(\eta \to 0)$

Then $\int_a^\infty \frac{da}{aA} > \varepsilon \int_a^\infty \frac{da}{a} = \varepsilon \Big|_a^\infty \ln a = \infty$,

contradicting the existence of the event horizon.

So we must have $\varepsilon = 0$ and $A_0 = \infty$, and $K \geq -\frac{1}{\eta}$ follows, which proves that the area of the EH is non-decreasing.

QED!