Problem set 9

Problem 1

There is only one ice cream cone worth talking about here in Norway: "Krone-is" with strawberries. All other types, including other types of "Krone-is" are abominations.

The relevant data are quite easy to find with a simple Google-search.

One "Krone-is" with strawberries gives you 1068 kJ per 100 g.

It weighs 83 g, and has a volume of 145 mL or $1{,}45 \cdot 10^{-4}\ m^3$

The relevant energy density therefore becomes

$$\rho_{ice} c^2 = \frac{0{,}83 \cdot 1068 \cdot 10^3\ J}{1{,}45 \cdot 10^{-4}\ m^3}$$

$$= 6{,}11 \cdot 10^9\ J\,m^{-3}$$

$$= 3{,}81 \cdot 10^{18}\ GeV\,m^{-3}$$

using standard conversion between J and eV.

The radiation energy density is

$$\rho c^2 = \frac{\pi^2}{30} g_* \frac{(k_B T)^4}{(\hbar c)^3},$$

and $T = T_0 (1+z)$, $T_0 = 2{,}725\ K$

Inserting this, using

$$k_B = 8,617 \cdot 10^{-5}\, eV/K$$

and $\hbar c = 197,327 \cdot 10^{-9}\, eV \cdot m$

gives

$$\rho c^2 = 1,302 \cdot 10^5\, g_* (1+z)^4\, eV\, m^{-3}$$

Equating this to $\rho_{ic} c^2$ gives $1\,GeV = 10^9\,eV$

$$1,302 \cdot 10^5\, g_* (1+z)^4\, eV\, m^{-3} = 3,81 \cdot 10^{27}\, eV\, m^{-3}$$

$$\Rightarrow \quad g_*^{1/4} (1+z) = \left(\frac{3,81 \cdot 10^{27}}{1,302 \cdot 10^5} \right)^{1/4} \simeq 4,14 \cdot 10^5$$

The corresponding thermal energy is

$$k_B T = k_B T_0 (1+z)$$

$$= k_B T_0 \cdot \frac{4,14 \cdot 10^5}{g_*^{1/4}}$$

$$= 8,617 \cdot 10^{-5}\, eV/K \cdot 2,725\, K \cdot \frac{4,14 \cdot 10^5}{g_*^{1/4}}$$

$$\simeq 97\, eV \cdot g_*^{-1/4}$$

Now, g_* varies from $\simeq 100$ to ~ 1,
so $g_*^{1/4}$ is in the range 1-3,
so we are at temperatures where only photons and neutrinos are relativistic, and after ν decoupling and e^+e^--annihilation.

So $T_\nu / T = \left(\frac{4}{11}\right)^{1/3}$ and

$$g_* = 2 + \frac{7}{8} \cdot 3 \cdot 2 \cdot 1 \cdot \left(\frac{4}{11}\right)^{4/3} \simeq 3,36$$

Therefore

$$1+z = \frac{4,14\cdot 10^5}{(3,36)^{1/4}} \simeq \underline{\underline{3,06\cdot 10^5}}$$

We can also find the corresponding time:

$$k_B T = 97\,\text{eV}\cdot (3,36)^{-1/4}$$

$$\simeq 72\,\text{eV} = 72\cdot 10^{-6}\,\text{MeV},$$

and using the $t-T$ relationship

$$t = 2,423\cdot (3,36)^{-1/2}\left(\frac{1\,\text{MeV}}{72\cdot 10^{-6}\,\text{MeV}}\right)^2 s$$

$$\simeq 2,55\cdot 10^8\, s \simeq 8\ \text{yrs},$$

so about 8 years after the Big Bang.

Alternative, "Desert island" solution

What to do when you have lost your connection and your calculator has been stolen:

- Recommended daily intake ~~of~~ for a male adult is ~ 2500 kcal.
- How many ice creams is that? They are pretty calory-dense, but it must surely be more than 1. I would guess more than 5 as well, so let's say 10. That is 250 kcal per ice cream

  Now I remember that 1 cal = 4 J, so one ice cream is 1000 kJ = 10^6 J
- What is the volume of an ice cream cone? The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base, and h is the height. I would guess that for an ice cream cone $h \sim 10$ cm and $r \sim 3$ cm, so

$$V = \frac{1}{3} \underbrace{\pi}_{\simeq 3} \cdot 3^2 \cdot 10 \text{ cm}^3$$

$$\simeq 90 \text{ cm}^3 \sim 10^2 \text{ cm}^3 = 10^2 \cdot (10^{-2} \text{ m})^3$$

$$\simeq 10^{-4} \text{ m}^3$$

- The energy density is therefore

$$\rho_{ice} c^2 \sim \frac{10^6 J}{10^{-4} m^3} = 10^{10} J m^{-3}$$

- The relativistic energy density is

$$\rho c^2 = \frac{\overset{\approx 10}{\pi^3}}{30} g_* \frac{(k_B T_0)^4}{(\hbar c)^3} (1+z)^4$$

$$\sim \frac{10}{30} g_* \frac{(10^{-23} J/K \cdot 3K)^4}{(10^{-34} J \cdot s \cdot 3 \cdot 10^8 m s^{-1})^3} (1+z)^4$$

$$\sim \frac{1}{3} \cdot \frac{3^4 \cdot 10^{-92} J^4}{3^3 \quad 10^{-78} J^3 m^3} g_* (1+z)^4$$

$$\sim 10^{-14} g_* (1+z)^4 J m^{-3}$$

- Equating the two:

$$10^{-14} g_* (1+z)^4 = 10^{10} J m^{-3}$$

$$\Rightarrow \quad g_*^{1/4} (1+z) = (10^{24})^{1/4} = 10^6$$

- To the accurace we are working at here, $g_*^{1/4} \sim 1$, so

$$\underline{\underline{1+z \sim 10^6}}.$$

Off by a factor of 2 or so, but not bad!

The corresponding time:

$$k_B T = k_B T_0 (1+z) \sim 10^{-4} eV/K \cdot 3K \cdot 10^6$$

$$\sim 3 \cdot 10^2 eV = 3 \cdot 10^{-4} MeV$$

$$t \sim 2 \cdot \underset{\approx 1}{g_*^{-1/2}} \left(\frac{1 MeV}{3 \cdot 10^{-4} MeV}\right)^2 s$$

$$\sim 2 \cdot \frac{1}{10} \cdot 10^8 s = 2 \cdot 10^7 s \sim 1 \text{ year}$$

Again slightly off, but not bad.

Problem 2

a) Neutrinos are fermions, so we have (see the lecture notes):

$$\rho c^2 = \frac{7}{8} \frac{\pi^2}{30} g_\nu \frac{(k_B T_\nu)^4}{(\hbar c)^3}$$

$$n = \frac{3}{4} \frac{\zeta(3)}{\pi^2} g_\nu \frac{(k_B T_\nu)^3}{(\hbar c)^3}$$

Using the definition

$$\langle p \rangle = \frac{1}{c} \frac{\rho c^2}{n} = \frac{1}{c} \frac{4\pi^2 \cancel{(\hbar c)^3}}{3\zeta(3) \cancel{g_\nu} \cancel{(k_B T_\nu)^3}} \frac{7}{8} \frac{\pi^2}{30} \cancel{g_\nu} \frac{(k_B T_\nu)^{\cancel{4}}}{\cancel{(\hbar c)^3}}$$

$$= \frac{7\pi^4}{180 \underbrace{\zeta(3)}_{\simeq 1,202}} \frac{k_B T_\nu}{c} \simeq \underline{\underline{3,15 \frac{k_B T_\nu}{c}}}$$

b) Transition from relativistic to non-relativistic:

$$\langle p \rangle = mc$$

$$\Rightarrow \quad 3,15 \frac{k_B T_\nu}{c} = mc^{\cancel{2}}$$

This will certainly happen after neutrino decoupling and e^+e^- annihilation, so

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T$$

and $T = T_0 (1+z)$, where

$$T_0 = 2,725 \text{ K}$$

Therefore,

$$3,15 \; k_B T_0 \left(\frac{4}{11}\right)^{1/3} (1 + z_{nr}) = mc^2$$

$$\Rightarrow \quad 1 + z_{nr} = \frac{mc^2}{3{,}15 k_B T_0 \left(\frac{4}{11}\right)^{1/3}}$$

$$= \frac{mc^2}{3{,}15 \cdot 8{,}617 \cdot 10^{-5}\,\mathrm{eV/K} \cdot 2{,}725\,\mathrm{K} \cdot \left(\frac{4}{11}\right)^{1/3}}$$

$$= \frac{mc^2}{5{,}4 \cdot 10^{-4}\,\mathrm{eV}} = \underline{\underline{\frac{mc^2}{0{,}54\,\mathrm{meV}}}}$$

c) Since the neutrinos are non-relativistic for $z < z_{nr}$,

$$\langle p \rangle = m v,$$

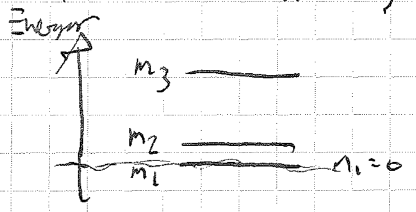
so

$$\frac{v}{c} = \frac{\langle p \rangle}{mc} = \frac{3{,}15 k_B T_\nu}{mc^2}$$

$$= \frac{3{,}15\, k_B T_0 \left(\frac{4}{11}\right)^{1/3} (1+z)}{mc^2}$$

$$= \frac{0{,}54\,\mathrm{meV}}{mc^2} (1+z) \quad (z < z_{nr})$$

d) ν oscillations



$$m_2^2 - m_1^2 = 75\,\mathrm{meV}^2$$

$$|m_3^2 - m_1^2| = 2500\,\mathrm{meV}^2$$

$m_1 = 0 \Rightarrow v_1 = c$, always, also today

$$m_3 > m_2 > m_1 = 0$$

$$\Rightarrow \quad m_3^2 = 2500\,\mathrm{meV}^2$$

$$m_2^2 = 75\,\mathrm{meV}^2$$

$z = 0$
$\Rightarrow$

$$\frac{v_2}{c} = \frac{0{,}54\,\mathrm{meV}}{\sqrt{75}\,\mathrm{meV}} \approx \underline{\underline{6{,}2 \cdot 10^{-2}}}$$

$$\frac{v_3}{c} = \frac{0{,}54\,\mathrm{meV}}{\sqrt{2500}\,\mathrm{meV}} \approx \underline{\underline{1{,}1 \cdot 10^{-2}}}$$

e) If m_{ν_e} is close to 1eV, so would $m_{1,2,3}$ have to be, so their masses would be much larger than their mass differences. If $m_i c^2 \sim 1$eV, then

$$\frac{v}{c}(z=0) \sim \frac{0{,}54\,\text{meV}}{10^3\,\text{meV}} \sim \underline{\underline{5{,}4\cdot 10^{-4}}}$$

Problem Set 9

Problem 1

See solution supplied with the old exam problems.

Problem 2

a) Since all the neutrinos have the same momentum p, they also have the same energy,
$\epsilon = \sqrt{p^2+m^2}$ $(c=1)$. The energy density is therefore

$$\rho = n\epsilon = n\sqrt{p^2+m^2},$$

where n is the number density of the neutrinos.

b) The equation of state parameter is given by

$$w = \frac{P}{\rho} = \frac{\frac{1}{3}\frac{np}{\sqrt{1+m^2/p^2}}}{n\sqrt{p^2+m^2}}$$

$$= \frac{1}{3}\frac{p}{\sqrt{p^2+m^2}\sqrt{1+\frac{m^2}{p^2}}}$$

$$w = \frac{1}{3}\frac{p}{p\sqrt{1+\frac{m^2}{p^2}}\sqrt{1+\frac{m^2}{p^2}}}$$

$$= \frac{1}{3}\frac{1}{1+\frac{m^2}{p^2}}$$

⌈ In general, for a distribution function $f(p)$:

$$n = \frac{g}{(2\pi\hbar)^3}\int f(p')\,d^3p'$$

$$P = \frac{g}{(2\pi\hbar)^3}\int \frac{(p'c)^2}{3\sqrt{p'^2c^2+m^2c^4}} f(p')\,d^3p'$$

Take $\hbar = c = 1$. All neutrinos have momentum $p \Rightarrow f(p') = A\,\delta(p'-p)$

$$n = \frac{g}{(2\pi)^3}\int 4\pi p'^2\,dp'\,A\,\delta(p-p')$$

$$= \frac{gA}{2\pi^2}p^2$$

$$\Rightarrow \quad A = \frac{2\pi^2 n}{g p^2}$$

$$\Rightarrow \quad P = \frac{g}{8\pi^3}\int 4\pi p'^2\,dp'\frac{p'^2}{3\sqrt{p'^2+m^2}}\frac{2\pi^2 n}{g p^2}\delta(p'-p)$$

$$= \frac{\cancel{g}}{\cancel{8\pi^3}}\cancel{4\pi p^2}\frac{p^2}{3\sqrt{p^2+m^2}}\frac{\cancel{2\pi^2}n}{\cancel{g p^2}}$$

$$= n\frac{p^2}{3\sqrt{p^2+m^2}} = n\frac{p^2}{3p\sqrt{1+m^2/p^2}}$$

$$= \frac{1}{3}\frac{np}{\sqrt{1+m^2/p^2}}$$ ⌋

c) With $p = p_0/a$, we find

$$w(a) = \frac{1}{3}\frac{1}{1+\frac{m^2}{(p_0/a)^2}} = \frac{1}{3}\frac{1}{1+\frac{a^2}{(p_0/m)^2}}$$

$$= \frac{1}{3}\frac{1}{1+a^2/f_0^2}, \quad f_0 \equiv \frac{p_0}{m}$$

d) The continuity equation (with $c=1$) is

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

$$\Rightarrow \frac{d\rho}{dt} + 3\cdot\frac{1}{a}\frac{da}{dt}(\rho + w\rho) = 0$$

$$\Rightarrow \frac{d\rho}{\rho} = -3\frac{da}{a}\left[1 + w(a)\right]$$

$$\Rightarrow \int_{\rho_0}^{\rho}\frac{d\rho'}{\rho'} = -3\int_1^a \frac{da'}{a'}\left[1 + w(a')\right]$$

$$\Rightarrow \ln\left(\frac{\rho}{\rho_0}\right) = -3\int_1^a\frac{da'}{a'} - 3\int_1^a\frac{da'}{a'}w(a')$$

$$= -3\ln a - 3I_w(a)$$

$$\Rightarrow \rho = \rho_0\, a^{-3}\exp\left[-3I_w(a)\right],$$

where $I_w(a) = \int_1^a \frac{da'}{a'}w(a')$

e) Our task is here to find $I_w(a)$:

$$I_w(a) = \int_1^a \frac{da'}{a'} \frac{1}{3} \frac{1}{1+\frac{a'^2}{f_0^2}}$$

$$= \frac{1}{3} f_0^2 \int_1^a \frac{da'}{a'} \frac{1}{f_0^2 + a'^2}$$

$$= \frac{1}{3} f_0^2 \int_1^a da' \frac{1}{f_0^2} \left(\frac{1}{a'} - \frac{a'}{f_0^2 + a'^2} \right)$$

using the hint

$$= \frac{1}{3} \left[\ln a - \frac{1}{2} \int_{f_0^2+1}^{f_0^2+a^2} \frac{du}{u} \right]$$

$$\boxed{\begin{aligned} u &= f_0^2 + a'^2 \\ du &= 2a' da' \end{aligned}}$$

$$= \frac{1}{3} \left[\ln a - \frac{1}{2} \ln \left(\frac{f_0^2 + a^2}{f_0^2 + 1} \right) \right]$$

$$= \frac{1}{3} \ln \left[a \sqrt{\frac{f_0^2+1}{f_0^2+a^2}} \right]$$

$$= \frac{1}{3} \ln \left[\sqrt{\frac{a^2(1+f_0^2)}{a^2+f_0^2}} \right]$$

So

$$\rho = \rho_0 a^{-3} \exp\left[-3 I_w(a) \right]$$

$$= \rho_0 a^{-3} \exp\left[-\ln \sqrt{\frac{a^2(1+f_0^2)}{a^2+f_0^2}} \right]$$

$$= \rho_0 a^{-3} \sqrt{\frac{a^2+f_0^2}{a^2(1+f_0^2)}} = \rho_0 \left(\frac{1+f_0^2 a^{-2}}{1+f_0^2} \right)^{1/2} a^{-3}$$

$$= \rho_0 \, g^{1/2}(a) \, a^{-3},$$

with $$g(a) = \frac{1 + f_0^2 a^{-2}}{1 + f_0^2}$$

f) FI with $k = 0$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho$$

$$\Rightarrow \quad \left(\frac{1}{a}\frac{da}{dt}\right)^2 = H_0^2 \underbrace{\frac{8\pi G}{3 H_0^2} \rho_0}_{= \frac{\rho_0}{\rho_{c0}} = 1, \text{ since } k=0} g^{1/2}(a) \, a^{-3}$$

$$= H_0^2 \, g^{1/2}(a) \, a^{-3}$$

$$\Rightarrow \quad \frac{1}{a}\frac{da}{dt} = H_0 \, g^{1/4}(a) \, a^{-3/2}$$

$$\Rightarrow \quad \frac{a^{1/2} \, da}{g^{1/4}(a)} = H_0 \, dt$$

$$\Rightarrow \quad \int_0^t H_0 \, dt' = H_0 t = \int_0^a \frac{(a')^{1/2} \, da'}{g^{1/4}(a')}$$

Next we need to carry out the integral:

$$\int_0^a \frac{(a')^{1/2}\,da'}{g^{1/4}(a')} = \int_0^a \left(\frac{1+f_0^2}{1+f_0^2(a')^{-2}}\right)^{1/4} (a')^{1/2}\,da'$$

$$= \int_0^a \left[\frac{(1+f_0^2)(a')^2}{(a')^{-2}\left((a')^2+f_0^2\right)}\right]^{1/4} da'$$

$$= \int_0^a \left[\frac{(1+f_0^2)(a')^4}{f_0^2+(a')^2}\right]^{1/4} da'$$

$$= (1+f_0^2)^{1/4} \int_0^a \frac{a'\,da'}{\left[f_0^2+(a')^2\right]^{1/4}}$$

$$= (1+f_0^2)^{1/4}\,\frac{1}{2}\int_{f_0^2}^{f_0^2+a^2} u^{-\frac{1}{4}}\,du$$

$u = f_0^2+(a')^2$

$du = 2a'\,da'$

$$= \frac{1}{2}(1+f_0^2)^{1/4} \Big|_{f_0^2}^{f_0^2+a^2} \frac{4}{3} u^{3/4}$$

$$= \frac{2}{3}(1+f_0^2)^{1/4}\left[(f_0^2+a^2)^{3/4} - f_0^{3/2}\right]$$

So

$$H_0 t = \frac{2}{3}(1+f_0^2)^{1/4}\left[(f_0^2+a^2)^{3/4} - f_0^{3/2}\right]$$

g) Go back to the EoS parameter

$$w(a) = \frac{1}{3}\,\frac{1}{1+\frac{a^2}{f_0^2}}$$

For $f_0 \to \infty$, we get

$w(a) \to \frac{1}{3}$, the EoS for relativistic particles

Next, rewrite $w^2(a)$:

$$w^2(a) = \frac{1}{3}\frac{f_0^2}{f_0^2+a^2},$$

and we see that $w(a) \to 0$ when $f_0 \to 0$, so in this limit the neutrinos behave like non-relativistic particles.

We found :

$$H_0 t = \frac{2}{3}(1+f_0^2)^{1/4}\left[(f_0^2+a^2)^{3/4} - f_0^{3/2}\right]$$

When $f_0 \to 0$:

$$H_0 t = \frac{2}{3} a^{3/2}$$

$a(t=t_0) = a_0 = 1$ gives

$$H_0 t_0 = \frac{2}{3} \Rightarrow t_0 = \frac{2}{3H_0}, \text{ and}$$

$$\frac{2}{3}\frac{t}{t_0} = \frac{2}{3} a^{3/2} \Rightarrow a = \left(\frac{t}{t_0}\right)^{2/3}$$

$\Rightarrow$ We recover the EdS solution, as we should in the limit of non-relativistic particles and $k=0$.

In the limit $f_0 \to \infty$, we can use an expansion of the solution for $f_0 >> 1$:

$$H_0 t \simeq \frac{2}{3} f_0^{1/2} \left[f_0^{3/2} \left(1 + \frac{a^2}{f_0^2}\right)^{3/4} - f_0^{3/2} \right]$$

$$\simeq \frac{2}{3} f_0^{1/2} \left[f_0^{3/2} \left(1 + \frac{3}{4}\frac{a^2}{f_0^2}\right) - f_0^{3/2} \right]$$

$$= \frac{1}{2} a^2$$

Again, $a(t_0) = a_0 = 1$, so

$$H_0 t_0 = \frac{1}{2} \Rightarrow t_0 = \frac{1}{2H_0}, \text{ and}$$

$$\frac{1}{2}\frac{t}{t_0} = \frac{1}{2} a^2$$

$$\Rightarrow a = \left(\frac{t}{t_0}\right)^{1/2},$$

and as we should expect, we recover the solution for a radiation - dominated universe with $k=0$.

h) In the general case,

$$H_0 t = \frac{2}{3}\left(1 + f_0^2\right)^{1/4} \left[\left(f_0^2 + a^2\right)^{3/4} - f_0^{3/2} \right]$$

$$\Rightarrow \underbrace{\frac{3 H_0}{2(1+f_0^2)^{1/4}}}_{\equiv \frac{1}{t_{fid}}} t = \left(f_0^2 + a^2\right)^{3/4} - f_0^{3/2}$$

$$\Rightarrow \left(f_0^2 + a^2\right)^{3/4} = \frac{t}{t_{fid}} + f_0^{3/2}$$

$$\Rightarrow f_0^2 + a^2 = \left(\frac{t}{t_{fid}} + f_0^{3/2}\right)^{4/3}$$

$$\Rightarrow \quad a(t) = \left[\left(\frac{t}{t_{fid}} + f_0^{3/2} \right)^{4/3} - f_0^2 \right]^{1/2}$$

i) We find the age by requiring $a(t_0) = a_0 = 1$:

$$1 = \left[\left(\frac{t_0}{t_{fid}} + f_0^{3/2} \right)^{4/3} - f_0^2 \right]^{1/2}$$

$$\Rightarrow \quad \frac{t_0}{t_{fid}} + f_0^{3/2} = \left(1 + f_0^2\right)^{3/4}$$

$$\Rightarrow \quad t_0 = \left[\left(1 + f_0^2\right)^{3/4} - f_0^{3/2} \right] t_{fid}$$

$$= \frac{2}{3H_0} \left(1 + f_0^2\right)^{1/4} \left[\left(1 + f_0^2\right)^{3/4} - f_0^{3/2} \right]$$

Even simpler : Take $t = t_0$, $a = a_0 = 1$ in the result from f).