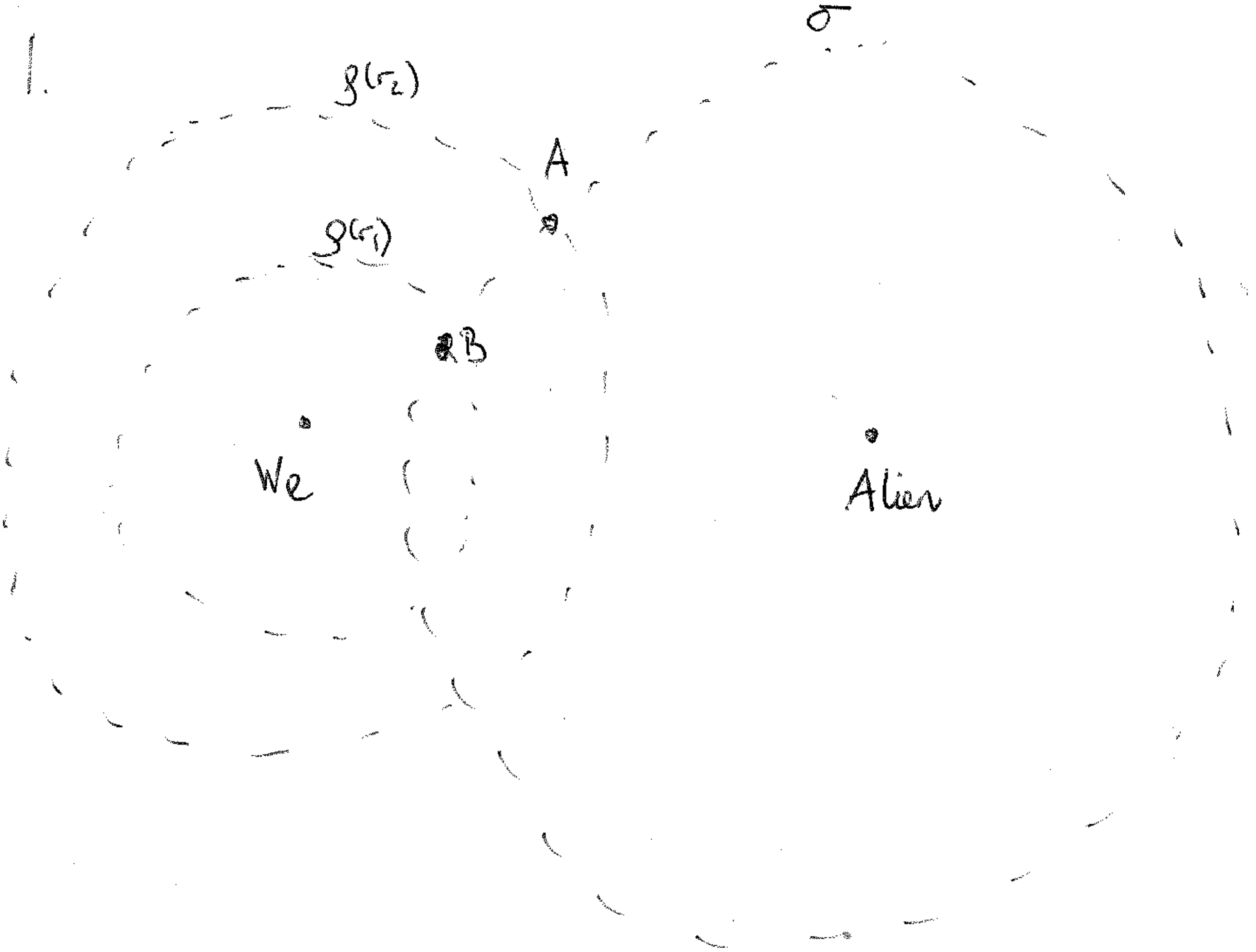
1.



Isotropy: Density $= \rho(r_1)$ on the whole shell of radius r_1, and ~~[illegible]~~ $= \rho(r_2)$ on the whole of shell of radius r_2

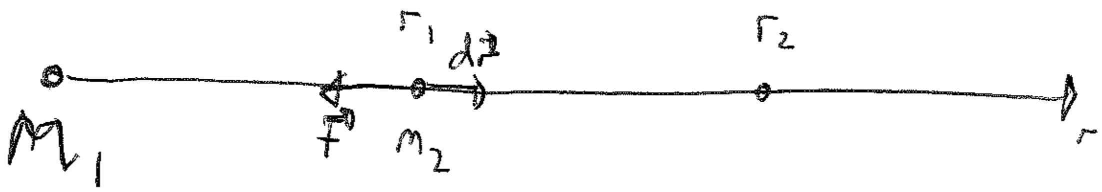
Copernican principle $\Rightarrow$ Alien also sees isotropy, density equal to ρ' across whole shell σ.

The density must be single-valued, so:

$$\left.\begin{array}{lll} \text{At } A & : & \rho(r_2) = \rho' \\ \text{at } B & : & \rho(r_1) = \rho' \end{array}\right\} \Rightarrow \rho(r_1) = \rho(r_2)$$

r_1 & r_2 arbitrary $\Rightarrow$ universe is homogeneous from our position. Copernican principle $\Rightarrow$ Homogeneous from any position $\Rightarrow$ Homogeneous.

2.



Work done by gravity in moving m_2 from r_1 to r_2:

$$W = \int_1^2 \vec{F} \cdot d\vec{r} = -\int_{r_1}^{r_2} \frac{G m_1 m_2}{r^2} dr$$

$$= -G m_1 m_2 \Big|_{r_1}^{r_2} -\frac{1}{r} = G m_1 m_2 \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

Potential energy = $-W$

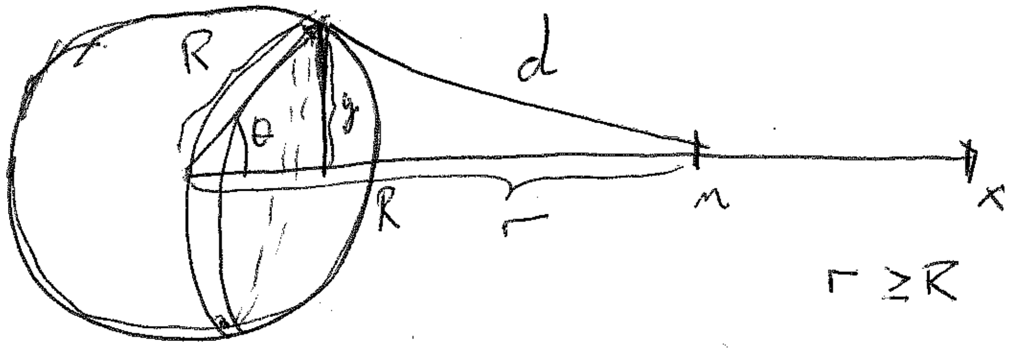
$$V = -W = -G m_1 m_2 \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

Convention: $r_2 = r$, $r_1 = \infty$ (zero pot. at infinity)

$$\Rightarrow \quad \underline{\underline{V = -\frac{G m_1 m_2}{r}}}$$

3.

$$M = 4\pi\sigma R^2$$



$r \geq R$

$$dM = \sigma \cdot 2\pi y \, R\, d\theta = 2\pi\sigma R^2 \sin\theta \, d\theta$$

$$dU = -\frac{G m \, dM}{d} = -2\pi G\sigma m R^2 \frac{\sin\theta\, d\theta}{\sqrt{R^2+r^2-2rR\cos\theta}}$$

$$U = -2\pi G\sigma m R^2 \int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{R^2+r^2-2rR\cos\theta}}$$

$$z = R^2 + r^2 - 2rR\cos\theta$$

$$dz = 2rR\sin\theta\, d\theta$$

$$\sin\theta\, d\theta = \frac{dz}{2rR}$$

$$\theta = 0 \Rightarrow z = R^2 + r^2 - 2rR = (R-r)^2 = (r-R)^2$$

$$\theta = \pi \Rightarrow z = R^2 + r^2 + 2rR = (r+R)^2$$

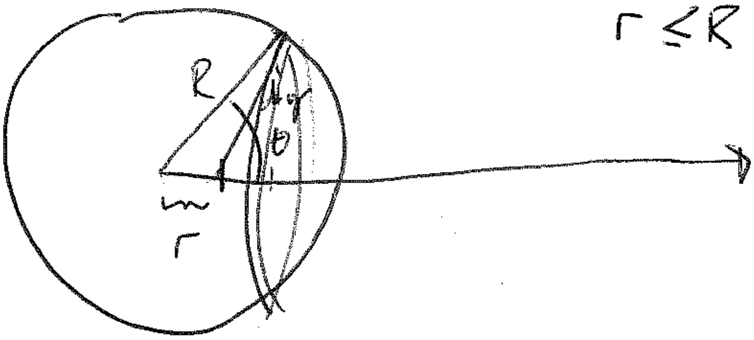
$$U = -\frac{2\pi G\sigma m R^2}{2rR} \int_{(r-R)^2}^{(r+R)^2} \frac{dz}{\sqrt{z}}$$

$$= -\frac{2\pi G\sigma m R^2}{rR} \Big|_{(r-R)^2}^{(r+R)^2} \sqrt{z}$$

$$= -\frac{2\pi G\sigma m R^2}{rR} \left[\cancel{r} + R - \cancel{r} + R \right]$$

$$= -\frac{4\pi\sigma R^2 G m}{r} = -\frac{GMm}{r} \rightarrow \vec{F} = -\nabla V = -\frac{GMm}{r^2}\vec{e}_r$$

4.



$r \leq R$

$$dM = 2\pi\sigma R^2 \sin\theta d\theta$$

$$dU = -\frac{GmdM}{d} = -2\pi G\sigma m R^2 \frac{\sin\theta d\theta}{\sqrt{R^2+r^2-2rR\cos\theta}}$$

$$U = -2\pi G\sigma m R^2 \int_0^{\pi} \frac{\sin\theta d\theta}{\sqrt{R^2+r^2-2rR\cos\theta}}$$

$$z = R^2+r^2-2rR\cos\theta$$

$$dz = 2rR\sin\theta$$

$$\theta = 0 \Rightarrow \quad z = (r-R)^2$$

$$\theta = \pi \Rightarrow \quad t = (r+R)^2$$

$$U = -\frac{2\pi G\sigma m R^2}{2rR} \int_{(r-R)^2}^{(r+R)^2} \frac{dz}{\sqrt{z}}$$

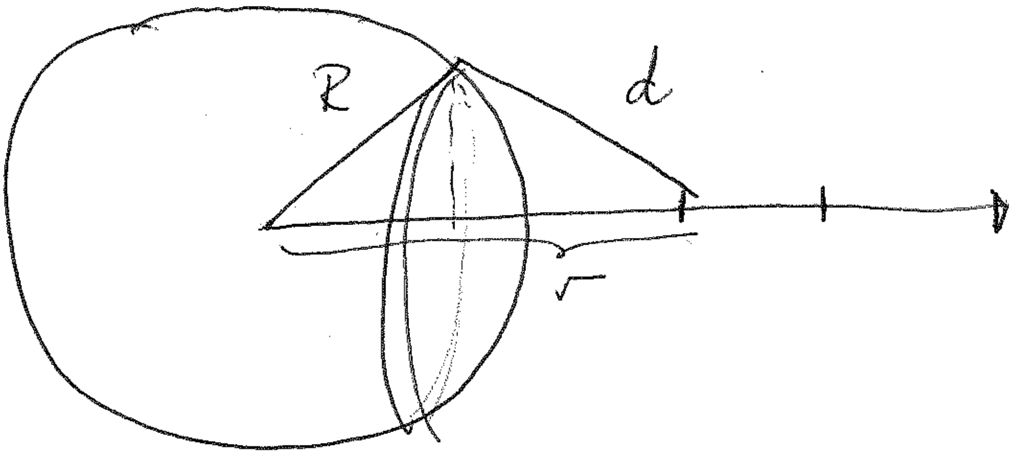
$$= -\frac{2\pi G\sigma m R^2}{rR} \left[\sqrt{(r+R)^2} - \sqrt{(r-R)^2}\right]$$

$$= -\frac{1}{2}\frac{GMm}{rR}\left[r+R-|r-R|\right]$$

$$= -\frac{1}{2}\frac{GMm}{rR}(r+R+r-R) = -\frac{GMm}{R} = \text{konstant}$$

$$\Rightarrow: \vec{F} = -\nabla U = \underline{\underline{0}}$$

5. $\cancel{U = \frac{1}{2}Kr^2}$ $U = \frac{1}{2} K M m r^2$



$$dM = 2\pi \sigma R^2 \sin\theta d\theta = \frac{1}{2} M \sin\theta d\theta$$

$$dU = \frac{1}{2} K m \, dM \, d^2$$

$$U = \int_0^\pi dU = \frac{1}{2} K m \frac{1}{2} M \int_0^\pi (R^2 + r^2 - 2rR\cos\theta) \sin\theta d\theta$$

$$= \frac{1}{4} K M m \frac{1}{2rR} \int_{(r-R)^2}^{(r+R)^2} z \, dz = \frac{KMm}{8rR} \Big|_{(r-R)^2}^{(r+R)^2} \frac{1}{2} z^2$$

$z = R^2 + r^2 - 2rR\cos t$

$dz = 2rR \sin\theta d\theta$

$$= \frac{KMm}{16rR} \left[(r+R)^4 - (r-R)^4 \right]$$

$$(r+R)^4 - (r-R)^4 = \left[(r+R)^2 - (r-R)^2\right]\left[(r+R)^2 + (r-R)^2\right]$$

$$= (\cancel{r^2} + 2rR + \cancel{R^2} - \cancel{r^2} + 2rR - \cancel{R^2})(r^2 + \cancel{2rR} + R^2 + r^2 - \cancel{2rR} + R^2)$$

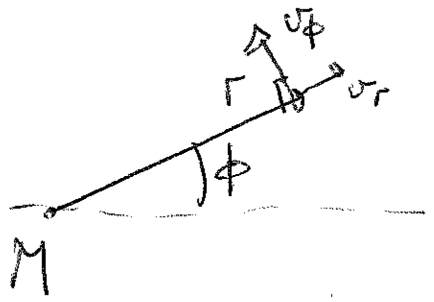
$$= 4rR \cdot 2(r^2+R^2) = 8rR(r^2+R^2)$$

$$U = \frac{KMm}{16rR} \cdot 8rR(r^2+R^2)$$

$$= \frac{1}{2} KMm(r^2+R^2)$$

$$F = -\frac{\partial U}{\partial r} = -KMmr$$ → Som om massen var samlet i sentrum

6.



a) $L = m r v_\phi = m r (r\dot\phi) = \underline{m r^2 \dot\phi}$

b) $E = \frac{1}{2} m v^2 - \frac{GMm}{r}$

$= \frac{1}{2} m (v_r^2 + v_\phi^2) - \frac{GMm}{r}$

$\Rightarrow \frac{E}{m} = \frac{1}{2}\dot r^2 + \frac{1}{2} r^2 \dot\phi^2 - \frac{GM}{r}$

$= \frac{1}{2}\dot r^2 + \frac{1}{2} r^2 \frac{L^2}{m^2 r^4} - \frac{GM}{r}$

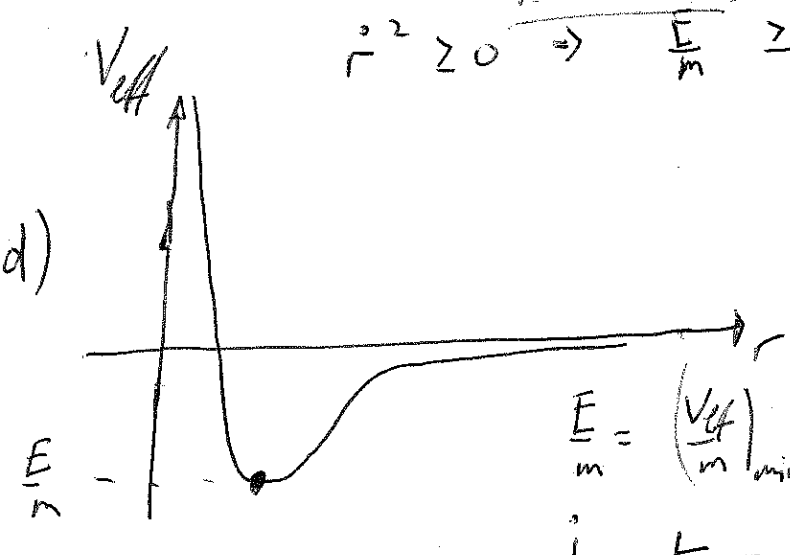
$= \underline{\frac{1}{2}\dot r^2 - \frac{GM}{r} + \frac{(L/m)^2}{2r^2}}$

c) b) $\Rightarrow$ $\frac{1}{2}\dot r^2 = \frac{E}{m} - \underline{\left[-\frac{GM}{r} + \frac{(L/m)^2}{2r^2}\right]}$

$= \underline{\frac{E}{m} - \frac{V_{eff}}{m}}$

$\dot r^2 \geq 0 \Rightarrow \frac{E}{m} \geq \frac{V_{eff}}{m}$

d)



$\frac{E}{m} = \left(\frac{V_{eff}}{m}\right)_{min} \Rightarrow$ only one value of r possible

ϕ can be anything

$\dot\phi = \frac{L}{mr^2} = \text{const} \Rightarrow$ Uniform circular motion.

e) Near minimum:

$$\frac{V_{eff}}{m}(r-r_0) \simeq \frac{V_{eff}(r_0)}{m} + \frac{V'_{eff}(r_0)}{m}(r-r_0) + \frac{1}{2}\frac{V''_{eff}(r_0)}{m}(r-r_0)^2$$

$$\underset{0}{\parallel}$$

so: like harm. osc. $\frac{V(x)}{m} = \frac{1}{2}\omega^2 x^2$

$$\Rightarrow \omega^2 = \frac{d^2}{dr^2}\left(\frac{V_{eff}}{m}\right)\Big|_{r=r_0}$$, r_0 = pos. of min

f) r_0 solution of

$$\frac{d}{dr}\frac{V_{eff}}{m} = 0,$$

so

$$\frac{d}{dr}\left[-\frac{GM}{r} + \frac{(L/m)^2}{2r^2}\right]_{r=r_0} = 0$$

$$+\frac{GM}{r_0^2} - \frac{(L/m)^2}{r_0^3} = 0$$

$$\Rightarrow \underline{\underline{GMr_0 = \left(\frac{L}{m}\right)^2}}$$

g) $$\omega^2 = \frac{d^2(V_{eff}/m)}{dr^2}\Big|_{r=r_0}$$

$$= \left(-\frac{2GM}{r^3} + 3\frac{(L/m)^2}{r^4}\right)\Big|_{r=r_0}$$

$$= -\frac{2GM}{r_0^3} + 3\frac{GMr_0}{r_0^4} = \frac{GM}{r_0^3}$$

$$\Rightarrow \omega = \sqrt{\frac{GM}{r_0^3}}$$

h) $\frac{L}{m} = r^2 \omega_\phi$

$$\Rightarrow \quad \frac{L}{m} = \sqrt{GMr_0} = r_0^2 \omega_\phi$$

$$\Rightarrow \quad \omega_\phi = \sqrt{\frac{GM}{r_0^3}}$$

i) $\frac{\omega}{\omega_\phi} = 1$, orbit is closed (after one radial osc., one angular osc is also completed → back at the same point)

j) $\frac{V(r)}{m} = \frac{1}{2} KMr^2$

$$\longrightarrow \quad \frac{E}{m} = \frac{1}{2}\dot{r}^2 + \frac{L^2}{2m^2r^2} + \frac{1}{2}KMr^2$$

$$\frac{1}{2}\dot{r}^2 = \frac{E}{m} - \underbrace{\left[\frac{1}{2}KMr^2 + \frac{(L/m)^2}{2r^2}\right]}_{V_{eff}/m}$$

$$\frac{d}{dr}\left(\frac{V_{eff}}{m}\right) = KMr - \frac{(L/m)^2}{r^3}$$

$$\frac{d}{dr}\left(\frac{V_{eff}}{m}\right)\Big|_{r=r_0} = 0 \Rightarrow KMr_0 = \frac{(L/m)^2}{r_0^3}$$

$$\Rightarrow \quad r_0^4 = \frac{(L/m)^2}{KM}, \quad \left(\frac{L}{m}\right)^2 = KMr_0^4$$

$$\frac{d^2}{dr^2}\left(\frac{V_{eff}}{m}\right) = KM + 3\,\frac{(L/m)^2}{r^4}$$

$$\Rightarrow \quad \omega^2 = \frac{d^2}{dr^2}\left(\frac{V_{eff}}{m}\right)\Big|_{r=r_0} = KM + 3\,\frac{KMr_0^4}{r_0^4} = 4KM$$

$$\Rightarrow \quad \underline{\underline{\omega = 2\sqrt{KM}}}$$

$$\frac{L}{m} = r^2\,\omega_\phi = r_0^2\,\omega_\phi$$

$$\left(\frac{L}{m}\right)^2 = KMr_0^4 = r_0^4\,\omega_\phi^2$$

$$\Rightarrow \quad \omega_\phi^2 = KM, \quad \omega_\phi = \sqrt{KM}$$

$$\Rightarrow \quad \frac{\omega}{\omega_\phi} = 2.$$ ~~Lukket bane~~ ~~Etter to hele~~ radielle ~~oscil~~

Closed orbit. Back at starting point after two radial oscillations.

k) $$\frac{V_{eff}}{m} = \frac{1}{3}Ar^3 + \frac{(L/m)^2}{2r^2}$$

$$\frac{dV_{eff}/m}{dr} = Ar^2 - \frac{(L/m)^2}{r^3} \Rightarrow r_0 \text{ given by } Ar_0^2 - \frac{(L/m)^2}{r_0^3} = 0$$

$$\Rightarrow \left(\frac{L}{m}\right)^2 = Ar_0^5$$

$$\frac{d^2(V_{eff}/m)}{dr^2}\Big|_{r=r_0} = 2Ar_0 + 3\,\frac{(L/m)^2}{r_0^4} = 2Ar_0 + 3Ar_0 = 5Ar_0 = \omega^2$$

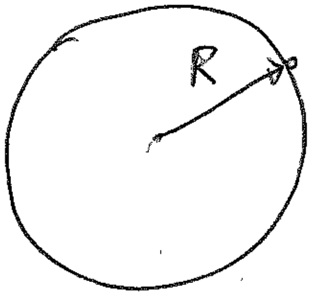
$$\omega_\phi^2 = \frac{1}{r_0^4}\left(\frac{L}{m}\right)^2 = Ar_0$$

$$\Rightarrow \frac{\omega}{\omega_\phi} = \sqrt{5}$$, ikke lukket bane.

Not closed orbit
Never full cycles of ϕ
after full osc. in r.
Vil aldri ~~bli gjennomført~~
et ~~helt antall~~ rotasjoner

7.

a)



Energy of test particle : $E = \frac{1}{2} m v^2 - \frac{GMm}{R}$

Escape velocity : Reaches $r = \infty$ with $v = 0 \Rightarrow E = 0$

Energy conserved, so

$$\frac{1}{2} m v_{esc}^2 = \frac{GMm}{R}$$

$$v_{esc}^2 = \frac{2GM}{R}$$

If $v_{esc} = c$:

$$c^2 = \frac{2GM}{R_s} \Rightarrow \underline{\underline{R_s = \frac{2GM}{c^2}}}$$

Radial motion :
$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} \qquad (*)$$

b) $v(R) = c$

$$(*) \Rightarrow \frac{dr}{dt}\frac{d^2 r}{dt^2} + \frac{dr}{dt}\frac{GM}{r^2} = 0$$

$$\Rightarrow \frac{d}{dt}\Big[\frac{1}{2}\underbrace{\dot{r}^2}_{v(r)} - \frac{GM}{r}\Big] = 0$$

$$\Rightarrow \frac{1}{2} v^2(r) - \frac{GM}{r} = C = \text{constant}$$

$r = R$:
$$\frac{1}{2} c^2 - \frac{GM}{R} = C$$

$$\Rightarrow \frac{1}{2} v^2(r) - \frac{GM}{r} = \frac{1}{2} c^2 - \frac{GM}{R}$$

$$\Rightarrow \quad v^2(r) = c^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$$

c) Max distance : $v^2(r_{max}) = 0$

$$\Rightarrow \quad c^2 + 2GM\left(\frac{1}{r_{max}} - \frac{1}{R}\right) = 0$$

$$\Rightarrow \quad \frac{2GM}{r_{max}} = \frac{2GM}{R} - c^2 \geq 0$$

$$\Rightarrow \quad R \leq \frac{2GM}{c^2} = R_S$$

If $R = R_S$: $\frac{2GM}{r_{max}} = 0 \Rightarrow r_{max} \to \infty$

For $R < R_S$:

$$\frac{2GM}{r_{max}} = c^2\left(\frac{2GM}{Rc^2} - 1\right) = c^2\left(\frac{R_S}{R} - 1\right)$$

$$\frac{1}{r_{max}} = \frac{c^2}{2GM}\left(\frac{R_S}{R} - 1\right) = \frac{1}{R_S}\left(\frac{R_S}{R} - 1\right)$$

$$\Rightarrow \quad r_{max} = \frac{R_S}{\frac{R_S}{R} - 1}$$

E.g. $R = \frac{1}{2} R_S \Rightarrow r_{max} = \frac{R_S}{2-1} = R_S = 2R$

d) Black hole : No light can move past R_S

Dark star : Light emitted from surface of star with $R = R_S$ reaches $r = \infty$

Also: If speed of light can vary, why assume it starts out with $v = c$?

e) ~~b)~~ d) Mean mass density of the Sun:

$$R_\odot = 7 \cdot 10^8 \, m$$

$$M_\odot = 2 \cdot 10^{30} \, kg$$

$$\rightarrow \rho_\odot = \frac{\cancel{3} M_\odot}{4 \cancel{\pi} R_\odot^3} = \frac{2 \cdot 10^{30} \, kg}{4 \cdot (7 \cdot 10^8 m)^3}$$

$$= \frac{2 \cdot 10^{30}}{4 \cdot 3{,}5 \, 10^2 \cdot 10^{24}} \, kg \, m^{-3}$$

$$= \frac{1}{7} \cdot 10^4 \, kg \, m^{-3} \simeq 10^3 \, kg \, m^{-3}$$

Mean density of a black hole / dark star:

$$R_s = \frac{2GM}{c^2}$$

$$\rho_{DS} = \frac{\cancel{3} M}{4 \cancel{\pi} R_s^3} = \frac{M}{4 \cdot \frac{8 G^3 M^3}{c^6}}$$

$$= \frac{1}{32} \frac{c^6}{G^3 M^2}$$

$$= 3 \cdot 10^{-2} \frac{c^6}{G^3 M_\odot^2} \left(\frac{M_\odot}{M}\right)^2$$

$$= 3 \cdot 10^{-2} \frac{(3 \cdot 10^8 \, m s^{-1})^6}{(7 \cdot 10^{-11} \, m^3 kg^{-1} s^{-2})^3 \, (2 \cdot 10^{30} \, kg)^2} \left(\frac{M_\odot}{M}\right)^2$$

$$= 3 \cdot 10^{-2} \frac{7 \cdot 10^2 \cdot 10^{48} \, m^6 s^{-6}}{3{,}5 \cdot 10^2 \cdot 10^{-33} \, m^9 kg^{-3} s^{-6} \cdot 4 \cdot 10^{60} \, kg^2} \left(\frac{M_\odot}{M}\right)^2$$

$$= 3 \cdot 10^{-3} \cdot \frac{1}{2} \cdot 10^{50+31-60} \left(\frac{M_\odot}{M}\right)^2 kg \, m^{-3}$$

$$\simeq 10^{19} \left(\frac{M_\odot}{M}\right)^2 kg \, m^{-3}$$

$$\rho_{DS} = \rho_\odot \Rightarrow 10^{19} \left(\frac{M_\odot}{M}\right)^2 = 10^3$$

$$\Rightarrow M = \underline{10^8 \, M_\odot} = 2 \cdot 10^{38} \, kg$$

$$R = 3 \left(\frac{M}{M_\odot}\right) km = 3 \cdot 10^8 \, km = \underline{\underline{2 \, AU}}$$