Problem 1

H-L : $v = H_0 d$,

$$H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1}$$

a) Non-relativistic Doppler:

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \quad \text{for } v \ll c$$

Definition of z:

$$1 + z = \frac{\lambda_0}{\lambda_e}, \quad \lambda_0 = \text{observed wavelength}, \; \lambda_e = \text{emitted} \text{ — " —}$$

$$\Rightarrow \quad z = \frac{\lambda_0}{\lambda_e} - 1 = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\Delta\lambda}{\lambda}$$

So

$$z = \frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

(If we interpret the cosmic redshift as a Doppler effect)

H-L: $v = H_0 d$

$$\Rightarrow \quad \underline{\underline{z = \frac{H_0 d}{c}}}$$

b) Want $v = 1000$ km/s

H-L: $10^3 \text{ km s}^{-1} = H_0 d = 70 \text{ km s}^{-1} \text{Mpc}^{-1} \cdot d$

$$\Rightarrow \quad d = \frac{10^3}{7 \cdot 10^1} \text{ Mpc} \simeq \underline{\underline{10 \text{ Mpc}}} \quad \text{(1 digit accuracy)}$$

Corresponding redshift:

$$z = \frac{v}{c} = \frac{10^3 \text{ km s}^{-1}}{3 \cdot 10^5 \text{ km s}^{-1}} \simeq \underline{\underline{3 \cdot 10^{-3}}}$$

c) Gravitational redshift:

$$1+z = \frac{1}{\sqrt{1-\frac{R_S}{R}}}$$

$$R_S = \frac{2GM}{c^2} = 3\,\frac{M}{M_\odot}\ \text{km}$$

Our galaxy : $M \sim 10^{12}\,M_\odot$ (including dark matter)

$$R \sim 5\cdot 10^4\ \text{ly}$$
$$\simeq 5\cdot 10^4\cdot 10^{16}\ \text{m}$$
$$= 5\cdot 10^{20}\ \text{m} = 5\cdot 10^{17}\ \text{km}$$

Schwarzschild radius :

$$R_S = 3\,\frac{10^{12}M_\odot}{M_\odot}\ \text{km} = 3\cdot 10^{12}\ \text{km}$$

So

$$\frac{R_S}{R} = \frac{3\cdot 10^{12}\ \text{km}}{5\cdot 10^{17}\ \text{km}} = 0.6\cdot 10^{-5} = 6\cdot 10^{-6} \ll 1$$

We can therefore use the Taylor expansion in the expression for the grav. redshift:

$$1+z = \left(1-\frac{R_S}{R}\right)^{-\frac{1}{2}} \simeq 1+\frac{1}{2}\frac{R_S}{R}$$

$$\Rightarrow \quad z \simeq \frac{1}{2}\frac{R_S}{R} = \frac{1}{2}\cdot 6\cdot 10^{-6} = \underline{\underline{3\cdot 10^{-6}}}$$

Unless we need extremely accurate redshifts this effect is nothing to worry about.

Problem 1

a) $$1+z = \sqrt{\frac{1+v/c}{1-v/c}} \quad , \quad z = 3{,}78$$

$$x \equiv v/c$$

$$(1+z)^2 = \frac{1+x}{1-x}$$

$$\Rightarrow \quad (1+z)^2 - x(1+z)^2 = 1 + x$$

$$\Rightarrow \quad (1+z)^2 - 1 = x\left[(1+z)^2 + 1\right]$$

$$\Rightarrow \quad x = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} \simeq 0{,}92$$

So $\underline{\underline{v = 0{,}92c}}$

b) $$v = H_0 d$$

$$\Rightarrow \quad d = \frac{v}{H_0} = \frac{0{,}92c}{H_0} = 0{,}92 \cdot \frac{3{,}00 \times 10^5 \text{ km s}^{-1}}{16 \text{ km s}^{-1} (\text{Mly})^{-1}}$$

$$\simeq 1{,}7 \times 10^4 \text{ Mly} = \underline{\underline{17 \text{ Gly}}}$$

c) Use Doppler effect from special relativity and interpret v as a real, physical speed

The redshift is an effect of the expansion. Light travels through curved spacetime.

d) $k = 0, \quad a = a_0 \left(\frac{t}{t_0}\right)^{2/3}$

$$d_p(t) = a(t)\, r \qquad (k=0)$$

$$\Rightarrow d_p(t_0) = a_0 r$$

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \frac{c t_0^{2/3}}{a_0} \int_{t_e}^{t_0} t^{-2/3} dt$$

$$= \frac{3 c t_0^{2/3}}{a_0} \left(t_0^{1/3} - t_e^{1/3} \right)$$

$$= \frac{3 c t_0}{a_0} \left[1 - \left(\frac{t_e}{t_0}\right)^{1/3} \right]$$

Redshift:

$$1+z = \frac{a_0}{a(t_e)} = \frac{a_0}{a_0 \left(\frac{t_e}{t_0}\right)^{2/3}} = \left(\frac{t_0}{t_e}\right)^{2/3}$$

$$\Rightarrow \left(\frac{t_e}{t_0}\right)^{2/3} = \frac{1}{\sqrt{1+z}}$$

Replace t_0 by H_0:

$$H(t) = \frac{\dot{a}}{a} = \frac{a_0 \frac{2}{3} \frac{t^{-1/3}}{t_0^{2/3}}}{a_0 \frac{t^{2/3}}{t_0^{2/3}}} = \frac{2}{3t}$$

$$\Rightarrow H_0 = H(t_0) = \frac{2}{3t_0}, \quad t_0 = \frac{2}{3H_0}$$

So

$$d_p = a_0 \cdot \frac{3c}{a_0} \frac{2}{3H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right)$$

$$= \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \simeq 1{,}09 \frac{c}{H_0} \simeq 2{,}0 \times 10^4 \text{ Mly}$$

$$= 20 \text{ Gly}$$

Problem 2

$$1+z = \frac{a(t)}{a(t_e)}$$ "Normally, $t = t_0$"

$t = t(t_e) \Rightarrow t_e = t_e(t)$

Because we look at the same object all the time

a) $$\frac{d}{dt}(1+z) = \frac{dz}{dt} = \frac{d}{dt}\frac{a(t)}{a(t_e(t))} = \frac{\dot{a}(t)a(t_e) - a(t)\dot{a}(t_e)\frac{dt_e}{dt}}{a^2(t_e)}$$

$$= \frac{\dot{a}(t)}{a(t_e)}\cancel{\frac{a(t_e)}{a(t_e)}} - \frac{a(t)}{a(t_e)}\frac{\dot{a}(t_e)}{a(t_e)}\frac{dt_e}{dt}$$

$$= \frac{a(t)}{a(t_e)}\frac{\dot{a}(t)}{a(t)} - \frac{a(t)}{a(t_e)}\frac{\dot{a}(t_e)}{a(t_e)}\frac{dt_e}{dt} \qquad \left(\dot{a}(t_e) = \frac{da(t_e)}{dt_e}\right)$$

b) $$r = \int_{t_e}^{t}\frac{c\,dt'}{a(t')}$$

r constant $\Rightarrow \frac{dr}{dt} = 0$

$a = a_0\left(\frac{t}{t_0}\right)^{2/3}$, $k=0$

$$r = \int_{t_e}^{t}\frac{c\,dt'}{a_0\left(\frac{t'}{t_0}\right)^{2/3}} = \frac{c t_0^{2/3}}{a_0}\int_{t_e}^{t}(t')^{-2/3}dt$$

$$= \frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - t_e^{1/3}\right)$$

$$\frac{dr}{dt} = \frac{3ct_0^{2/3}}{a_0}\left(\frac{1}{3}t^{-2/3} - \frac{1}{3}t_e^{-2/3}\frac{dt_e}{dt}\right) = 0$$

$$\Rightarrow \frac{dt_e}{dt} = \left(\frac{t_e}{t}\right)^{2/3}$$

⌜ In general (not asked for in the problem)

$$\frac{dr}{dt} = \frac{d}{dt}\int_{t_e}^{t} \frac{c\,dt'}{a(t')}$$

$t_e < x < t$

$$= \frac{d}{dt}\int_{t_e}^{x} \frac{c\,dt'}{a(t')} + \frac{d}{dt}\int_{x}^{t} \frac{c\,dt'}{a(t')}$$

$$= \frac{dt_e}{dt}\frac{d}{dt_e}\int_{t_e}^{x} \frac{c\,dt'}{a(t')} + \frac{c}{a(t)}$$

$$= -\frac{dt_e}{dt}\frac{c}{a(t_e)} + \frac{c}{a(t)} = 0$$

$$\Rightarrow \quad \frac{dt_e}{dt} = \frac{a(t_e)}{a(t)} = \frac{1}{1+z}$$ ⌟

$$1+z = \frac{a(t)}{a(t_e)} = \frac{a_0\left(\frac{t}{t_0}\right)^{2/3}}{a_0\left(\frac{t_e}{t_0}\right)^{2/3}} = \left(\frac{t}{t_e}\right)^{2/3}$$

$$\Rightarrow \qquad \frac{dt_e}{dt} = \frac{1}{1+z}$$

c) $$\frac{dz}{dt} = \frac{a(t)}{a(t_e)}\frac{\dot a(t)}{a(t)} - \frac{a(t)}{a(t_e)}\frac{\dot a(t_e)}{a(t_e)}\frac{dt_e}{dt}$$

$$= (1+z)\,H(t) - (1+z)\,H(t_e)\frac{1}{1+z}$$

$$= (1+z)H(t) - H(t_e)$$

d) $$H(t) = \frac{\dot a}{a} = \frac{\frac{2}{3}\cancel{a_0}\cancel{\frac{1}{t_0^{2/3}}}\,t^{-1/3}}{\cancel{a_0}\cancel{\left(\frac{t}{t_0}\right)^{2/3}}} = \frac{2}{3t}$$

$$1+z = \left(\frac{t}{t_e}\right)^{2/3} \Rightarrow t = t_e(1+z)^{3/2}$$

$$t = t_0 \quad \Rightarrow \qquad t_0 = t_e(1+z)^{3/2}$$

$$\left.\frac{dz}{dt}\right|_{t=t_0} = (1+z)\frac{2}{3t_0} - \frac{2}{3t_e}$$

$$= \frac{2}{3t_0}\left[1+z - \frac{t_0}{t_e}\right]$$

$$= \frac{2}{3t_0}\left[1+z-(1+z)^{3/2}\right] \quad \left(= -\frac{4,12}{t_0}\right)$$

$t_0 \sim 10^{10}$ añ

$$\simeq H_0\left(5-5^{3/2}\right) \simeq -6,18\,H_0$$

e) $$a(t) = a_0 e^{H_0(t-t_0)}$$

$$\Rightarrow \quad \frac{\dot{a}}{a} = \frac{H_0 a_0 e^{H_0(t-t_0)}}{a_0 e^{H_0(t-t_0)}} = H_0 = \text{constant}$$

So

$$\frac{dz}{dt} = (1+z)H_0 - H_0 = zH_0 = 4H_0 > 0.$$

In this case, $dz/dt > 0$, redshift increases with time.

(Redshift measures the amount of expansion during the time between emission and reception of the light from the object. Negative dz/dt means that the amount of expansion between the two times decreases $\Rightarrow$ decelerating expansion.)

Problem 3

$$q = -\frac{\ddot{a}\,a}{\dot{a}^2}$$

Note: $q < 0$ for $\ddot{a} > 0$. Historical reasons for why q was defined in this way

a) $$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}$$

$$\dot{a} = \frac{2}{3} a_0 \frac{t^{-1/3}}{t_0^{2/3}}$$

$$\ddot{a} = -\frac{2}{9} \frac{a_0}{t_0^{2/3}} t^{-4/3}$$

$$\rightarrow \quad q = -\frac{\left(-\frac{2}{\cancel{9}}\right) \cancel{\frac{a_0}{t_0^{2/3}}} t^{-4/3} \; \cancel{a_0} \cancel{\frac{t^{2/3}}{t_0^{2/3}}}}{\frac{4}{\cancel{9}} \cancel{a_0^2} \cancel{\frac{t^{-2/3}}{t_0^{4/3}}}}$$

$$= +\frac{1}{2}$$

b) $$H = \frac{\dot{a}}{a}, \text{ so}$$

$$\frac{d}{dt}\frac{1}{H} = \frac{d}{dt}\frac{a}{\dot{a}} = \frac{\dot{a}^2 - a\ddot{a}}{\dot{a}^2} = 1 - \frac{\ddot{a}a}{\dot{a}^2} = \underline{\underline{1+q}}$$

c) $1+z = \frac{a_0}{a} \Rightarrow \frac{dz}{da} = -\frac{a_0}{a^2} = -\frac{1}{a_0}\left(\frac{a_0}{a}\right)^2$

$= -\frac{1}{a_0}(1+z)^2$

(Note: Here we are not looking at a fixed object. z can be seen as just an alternative time variable)

$$\frac{d}{dt}\frac{1}{H} = \frac{d}{da}\frac{1}{H}\frac{da}{dt} = \frac{d}{dz}\frac{1}{H}\frac{dz}{da}\frac{da}{dt}$$

$$= -\frac{1}{H^2}\frac{dH}{dz}\frac{dz}{da}\, a \underbrace{\frac{1}{a}\frac{da}{dt}}_{=H}$$

$$= -\frac{1}{H}\frac{dH}{dz}\, a\frac{dz}{da} = -\frac{1}{H}\frac{dH}{dz}\left[-\underbrace{\frac{a}{a_0}}_{=\frac{1}{1+z}}(1+z)^2\right]$$

$$= \frac{1+z}{H}\frac{dH}{dz}$$

d) b) & c) $\Rightarrow \quad \frac{1+z}{H}\frac{dH}{dz} = 1+q(z)$

$$\Rightarrow \quad \frac{dH}{H} = \frac{1+q(z)}{1+z}dz$$

$$\Rightarrow \quad \int_{H_0}^{H}\frac{dH}{H} = \int_0^z \frac{1+q(z')}{1+z'}dz'$$

$$\Rightarrow \quad \ln\left(\frac{H}{H_0}\right) = \int_0^z \frac{1+q(z')}{1+z'}dz'$$

$$\Rightarrow \quad H(z) = H_0 \exp\left[\int_0^z \frac{1+q(z')}{1+z'}dz'\right]$$

e) $q = q_0 = \text{constant}$

$$\Rightarrow \quad H(z) = H_0 \exp\left[\int_0^z \frac{1+q_0}{1+z'} dz' \right]$$

$$= H_0 \exp\left[(1+q_0) \ln(1+z) \right]$$

$$= H_0 \exp\left[\ln(1+z)^{1+q_0} \right]$$

$$= \underline{\underline{H_0 (1+z)^{1+q_0}}}$$