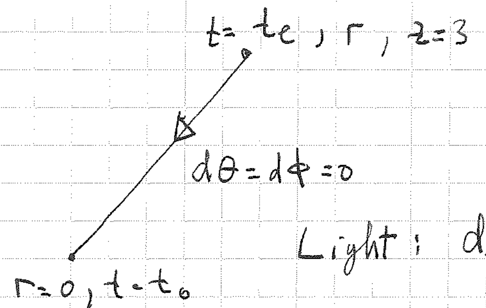
Problem 1

$$k=0, \quad a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} \qquad (EdS)$$

a)



Light: $ds^2 = 0$

RW with $k=0$ gives

$$ds^2 = c^2 dt^2 - a^2(t) dr^2 = 0$$

$$\Rightarrow \quad dr = -\frac{c\,dt}{a(t)}$$ (− because light is travelling towards $r=0$)

$$\Rightarrow \quad \int_0^r dr = -\int_{t_0}^{t_e} \frac{c\,dt}{a(t)}$$

$$\Rightarrow \quad r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \frac{c}{a_0} t_0^{2/3} \int_{t_e}^{t_0} t^{-2/3}\,dt$$

$$= \frac{3ct_0^{2/3}}{a_0}\left(t_0^{1/3} - t_e^{1/3}\right)$$

$$= \frac{3ct_0}{a_0}\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right]$$

From the definition of redshift:

$$1+z = \frac{a_0}{a(t_e)} = \frac{a_0}{a_0\left(\frac{t_e}{t_0}\right)^{2/3}} = \left(\frac{t_0}{t_e}\right)^{2/3}$$

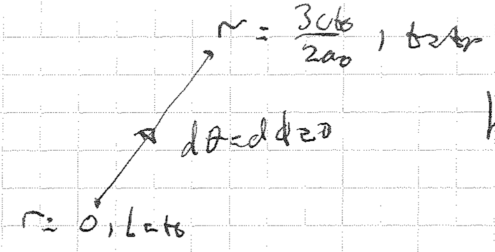
$$\Rightarrow \quad \left(\frac{t_e}{t_0}\right)^{1/3} = \frac{1}{\sqrt{1+z}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

Therefore $$\underline{\underline{r = \frac{3ct_0}{2a_0}}}$$

Proper distance: $$d_p(t_0) = a_0 r = \underline{\underline{\frac{3ct_0}{2}}}$$

c) Now we are sending light outwards.

Starts at $t = t_0$, received at $t = t_r = ?$



Key: The comoving coordinate of an object which follows the expansion is constant.

So:

$$ds^2 = 0$$

$$\Rightarrow dr = + \frac{c\,dt}{a(t)}$$ (light moves outwards from $r=0$)

$$\Rightarrow \int_0^r dr = \int_{t_0}^{t_r} \frac{c\,dt}{a(t)}$$

$$\Rightarrow r = \frac{3ct_0}{2a_0} = \frac{ct_0^{2/3}}{a_0} \int_{t_0}^{t_r} t^{-2/3}\,dt$$

$$= \frac{3ct_0}{a_0}\left[\left(\frac{t_r}{t_0}\right)^{1/3} - 1\right]$$

$$\Rightarrow \left(\frac{t_r}{t_0}\right)^{1/3} - 1 = \frac{1}{2}$$

$$\Rightarrow \frac{t_r}{t_0} = \left(\frac{3}{2}\right)^3 = \underline{\underline{\frac{27}{8}}}$$ Must be patient!!

Problem 2

a) $p = \rho = 0, \quad k = -1, \quad a(t) = ct \Rightarrow \dot{a} = c$

$\ddot{a} = 0$

FI : $\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho$

Inserting :

$$c^2 + (-1)c^2 = \frac{8\pi G}{3} \cdot 0$$

$$0 = 0, \quad \text{OK.}$$

FII : $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

$$\frac{0}{ct} = -\frac{4\pi G}{3}\left(0 + \frac{3 \cdot 0}{c^2}\right)$$

$$0 = 0, \quad \text{OK}$$

b) Proper distance : $d_p(t_0) = a_0 \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}$

Angular diameter distance :

$$d_A = \frac{a_0 r}{1+z}$$

For $k = -1$,

$$r = \sinh\left[\int_{t_e}^{t_0} \frac{c\,dt}{a(t)}\right]$$

Calculate the integral first :

$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_{t_e}^{t_0} \frac{c\,dt}{ct} = \int_{t_e}^{t_0} \frac{dt}{t} = \ln\left(\frac{t_0}{t_e}\right)$$

Definition of z :

$$1 + z = \frac{a_0}{a(t_e)} = \frac{ct_0}{ct_e} = \frac{t_0}{t_e}$$

So

$$d_p = a_0 \ln(1+z) = ct_0 \ln(1+z)$$

and

$$r = \sinh\left[\ln(1+z)\right]$$

$$= \frac{1}{2}\left[e^{\ln(1+z)} - e^{-\ln(1+z)}\right]$$

= [illegible]

$$= \frac{1}{2}\left[1+z - \frac{1}{1+z}\right]$$

Finally,

$$d_A = a_0 r/(1+z) = \frac{a_0}{2}\left[1 - \frac{1}{(1+z)^2}\right]$$

$$= \underline{\underline{\frac{ct_0}{2}\left[1 - \frac{1}{(1+z)^2}\right]}}$$

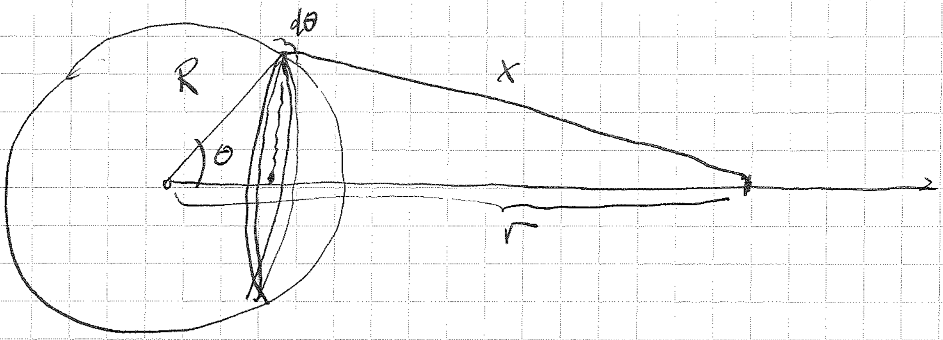
Problem 3

Modify Newtonian gravity with extra term

$$V(r) = -\frac{\alpha G m_1 m_2}{r} e^{-r/\lambda}$$

α is dimensionless, $[\lambda] = m$ (length)

a)



Mass per area of the shell : $\frac{M}{4\pi R^2}$

Mass of ring with radius $R\sin\theta$, with $Rd\theta$:

$$dM = \frac{M}{4\pi R^2} \cdot 2\pi R \sin\theta \cdot R d\theta$$

$$= \frac{M}{2} \sin\theta d\theta$$

Contribution of ring to grav. pot. at r : $(m=1)$

$$dV = -\frac{\alpha G dM}{x} e^{-x/\lambda}$$

Law of cosines ⇒ $x^2 = R^2 + r^2 - 2Rr\cos\theta$

$$\Rightarrow 2x dx = 2Rr \sin\theta d\theta$$

$$\Rightarrow \sin\theta d\theta = \frac{x}{Rr} dx$$

So

$$dV = -\frac{\alpha G}{x} \cdot \frac{M}{2} \sin\theta d\theta \, e^{-x/\lambda}$$

$$= -\frac{\alpha G M}{2} \frac{1}{x} \frac{x}{Rr} dx \, e^{-x/\lambda}$$

$$= -\frac{\alpha G M}{2Rr} e^{-x/\lambda} dx$$

Integrate from $\theta = 0 \Rightarrow x^2 = R^2 + r^2 - 2Rr$

$= (r-R)^2 \quad (r > R)$

$x = r - R$

for $\theta = \pi \Rightarrow x^2 = R^2 + r^2 + 2Rr$

$= (r+R)^2$

$x = r + R$

J:

$$V(r) \underset{r>R}{=} - \frac{\alpha GM}{2Rr} \int_{r-R}^{r+R} e^{-x/\lambda} \, dx$$

$$= - \frac{\alpha GM}{2Rr} \Big| -\lambda e^{-x/\lambda} \Big|_{r-R}^{r+R}$$

$$= - \frac{\alpha \lambda GM}{Rr} \frac{1}{2} \left[e^{-(r-R)/\lambda} - e^{-(r+R)/\lambda} \right]$$

$$= - \frac{\alpha \lambda GM}{rR} \cdot \frac{1}{2} \left[e^{-r/\lambda} e^{R/\lambda} - e^{-r/\lambda} e^{-R/\lambda} \right]$$

$$= - \frac{\alpha \lambda GM}{rR} \frac{1}{2} e^{-r/\lambda} \left[e^{R/\lambda} - e^{-R/\lambda} \right]$$

$$\underline{\underline{= - \frac{\alpha \lambda GM}{rR} \sinh\left(\frac{R}{\lambda}\right) e^{-r/\lambda}}}$$

For points with $r < R$ (inside the shell), the argument is the same, but here the lower limit on the integral will be $x = R - r$. We find

$$V(r)_{r<R} = -\frac{\alpha G M}{2Rr} \int_{R-r}^{R+r} e^{-x/\lambda} \, dx$$

$$= -\frac{\alpha G M}{2Rr} \Big|_{R-r}^{R+r} -\lambda e^{-x/\lambda}$$

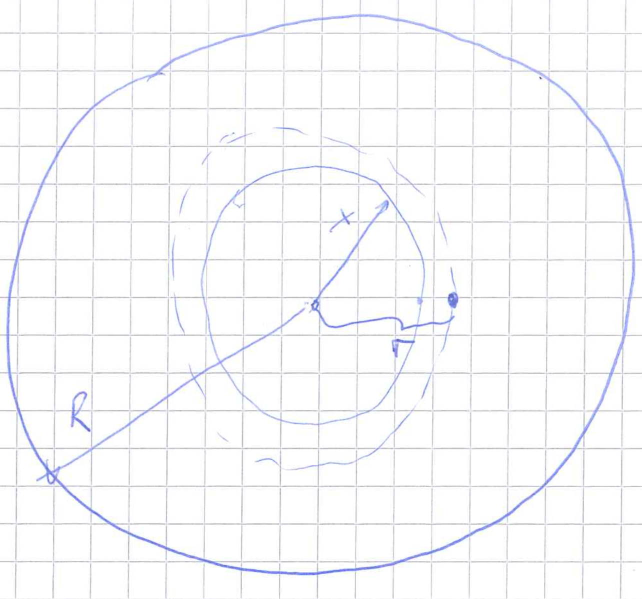
$$= -\frac{\lambda \alpha G M}{rR} \cdot \frac{1}{2} \left[e^{-(R-r)/\lambda} - e^{-(R+r)/\lambda} \right]$$

$$= -\frac{\lambda \alpha G M}{rR} \cdot \frac{1}{2} \left[e^{-R/\lambda} e^{r/\lambda} - e^{-R/\lambda} e^{-r/\lambda} \right]$$

$$= -\frac{\lambda \alpha G M}{rR} e^{-R} \frac{1}{2} \left(e^{r/\lambda} - e^{-r/\lambda} \right)$$

$$= -\frac{\lambda \alpha G M}{rR} e^{-R} \sinh\left(\frac{r}{\lambda}\right)$$

b)



$$M = \frac{4\pi}{3} \varsigma R^3$$

Contribution from shells with radius $x < r$

$$dV_{x<r} = -\frac{\alpha\lambda G dM}{r x} \sinh\left(\frac{x}{\lambda}\right) e^{-r/\lambda}$$

$$dM = 4\pi x^2 dx \rho$$

$$\rightarrow \quad dV_{x<r} = -\frac{\alpha\lambda G \cdot 4\pi x^2 dx \rho}{r x} \sinh\left(\frac{x}{\lambda}\right) e^{-r/\lambda}$$

$$= -\frac{4\pi\rho\alpha\lambda G}{r} dx \, x \sinh\left(\frac{x}{\lambda}\right) e^{-r/\lambda}$$

$$= -\frac{4\pi\rho\alpha\lambda G}{r} e^{-r/\lambda} \, x \sinh\left(\frac{x}{\lambda}\right) dx$$

$$\rightarrow \quad V_{x<r} = -\frac{4\pi\rho\alpha\lambda G}{r} e^{-r/\lambda} \int_0^r x \sinh\left(\frac{x}{\lambda}\right) dx$$

Use partial integration :

$$\int_0^r x \sinh\left(\frac{x}{\lambda}\right) dx \underset{P}{=} \quad \boxed{\begin{matrix} u = x & v = \lambda \cosh\left(\frac{x}{\lambda}\right) \\ du = dx & dv = \sinh\left(\frac{x}{\lambda}\right) dx \end{matrix}}$$

$$= \Big|_0^r \lambda x \cosh\left(\frac{x}{\lambda}\right) - \lambda \int_0^r \cosh\left(\frac{x}{\lambda}\right) dx$$

$$= \lambda r \cosh\left(\frac{r}{\lambda}\right) - \lambda \int_0^r \lambda \sinh\left(\frac{x}{\lambda}\right)$$

$$= \lambda r \cosh\left(\frac{r}{\lambda}\right) - \lambda^2 \sinh\left(\frac{r}{\lambda}\right)$$

So

$$V_{x<r} = -\frac{4\pi\rho\alpha\lambda G}{r} e^{-r/\lambda}\left[\lambda r \cosh\left(\frac{r}{\lambda}\right) - \lambda^2 \sinh\left(\frac{r}{\lambda}\right)\right]$$

Similarly, the contribution from shells with $r < x < R$ is

$$dV_{x>r} = -\frac{\alpha\lambda G dM}{rx} e^{-x/\lambda} \sinh\left(\frac{r}{\lambda}\right)$$

$$= -\frac{\alpha\lambda G}{rx} 4\pi x^2 \rho dx\, e^{-x/\lambda} \sinh\left(\frac{r}{\lambda}\right)$$

$$= -\frac{4\pi\rho\alpha\lambda G}{r} \sinh\left(\frac{r}{\lambda}\right) e^{-x/\lambda} x dx$$

$$\Rightarrow V_{x>r} = -\frac{4\pi\rho\alpha\lambda G}{r} \sinh\left(\frac{r}{\lambda}\right) \int_r^R x e^{-x/\lambda} dx$$

Again, use integration by parts:

$$\int_r^R x e^{-x/\lambda} dx = \Big|_r^R -\lambda x e^{-x/\lambda} + \lambda \int_r^R e^{-x/\lambda} dx$$

$$\boxed{\begin{array}{ll} u = x & v = -\lambda e^{-x/\lambda} \\ du = dx & dv = e^{-x/\lambda} dx \end{array}}$$

$$= -\lambda R e^{-R/\lambda} + \lambda r e^{-r/\lambda} + \lambda \Big|_r^R -\lambda e^{-x/\lambda}$$

$$= -\lambda R e^{-R/\lambda} + \lambda r e^{-r/\lambda} - \lambda^2 e^{-R/\lambda} + \lambda^2 e^{-r/\lambda}$$

So

$$V_{x>r} = -\frac{4\pi\rho\alpha\lambda G}{r} \sinh\left(\frac{r}{\lambda}\right)\left[\lambda r e^{-r/\lambda} + \lambda^2 e^{-r/\lambda} - \lambda R e^{-R/\lambda} - \lambda^2 e^{-R/\lambda}\right]$$

Then

$$V(r) = V_{x<r} + V_{x>R}$$

$$= -\frac{4\pi\rho\alpha\lambda G}{r}\Big\{ e^{-r/\lambda}\lambda r\cosh\left(\frac{r}{\lambda}\right) - \cancel{\lambda^2 e^{-r/\lambda}\sinh\left(\frac{r}{\lambda}\right)}$$

$$+ \sinh\left(\frac{r}{\lambda}\right)\lambda r e^{-r/\lambda} + \cancel{\sinh\left(\frac{r}{\lambda}\right)\lambda^2 e^{-r/\lambda}}$$

$$- \sinh\left(\frac{r}{\lambda}\right)\lambda R e^{-R/\lambda} - \sinh\left(\frac{r}{\lambda}\right)\lambda^2 e^{-R/\lambda}\Big\}$$

Look at

$$\lambda e^{-r/\lambda} r\cosh\left(\frac{r}{\lambda}\right) + \lambda e^{-r/\lambda} r\sinh\left(\frac{r}{\lambda}\right)$$

$$= \lambda r e^{-r/\lambda}\left[\cosh\left(\frac{r}{\lambda}\right) + \sinh\left(\frac{r}{\lambda}\right)\right]$$

$$= \lambda r e^{-r/\lambda}\left[\frac{1}{2}e^{r/\lambda} \cancel{+\frac{1}{2}e^{-r/\lambda}} + \frac{1}{2}e^{r/\lambda} \cancel{-\frac{1}{2}e^{-r/\lambda}}\right]$$

$$= \lambda r e^{-r/\lambda} \cdot e^{r/\lambda} = \lambda r$$

So

$$V(r) = -4\pi\rho\alpha\lambda^2 G + \frac{4\pi\rho\alpha\lambda^2 G}{r} e^{-R/\lambda}\left[R\sinh\left(\frac{r}{\lambda}\right) + \lambda\sinh\left(\frac{r}{\lambda}\right)\right]$$

The first term is a constant.
For fixed r, the factor $e^{-R/\lambda}$ makes sure the second term $\to 0$ as $R \to \infty$.
So, in the limit of an infinite universe, the potential becomes a constant, and hence this extra term does not give rise to an additional force, and therefore it will not influence the evolution of the Universe.

Problem 4

Friedmann equations w/ cosmological constant

$$\text{FI} \qquad \dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{3}a^2$$

$$\text{FII} \qquad \ddot{a} = -\frac{4\pi G}{3}\rho a + \frac{\Lambda}{3}a$$

ρ is the density of NR matter with $p = 0$ $(w = 0)$. The solution of the CE gives $\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3}$

The Einstein model is defined by

$$a = a_0 = \text{constant}, \quad \rho = \rho_0 = \text{constant}$$

FII then gives

$$0 = -\frac{4\pi G}{3}\rho_0 a_0 + \frac{\Lambda}{3}a_0$$

$$\Rightarrow \quad \Lambda = 4\pi G\rho_0$$

FI gives in turn

$$kc^2 = \frac{8\pi G}{3}\rho_0 a_0^2 + \frac{4\pi G}{3}\rho_0 a_0^2 = 4\pi G\rho_0 a_0^2 > 0$$

$$\Rightarrow \quad k = +1, \quad a_0 = \frac{c}{\sqrt{4\pi G\rho_0}} = \frac{c}{\sqrt{\Lambda}}$$

A small, time-dependent perturbation around this solution:

$$a(t) = a_0 + \eta(t) \quad (\eta \ll a_0)$$

$$\Rightarrow \quad \ddot{a} = \ddot{\eta}$$

The density can be written as

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3} = \rho_0\left(1 + \frac{\eta}{a_0}\right)^{-3} \simeq \rho_0\left(1 - 3\frac{\eta}{a_0}\right)$$

$$= \rho_0 - 3\frac{\eta}{a_0}\rho_0 \quad , \quad \eta/a_0 \ll 1$$

We now insert this in F II:

$$\ddot{\eta} = -\frac{4\pi G}{3}\left(\rho_0 - 3\frac{\eta}{a_0}\rho_0\right)(a_0+\eta) + \frac{\Lambda}{3}(a_0+\eta)$$

$$\approx -\frac{4\pi G}{3}\rho_0 a_0 - \frac{4\pi G}{3}\rho_0\eta + 4\pi G\rho_0\eta + \frac{\Lambda}{3}a_0 + \frac{\Lambda}{3}\eta$$

(neglect terms of order η^2)

$$= \underbrace{-\frac{4\pi G}{3}\rho_0 a_0 + \frac{\Lambda}{3}}_{=0 \text{ for the Einstein model}} + \frac{8\pi G}{3}\rho_0\eta + \frac{\Lambda}{3}\eta$$

$$= \frac{8\pi G}{3}\rho_0\eta + \frac{\overset{\Lambda}{\overset{\shortparallel}{4\pi G\rho_0}}}{3}\eta = 4\pi G\rho_0\eta$$

$$= \Lambda\eta$$

Since $\Lambda > 0$ in this model, this equation is on the form

$$\ddot{\eta} = \omega^2\eta, \quad \text{with } \omega^2 = \Lambda$$

Solution:

$$\eta(t) = Ae^{\omega t} + Be^{-\omega t}$$

Unless the initial conditions are fine-tuned to make $A = 0$, the exponentially increasing part will rapidly take over and take us away from the static solution. Hence, the Einstein universe is unstable.

Problem 5

a) Energy of a photon:

$$\epsilon = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\epsilon} \sim \frac{\hbar c}{\epsilon} \quad \left(\hbar = \frac{h}{2\pi}, \; 2\pi \sim 1 !\right)$$

In thermal equilibrium the energy of a typical photon is of order $k_B T$, so $\epsilon \sim k_B T$ and

$$\underline{\lambda \sim \frac{\hbar c}{k_B T}}$$

A single photon then occupies a volume $\sim \lambda^3$, so

$$\underline{n_\gamma \sim \frac{1}{\lambda^3} \sim \left(\frac{k_B T}{\hbar c}\right)^3}$$

and the energy density is

$$\rho_\gamma c^2 \sim n_\gamma \epsilon \sim n_\gamma k_B T \sim \underline{\frac{(k_B T)^4}{(\hbar c)^3}}$$

b) We have

$$\rho_\gamma c^2 \propto T^4 \quad \text{and} \quad \rho_\gamma c^2 \propto a^{-4},$$

so $T^4 \propto a^{-4} \Rightarrow T \propto \frac{1}{a}$

and normalized for the CMB temperature today we can write

$$T = T_0 \frac{a_0}{a}$$

Now from the definition of the redshift, $1 + z = \frac{a_0}{a}$, so

$$\underline{\underline{T = T_0 (1 + z)}}$$

c) The total energy of the CMB within the observable universe must be

$$E_\gamma = \rho_\gamma c^2 \cdot V$$

$$\approx \frac{(k_B T)^4}{(\hbar c)^3} \cdot \frac{4\pi}{3} \underbrace{\left(\frac{c}{H_0}\right)^3 \left(\frac{a}{a_0}\right)^3}_{\text{Radius increases } \propto a}$$

$$\sim \frac{(k_B T_0)^4}{(\hbar c)^3} \left(\frac{a_0}{a}\right)^4 \cdot V_0 \left(\frac{a}{a_0}\right)^3$$

$$= V_0 \frac{(k_B T_0)^4}{(\hbar c)^3} \left(\frac{a_0}{a}\right) \equiv \underline{\underline{E_{\gamma,0} \frac{a_0}{a}}}$$

with $V_0 \equiv \frac{4\pi}{3} \left(\frac{c}{H_0}\right)^3$

d) We have

$$\dot{E}_\gamma = \frac{d}{dt} E_\gamma = E_{\gamma,0} a_0 \frac{d}{dt}\left(\frac{1}{a}\right)$$

$$= - E_{\gamma,0} a_0 \frac{\dot{a}}{a^2} = - \underbrace{E_{\gamma,0} \frac{a_0}{a}}_{= E_\gamma} \cdot \underbrace{\frac{\dot{a}}{a}}_{= H} = \underline{\underline{- E_\gamma H}}$$

e) With $H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1}$:

$$H_0 = 7 \cdot 10^1 \cdot \frac{10^3 \text{ m s}^{-1}}{10^6 \cdot 3 \cdot 10^{16} \text{ m}} \sim 2 \cdot 10^{-18} \text{ s}^{-1}$$

$$E_{\gamma,0} = \frac{4\pi}{3} \left(\frac{c}{H_0}\right)^3 \frac{(k_B T_0)^4}{(\hbar c)^3}$$

$$\sim 4 \cdot \left(\frac{3 \cdot 10^8 \text{ m s}^{-1}}{2 \cdot 10^{-18} \text{ s}^{-1}}\right)^3 \cdot \frac{(10^{-23} \text{ J/K} \cdot 3 \text{ K})^4}{(10^{-34} \text{ Js} \cdot 3 \cdot 10^8 \text{ m s}^{-1})^3}$$

$$\sim 5 \cdot 10^{65}\,\mathrm{J} = 5 \cdot 10^{65}\,\mathrm{Ws} \approx \frac{5 \cdot 10^{65}}{4 \cdot 10^{3}}\,\mathrm{Wh}$$

$(1\,\mathrm{h} \sim 4 \cdot 10^{3}\,\mathrm{s})$

$$\sim 10^{62}\,\mathrm{Wh} \sim \underline{\underline{10^{50}\,\mathrm{TWh}}}$$

(Total energy consumption in Norway in 2021 was 326 TWh (SSB), out of which 138 TWh was electricity)

f) Energy loss rate today, at $t = t_0$:

$$\dot{E}_{\gamma,0} = -E_{\gamma,0} H_0$$

$$\approx -5 \cdot 10^{65}\,\mathrm{Ws} \cdot 2 \cdot 10^{-18}\,\mathrm{s}^{-1}$$

$$= \underline{\underline{-10^{48}\,\mathrm{W} = -10^{36}\,\mathrm{TW}}}$$

g) Energy lost since recombination:

$$\Delta E = E_{\gamma,\mathrm{rec}} - E_{\gamma,0}$$

$$= E_{\gamma,0} \frac{a_0}{a_{\mathrm{rec}}} - E_{\gamma,0}$$

$$= E_{\gamma,0} (1 + z_{\mathrm{rec}}) - E_{\gamma,0}$$

$$= z_{\mathrm{rec}} E_{\gamma,0} \sim 10^{3} \cdot 10^{50}\,\mathrm{TWh}$$

$$= \underline{\underline{10^{53}\,\mathrm{TWh}}}$$ (What a waste!)