Problem 1: The Copernican principle

The Copernican principle states that our location in the Universe is not special. This principle can be everything from reasonable to obviously untrue, depending on what is meant by ‘special’. Here we will take it to mean that any observer in any other galaxy would find the Universe to have the same large-scale properties as we do. Assume that the principle is valid, and that the mass distribution in the Universe is isotropic as observed from our location. Show that the mass distribution must then also be homogeneous.

Problem 2: Newtonian gravitational potential energy of point masses

Look back at notes from your earlier courses and review the derivation of the gravitational potential energy of two point masses, m_1 and m_2, separated by a distance r:

$$V(r) = -\frac{Gm_1m_2}{r}.$$

Problem 3: The gravitational force of a sphere on a point particle

Show that the gravitational force exerted on a point particle of mass m by a spherical mass distribution (of finite spatial extent) with mass M is the same as if the whole mass M was concentrated in the centre. (Hint 1: Divide the mass distribution into spherical shells and show the result first for one such shell. Hint 2: Work with the potential energy instead of the force, and use $\vec{F} = -\nabla V$ to derive the force). What would the force between two spherical mass distributions be? Why is this result important?

Problem 4: The gravitational force inside a spherical shell

Show that a spherical shell of mass M exerts no force on a point mass m_1 located inside it.

Problem 5: An alternative law of gravity

Assume that you live in a universe where the gravitational potential energy of two point masses m_1 and m_2 is given by

$$V(r) = \frac{1}{2} K m_1 m_2 r^2,$$

where r is their separation, and K is a constant. Show that the result from Problem 3 still holds: The force between a spherical mass and a point particle is the same as if the whole mass of the sphere was concentrated in its centre.

Problem 6: Closed planetary orbits

In this problem you will look at a planet of mass m orbiting our Sun with mass M. Because the angular momentum is conserved in a central force field, the motion is confined to a plane and you can use polar coordinates (r, ϕ) where r is the distance bewteen the planet and the Sun.

a) Write down an expression for the angular momentum L of the planet.

b) Write down an expression for the total energy per mass, E/m of the planet.

c) Show that

$$\frac{1}{2}\dot{r}^2 = \frac{E}{m} - \frac{V_{\text{eff}}}{m},$$

where $\dot{r} = dr/dt$, and

$$\frac{V_{\text{eff}}(r)}{m} = -\frac{GM}{r} + \frac{(L/m)^2}{2r^2}$$

is known as the effective potential.

The square of the radial velocity must obviously be non-negative, so the energy per mass must be at least equal to the minimum of the effective potential V_{eff}/m.

d) What will the orbit look like if E/m equals the minimum value of V_{eff}/m?

For small deviations from the minimum, we can describe the motion of the planet as a combination of radial and angular oscillations. The frequency of the radial oscillations can be derived by approximating the effective potential by a quadratic funtion near its minimum.

e) Show that the radial oscillation frequency is given by

$$\omega^2 = \left(\frac{d^2(V_{\text{eff}}/m)}{dr^2}\right)_{r=r_0},$$

where r_0 is the position of the minimum of the effective potential.

f) Show that L/m is given in terms of r_0 by

$$\left(\frac{L}{m}\right)^2 = GMr_0$$

g) Find the frequency of the radial oscillations in terms of G, M, and r_0.

h) Use the fact that $L/m = r^2\omega_\phi$, where ω_ϕ is the frequency of the angular oscillations, to find ω_ϕ.

i) What is the ratio of the two frequencies? Is the orbit closed (i.e. the planet returns to the same position after a finite number of oscillations)?

j) Carry out the same analysis if we replace the Newtonian gravitational potential by $V(r)/m = \frac{1}{2}KMr^2$, where K is a constant. Do we get closed orbits in this case?

k) Repeat the analysis with $V(r)/m = \frac{1}{3}Ar^3$. Are the orbits closed?

It can be shown that the only two potential energy functions for which all bounded orbits are also closed, are $V \propto 1/r$ and $V \propto r^2$. This result is known as Bertrand's theorem. It can also be shown that the same two potentials are the only ones with the property that the field outside a spherical mass distribution is as if all its mass were concentrated in the centre. One lesson to learn from this is that we can add a term linear in r (which gives a potential energy quadratic in r) to Newton's law of gravity without screwing up important properties of gravity. This will be relevant when we will later learn about the cosmological constant.

Problem 7: Dark stars vs. black holes

a) Show that the radius of a star whose escape velocity from the surface equals the speed of light c is given by

$$R \equiv R_s = \frac{2GM}{c^2},$$

and (without using a calculator) that this result can also be written as

$$R_s = 3\frac{M}{M_\odot}\ \mathrm{km}$$

where $M_\odot$ is the mass of the Sun.

By coincidence, the result in a) is identical with the Schwarzschild radius, which sets the size of a black hole. This coincidence is often exploited in popular accounts of black holes to explain them as objects that have escape velocity equal to the speed of light. Such objects were first speculated about by John Michell in a paper in 1784 with the catchy title *On the Means of Discovering the Distance, Magnitude, & c. of the Fixed Stars, in Consequence of the Diminution of the Velocity of Their Light, in Case Such a Diminution Should be Found to Take Place in any of Them, and Such Other Data Should be Procured from Observations, as Would be Farther Necessary for That Purpose.* He called these objects *dark stars.*

An important assumption in Michell's paper (and in a)) is that motion of light in a gravitational field can be treated like the motion of any other particle, so the equation for purely radial motion of light is

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$

b) Assume that light starts out at the surface of the star (with radius R, not necessarily equal to R_s) with speed c and show that its speed at a distance r from the centre of the star is given by

$$v^2(r) = c^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right).$$

c) Show that for $R < R_s$ the light reaches a maximum distance from the center of the star given by

$$r_{\max} = \frac{R_s}{\frac{R_s}{R} - 1}.$$

d) Michell's original dark star had the same mean mass density as the Sun. Without using a calculator, estimate the mass and radius of this dark star.

e) Based on this problem, would you say that the popular account of black holes as objects with escape velocities exceeding the speed of light is a good one?