Problem 1

One-digit accuracy suffices in the following questions, so try to not use a calculator, just pen and paper.

Hubble-Lemaitre's law at our present time is

$$v = H_0 d,$$

where v is the radial velocity, d is the distance, and H_0 is the Hubble constant, the present value of the Hubble parameter $H(t) = \dot{a}/a$. We will use the value $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.

a) Use the non-relativistic Doppler formula to show that the redshift of a galaxy at a distance d from us is

$$z = \frac{H_0 d}{c}.$$

Galaxies often belong to a cluster of galaxies. Within a cluster gravitational forces from the other galaxies may give a galaxy a significant velocity. This may give an additional contribution to its redshift, and observationally it is impossible to separate this redshift from the redshift caused by the expansion of the Universe. The velocity caused by the other galaxies is typically of order 100 km/s.

b) At what distance does this effect become insignificant, in the sense that the expansion velocity is ten times larger? What redshift does this correspond to?

Another effect one might worry about is the gravitational redshift. According to general relativity, the wavelength λ_0 observed far away from a spherical source emitting radiation of wavelength λ_e from its surface is given by

$$1 + z = \frac{\lambda_0}{\lambda_\mathrm{e}} = \frac{1}{\sqrt{1 - \frac{R_\mathrm{s}}{R}}},$$

where R is the radius of the source, and $R_\mathrm{s} = 2GM/c^2$ is its Schwarzschild radius.

c) Find reasonable values for the mass and radius of a typical galaxy (you can, for example, use the values for the Milky Way system) (don't worry about the fact that they are usually not spherical), and estimate the gravitational redshift.

Problem 2 (variation on a problem from the exam in AST4220, 2006)

I found the following problem in a physics textbook for the upper secondary school: 'A quasar is observed at redshift $z = 3.78$. In special relativity the relationship between z and the speed at which the quasar is moving away from us is given by

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

a) Calculate the quasar's speed.

b) Use Hubble's law $v = H_0 d$ to calculate the distance d to the quasar. Let $H_0 = 16$ kilometers per second per million lightyears.'

Pretend that you are a gullible student and answer question a) and b).

c) Which serious error are the authors of this textbook guilty of? What would you consider to be a suitable punishment for them? (Please don't answer: 'To attend Øystein's lectures.')

d) Assume that space is flat, and that the scale factor is given by $a = a_0(t/t_0)^{2/3}$ and calculate the proper distance d_P to the quasar. Use the same value of H_0 as in b) and compare the results.

Problem 3 (variation on a problem from the exam in AST4220, 2007)

Light emitted from an object at time t_e and received by an observer at a later time t has a cosmological redshift

$$1 + z = \frac{a(t)}{a(t_\mathrm{e})}.$$

In this exercise you will calculate how this redshift changes with the time of observation t.

a) Show that

$$\frac{dz}{dt} = \frac{a(t)}{a(t_\mathrm{e})}\frac{\dot{a}(t)}{a(t)} - \frac{a(t)}{a(t_\mathrm{e})}\frac{\dot{a}(t_\mathrm{e})}{a(t_\mathrm{e})}\frac{dt_\mathrm{e}}{dt}.$$

b) The comoving radial coordinate of the object is given by

$$r = \int_{t_e}^{t} \frac{c dt'}{a(t')}.$$

Use the fact that $dr/dt = 0$ (why?), assume the Universe to be spatially flat and the scale factor to be given by $a(t) = a_0(t/t_0)^{2/3}$, and show that

$$\frac{dt_e}{dt} = \frac{1}{1+z}.$$

This result can be shown to be valid in general.

c) Use the results in a) and b) to show that

$$\frac{dz}{dt} = (1+z)H(t) - H(t_e).$$

d) For the model in b), what is the value of dz/dt at the present epoch ($t = t_0$) for an object with $z = 4$?

e) For the same object, calculate dz/dt at $t = t_0$ if the Universe is described by a spatially flat model with $a = a_0 e^{H_0(t-t_0)}$. Compare with the result in d).

Problem 4 (from the midterm exam in AST4220, 2007)

The so-called deceleration parameter is defined by

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2}.$$

a) Find q for a spatially flat universe with $a(t) = a_0(t/t_0)^{2/3}$.

b) Show that

$$\frac{d}{dt}\frac{1}{H} = 1 + q.$$

c) Show that

$$\frac{d}{dt}\frac{1}{H} = \frac{1+z}{H}\frac{dH}{dz}.$$

Hint: Use the chain rule, first to replace d/dt with d/da, then to replace d/da with d/dz.

d) Use b) and c) to show that

$$H(z) = H_0 \exp\left[\int_0^z \frac{1+q(z')}{1+z'}dz'\right].$$

e) Find an expression for $H(z)$ when $q = q(z=0) = q_0 =$ constant.