Problem 1

Assume a spatially flat universe ($k = 0$) with scale factor given by

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}.$$

Here t_0 is the present cosmic time, and a_0 is the present value of the scale factor. We observe an object at cosmic redshift $z = 3$.

a) Calculate the comoving coordinate r of the object and its proper distance from us at $t = t_0$.

b) The radiation we receive from the object contains a message from an advanced civilization. We wish to send a radio signal back to them. If we send it at t_0, at what time (in units of t_0) will our signal reach them?

Problem 2

a) Show that $a(t) = ct$ is a solution of the Friedmann equations for a completely empty universe ($\rho = p = 0$) if $k = -1$.

b) Find expressions for the proper distance d_p and the angular diameter distance d_A as functions redshift z for this model.

Problem 3

Suppose we modify Newtonian gravity by adding a term to the potential energy of two point masses m_1 and m_2 of the form

$$V(r) = -\frac{\alpha G m_1 m_2}{r} e^{-r/\lambda}$$

where r is the distance between m_1 and m_2, and α and λ are positive constants (they set the strength and the range of the correction, respectively).

a) Show that the contribution to potential energy from to this term from a spherical shell of mass M and radius R, and a point mass of unit mass at a distance r from the centre of the shell is given by

$$V(r) = -\frac{\alpha\lambda G M}{rR} \sinh\left(\frac{R}{\lambda}\right) e^{-r/\lambda}$$

if $r > R$, and by

$$V(r) = -\frac{\alpha\lambda GM}{rR} e^{-R/\lambda} \sinh\left(\frac{r}{\lambda}\right)$$

if $r < R$.

b) Let us make a Newtonian cosmological model where the Universe is a sphere of uniform density ρ and radius R. Use the results from a) to show that in the limit of an infinite universe, $R \to \infty$, the additional term in the potential energy makes no difference to the dynamics. Hint: Calculate the potential energy of a unit mass at a point r within the sphere by dividing the sphere into spherical shells.

Problem 4

Use the Friedmann equation for $\ddot{a}$ with a cosmological constant to find the equation for the time evolution of a small, time dependent perturbation η around the Einstein static solution $a = a_0 = c/\sqrt{\Lambda}$, and use this equation to show that the Einstein model is unstable.

Problem 5

As I mentioned in the lectures, energy is in general not conserved in an expanding universe. In particular, the cosmic microwave background (CMB) loses energy. In this problem you will estimate roughly the rate at which it is presently losing energy, and how much energy it has lost since it decoupled from matter in the epoch called recombination (more on that later in the course).

First, we need to know the energy density of the CMB. We assume that the photons are in thermal equilibrium at a temperature T which may vary with time.

a) Explain why the wavelength of a typical photon is $\lambda \sim \frac{\hbar c}{k_B T}$ where k_B is Boltzmann's constant. Argue from this that the number density of photons will roughly be

$$n_\gamma \sim \left(\frac{k_B T}{\hbar c}\right)^3$$

and that the energy density is

$$\rho_\gamma c^2 \sim \frac{(k_{\rm B}T)^4}{(\hbar c)^3}$$

b) The solution of the continuity equation for radiation ($w = \frac{1}{3}$) is $\rho_\gamma \propto a^{-4}$. Combine this with the result from a) and show that

$$T = T_0 \frac{a_0}{a} = T_0(1+z),$$

where $T_0 \sim 3$ K is the present temperature of the CMB.

c) Let us take the radius of the observable universe today to be $\sim c/H_0$. Show that the total energy of the CMB contained in the observable universe is

$$E_\gamma \sim V_0 \frac{(k_{\rm B}T_0)^4}{(\hbar c)^3}\frac{a_0}{a} \equiv E_{\gamma,0}\frac{a_0}{a}$$

where

$$V_0 = \frac{4\pi}{3}\left(\frac{c}{H_0}\right)^3$$

d) Show that

$$\dot{E}_\gamma = -E_\gamma H$$

e) Calculate $E_{\gamma,0}$. Use $H_0 = 70\ {\rm km\,s^{-1}\,Mpc^{-1}}$. To give the result a contemporary spin, give the result in units of TWh.

f) Calculate the rate at which the CMB is currently losing energy.

g) The CMB started to propagate freely during an epoch called recombination, which took place at a redshift $z_{\rm rec} \sim 10^3$. How much energy has the CMB lost between that time and today?