Problem 1: Alternative form of the Hubble-Lemaitre law

Consider light emitted by a galaxy at comoving radial coordinate r at time t_e, received by us at the origin at time t_0.

a) Show that the motion of the light satisfies the equation

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_{t_e}^{t_0} \frac{cdt}{a(t)}.$$

b) Assume $r \ll 1$ and $(t_0 - t_e)/t_0 \ll 1$, and show that the equation in a) becomes

$$r \approx \frac{c(t_0 - t_e)}{a(t_e)}.$$

c) Taylor expand $a(t)$ around t_0 and show that

$$a(t) \approx a(t_0)[1 - H_0(t_0 - t_e)].$$

d) Using c), show that the redshift of the galaxy satisfies

$$z \approx H_0(t_0 - t_e).$$

e) Combine the results above to show that

$$cz \approx H_0 d_{\mathrm{P}}(t_0)$$

for $z \ll 1$.

f) Finally, argue that in this limit $d_{\mathrm{P}}(t_0) \approx d_{\mathrm{L}}$. Hence, for low redshift the Hubble-Lemaitre law can be formulated in terms of measurable quantities, without any reference to the speed of a galaxy.

Problem 2: The Einstein Universe

If the density of matter ρ_0 is $10^{-27}\ \mathrm{kg\,m^{-3}}$, how long would it take a ray of light to travel once around Einstein's static universe model?

Problem 3: A negative cosmological constant

For the case $\Lambda < 0$ show that the first Friedmann equation has solutions only if $k = -1$. Without solving the equation, show that the Universe will expand to a maximum size, and then collapse. Find the solution $a = a(t)$. Make an informed guess about how old a universe must be in order to contain intelligent life, and use this guess to find an upper bound on $|\Lambda|$.

Problem 4: Exploding universe?

'Phantom energy', a substance with equation of state parameter $w < -1$, has been proposed as an alternative to the cosmological constant for explaining the present accelerated phase of expansion. Assume that we live in a spatially flat universe, dominated by phantom energy with $w = -2$.

a) Determine how the energy density of this component varies with the scale factor a.

b) Integrate the Friedmann equation for $\dot{a}/a$ from our present epoch t_0 ($a(t_0) = a_0$) and into the future to find $a(t)$ for $t > t_0$.

c) What happens as $t - t_0 \to \frac{2}{3H_0}$? Can you see why this is called 'the Big Rip'?

Problem 5 (just for fun, no need to do it if you don't find it interesting)

The Friedmann equations are probably not valid at very early times. When the density becomes very high, we really need a quantum theory of gravity to describe the situation. Although there are candidates, like string theory, for such a theory, they are not understood in detail and lack a solid empirical foundation. However, that doesn't stop theorists from speculating about what a quantum version of the history of the Universe might look like. In this problem we will look at a simplified version of a speculation of this sort.

We will use units where $\hbar = c = 1$. This may be unfamiliar to you, but it is a very convenient choice for the calculations we are going to carry out here.

We will stick to one specific model of the Universe where the only contribution to the energy density comes from the cosmological constant Λ, and where the spatial curvature $k = +1$. The first Friedmann equation can then be written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{\Lambda}{3}.$$

We will first consider some of the properties of this model.

a) Explain why the scale factor can never be smaller than

$$a_\Lambda = \sqrt{\frac{3}{\Lambda}}.$$

b) How old can the Universe be in this model?

c) Show that the scale factor can be written as

$$a(t) = a_\Lambda \cosh\left(\frac{t}{a_\Lambda}\right).$$

Quantum gravity was studied long before string theory came along. One early approach to the problem is the so-called Wheeler-De Witt equation. This is a Schrödinger-like equation for the wave function of the Universe. The wave function of the Universe gives the probability density for observing different metrics. If we restrict the space of possible metrics to those of the Robertson-Walker form with a given spatial curvature k, the wave function gives the probability amplitude for observing the Universe in a state with a particular value of the scale factor a. There are many serious issues that can be raised at this point, for example whether it makes sense at all to talk about a wave function for the entire Universe. We will leave this and other questions aside here.

The Wheeler-De Witt equation for the wave function $\psi(a)$ for the model we considered in a), b) and c) is

$$-\frac{d^2\psi(a)}{da^2} + \frac{9\pi^2}{4G^2}\left(a^2 - \frac{\Lambda}{3}a^4\right)\psi(a) = 0.$$

This looks exactly like the one-dimensional, time-independent Schrödinger equation of a particle moving along the a-axis in a potential

$$V(a) = \frac{9\pi^2}{4G^2}\left(a^2 - \frac{\Lambda}{3}a^4\right) = \frac{9\pi^2 a_\Lambda^2}{4G^2}\left[\left(\frac{a}{a_\Lambda}\right)^2 - \left(\frac{a}{a_\Lambda}\right)^4\right].$$

Note that understood this way, the equation corresponds to an energy eigenvalue equal to zero, making the wave function time-independent. Therefore, time does not appear in the wave function of the Universe. This is not specific to the model we are considering, it is a general fact that the Wheeler-De Witt equation does not contain time. This aspect of the formalism is not easy to understand, but we will again leave this aside and carry out some calculations instead.

d) Make a sketch of the potential V as a function of a. Mark classically allowed and forbidden regions for the scale factor.

e) Change variable in the equation, from a to $x = a/a_\Lambda$.

From quantum mechanics we know that a particle has a non-zero probability for tunneling through a classically forbidden region. Analogously, the Universe can tunnel from a state where the scale factor is zero to a state where it is greater than a_Λ. The potential above is a bit complicated to treat analytically, but we can capture the most important qualitative aspects by looking at a simplified model where the potential is given by

$$\begin{aligned} V(x) &= \infty, \ x < 0 \\ &= 0, \ 0 < x < 1 \\ &= V_0, \ 1 < x < 2 \\ &= -V_1, \ x > 2 \end{aligned}$$

where

$$V_0 = \frac{9\pi^2 a_\Lambda^4}{16G^2},$$

and V_1 is a positive constant.

e) Find the general solution of the equation for ψ in the four regions.

f) Use the conditions that ψ and its first derivative have to be continuous to find equations for the unknown constants of integration.

g) Find an expression for the transmission probability amplitude, that is, the probability amplitude for tunnelling from the region $0 < x < 1$ to $x > 2$.

h) Let $V_1 \to \infty$ and show that the probability of tunneling is proportional to $e^{2\sqrt{V_0}}$.

i) Why do you think the probability in h) is sometimes called 'the probability of the Universe appearing from nothing'?