Problem 1

You will sometimes hear or read that the accelerated expansion of the Universe means that the Hubble parameter increases with time.

a) Consider a model where $k = 0$ and $a(t) = Ct^2$, where $C > 0$ is a constant. Does the expansion accelerate in this model? Does the Hubble parameter increase with time?

b) Consider the de Sitter solution, $a = a_0 \exp[H_0(t - t_0)]$. Does the expansion accelerate? Does the Hubble parameter increase with time?

c) Prove: In any spatially flat model with $\rho + p/c^2 > 0$, the Hubble parameter decreases with time. (Hint: Consider $dH/dt = \frac{d}{dt}\frac{\dot{a}}{a}$ and use both the first and the second Friedmann equation.)

Problem 2

a) Solve the first Friedmann equation for an empty universe with $k = -1$.

b) Do the same for a spatially flat universe filled with a substance with equation of state parameter $w = -\frac{1}{3}$. Compare with a).

c) Show that it is possible to distinguish between the two models by measuring luminosity distances.

Problem 3

a) Show that the age of the Universe in models with non-relativistic matter and spatial curvature can be written as

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{\mathrm{m0}}(1+z)^3 + (1-\Omega_{\mathrm{m0}})(1+z)^2}}.$$

b) Use this result to show that

$$t_0 < \frac{1}{H_0}$$

for $\Omega_{\mathrm{m0}} > 0$. (Hint: Show that the age is a strictly decreasing function of Ω_{m0}.)

Problem 4

a) Show that the first Friedmann equation for models with non-relativistic matter, a cosmological constant, and spatial curvature can be written as

$$\left(\frac{dx}{d\tau}\right)^2 + \left[-\left(\frac{\Omega_{\mathrm{m0}}}{x} + \Omega_{\Lambda 0} x^2\right)\right] = \Omega_{\mathrm{k0}} = 1 - \Omega_{\mathrm{m0}} - \Omega_{\Lambda 0} = \text{constant}$$

where $x = a/a_0$ and $\tau = H_0 t$.

b) Explain why this means that

$$U(x) = -\left(\frac{\Omega_{\mathrm{m0}}}{x} + \Omega_{\Lambda 0} x^2\right) \leq \Omega_{\mathrm{k0}}$$

c) Plot $U(x)$ for the following choices of parameters:

1. $\Omega_{\mathrm{m0}} = 0.3$, $\Omega_{\Lambda 0} = 0.7$
2. $\Omega_{\mathrm{m0}} = 2.0$, $\Omega_{\Lambda 0} = -0.5$
3. $\Omega_{\mathrm{m0}} = 0.5$, $\Omega_{\Lambda 0} = 3.0$

Use the graphs to discuss the expansion history of the Universe in each case.

d) In the so-called DGP model the behaviour of gravity is modified by the presence of a large extra dimension. The analogue of the first Friedmann equation is given by

$$\frac{H^2(z)}{H_0^2} = \left(\sqrt{\Omega_{\mathrm{m0}}(1+z)^3 + \Omega_{\mathrm{r_c}}} + \sqrt{\Omega_{\mathrm{r_c}}}\right)^2 + \Omega_{\mathrm{k0}}(1+z)^2$$

The parameter $\Omega_{\mathrm{r_c}}$ does not represent an energy density, but is a new parameter related to the distance scale at which the large extra dimension starts to make its presence felt.

Take $z = 0$, and find the relation between Ω_{k0}, Ω_{m0}, and $\Omega_{\mathrm{r_c}}$.

e) With $x = a/a_0$, $\tau = H_0 t$, show that

$$\left(\frac{dx}{d\tau}\right)^2 + \left[-x^2\left(\sqrt{\Omega_{\mathrm{m0}} x^{-3} + \Omega_{\mathrm{r_c}}} + \sqrt{\Omega_{\mathrm{r_c}}}\right)^2\right] = \Omega_{\mathrm{k0}}$$

f) Following the same approach as in c), see if you find combinations of Ω_{m0} and $\Omega_{\mathrm{r_c}}$ that gives expansion histories similar to those you found in c).