Problem 1

a) Prove Mattig's formula:

$$d_{\rm L} = \frac{2c}{H_0\Omega_{\rm m0}^2}\left[\Omega_{\rm m0}z + (\Omega_{\rm m0}-2)(\sqrt{1+\Omega_{\rm m0}z}-1)\right]$$

for any $\Omega_{\rm m0}$ in a model with just non-relativistic matter. (Hint: This involves quite a bit of work, but in the case $\Omega_{\rm m0} = 1$, you just need to check that the formula reduces to the result for the EdS model. Next, you may want to look at $\Omega_{\rm m0} < 1$ first. Rewrite the integrand slightly and use the substitution $u = \sqrt{1+\Omega_{\rm m0}z}$. Finally, you should be able to argue that the result for $\Omega_{\rm m0} > 1$ follows quite straightforwardly from the result for $'\Omega_{\rm m0} < 1$.)

b) Show that in the limit $\Omega_{\rm m0} \to 0$ we get the result for the Milne universe.

Problem 2

a) Specialize the expression we derived for the luminosity distance in the lectures to a spatially flat universe containing non-relativistic matter and vacuum energy and show that it can be written as

$$d_{\rm L} = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{\rm m0}(1+z')^3 + 1 - \Omega_{\rm m0}}},$$

where $\Omega_{\rm m0}$ is the density parameter for non-relativistic matter (dust).

b) Evaluate the integral for i) $\Omega_{\rm m0} = 1$ and ii) $\Omega_{\rm m0} = 0$.

c) Show that in the limit $z \ll 1$ $d_{\rm L}$ is approximately given by

$$d_{\rm L} \approx \frac{cz}{H_0},$$

and is therefore independent of $\Omega_{\rm m0}$.

d) Introduce $s = [(1-\Omega_{\rm m0})/\Omega_{\rm m0}]^{1/3}$, and use the substitution $u = s/(1+z')$ to show that the integral in a) can be written as

$$\frac{H_0 d_L}{c} = \frac{1+z}{\sqrt{s\Omega_{\rm m0}}}\left[T(s) - T\left(\frac{s}{1+z}\right)\right]$$

where

$$T(s) = \int_0^s \frac{du}{\sqrt{u^4 + u}}$$

Fluxes in astronomy are (sadly) usually quoted in terms of *magnitudes*. Magnitudes are related to fluxes via $m = -\frac{5}{2}\log(F) + \text{constant}$, where log denotes the logarithm with base 10. The *apparent magnitude* is the flux we observe here on Earth, whereas the *absolute magnitude* is the flux we would receive if the source was at a distance of 10 pc from us. They are related by

$$m - M = 5\log\left(\frac{d_{\rm L}}{10\ {\rm pc}}\right),$$

If we know both m and M for a source, we can infer its luminosity distance. Objects of known M are called *standard candles*. Supernovae of type Ia are believed to be standard candles.

e) Supernova 1997ap was found at redshift $z = 0.83$ with apparent magnitude $m = 24.32$, and Supernova 1992P was observed at low redshift $z = 0.026$ with apparent magnitude $m = 16.08$. Assuming they both have the same absolute magnitude, show that the luminosity distance to Supernova 1997ap is given by

$$d_{\rm L}(z = 0.83) = 1.16\frac{c}{H_0}.$$

f) Compare the result from d) with the two models considered in b).

g) Pen (U.-L. Pen, ApJS 120 (1999) 49) showed that a good approximation to the function T in d) is given by

$$T(x) \approx \frac{2\sqrt{x}}{(1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)^{1/8}},$$

with $a_1 = -0.1540$, $a_2 = 0.4304$, $a_3 = 0.19097$, and $a_4 = 0.066941$. Use this to write a code which calculates the luminosity distance in the spatially flat ΛCDM model for any $\Omega_{\rm m0}$ and z, and find the value of $\Omega_{\rm m0}$ that reproduces the measured luminosity distance to Supernova 1997ap.

h) Use Mattig's formula from problem 1 to investigate whether there is any value of $\Omega_{\rm m0}$ in a universe without a cosmological constant that will reproduce the luminosity distance to Supernova 1997ap. It may be a good idea to do this numerically.

h) What does the empty Milne model predict for the value of $H_0 d_{\rm L}(z = 0.83)/c$?

i) Discuss: Based on this single measurement of $d_{\rm L}$, which model should we prefer? Can we conclude that we need the cosmological constant?

Problem 3

Consider universe models with dust (non-relativistic matter) and a cosmological constant. Give a graphical argument for why, for fixed H_0, increasing $\Omega_{\rm m0}$ will decrease t_0, the present age of the Universe, while increasing $\Omega_{\Lambda 0}$ will increase it. Hint: The second Friedmann equation is useful here.

Problem 4: Another bonus problem for those who are interested

In the lectures we derived the de Sitter solution,

$$a(t) = a_0 e^{H_0(t-t_0)},$$

for a spatially flat universe dominated by the cosmological constant ($\Omega_{\Lambda 0} = 1$). In this model, $a(t) \to 0$ only in the limit $t \to -\infty$, so this universe seems to be infinitely old and has no beginning in time. But is this really so? We will now go through the steps of an argument which shows that the answer is no.

In the following we will only consider models with $k = 0$, so the RW line element is given by

$$ds^2 = c^2 dt^2 - a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2).$$

The spatial part of the line element is now the line element of 3D Euclidean space, written in spherical coordinates. We can just as well write it in Cartesian coordinates, and this will simplify the calculations in this problem. Furthermore, to save some writing we will work in units where $c = 1$. The line element is therefore

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) = dt^2 - a^2(t)d\mathbf{x} \cdot d\mathbf{x}. \tag{1}$$

Note that $\mathbf{x} = (x^1, x^2, x^3) = (x, y, z)$ are still co-moving coordinates, and hence constant for observers who follow the expansion.

a) *Proper time*, τ, for an observer is the time measured on a clock following her/his motion. Explain why $\tau = t$ for an observer who follows the expansion.

Co-moving observers follow *geodesics*: trajectories of particles falling freely (i.e., not subject to non-gravitational forces). These are, however, not the only type of geodesics. We will now set up the geodesic equation, an equation that describes freely falling particles and is the analogue in general relativity to Newton's second law in classical mechanics. First, I introduce the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix} \tag{2}$$

and write $dx^\mu = (dt, dx, dy, dz) = (dt, d\mathbf{x})$, where the indices μ and ν can have the values $0, 1, 2, 3$. This means that $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -a^2(t)$, and $g_{\mu\nu} = 0$ when $\mu \neq \nu$. We will also use the Einstein summation convention which says that repeated indices are to be summed over, so that, e.g.,

$$g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu.$$

b) Show that $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$.

Let's look at Newtonian mechanics for a while. A particle of mass m moves along the x-axis and has potential energy $V(x)$. Its kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$. We now introduce *the Lagrangian*

$$L = L\left(x, \frac{dx}{dt}\right) = T - V,$$

where x and $\frac{dx}{dt}$ are to be considered as independent variables.

c) Show that *the Lagrange equation*

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx}{dt}\right)}\right) = 0$$

gives the Newtonian equation of motion

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}.$$

The equation of motion of freely falling particles, the geodesic equation, in GR can also be derived from a Lagrangian. Recall that in GR, gravity is not considered to be a force, so $V = 0$ in this case. Therefore $L = T$, but now T is given by

$$T = \frac{1}{2}mg_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau},$$

where τ is proper time along the geodesic. The Lagrange equation is

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\right) = 0,$$

so there will an equation for each value of $\mu = 0, 1, 2, 3$, in total four equations. The solution will be of the form $x^\mu = x^\mu(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$, i.e., each of the four coordinates given as a function of proper time.

d) For the RW metric, show that

$$L = \frac{1}{2}m\left\{\left(\frac{dt}{d\tau}\right)^2 - a^2(t)\left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right]\right\}.$$

e) Show that the geodesic equations are

$$\frac{d^2t}{d\tau^2} + a\dot{a}\frac{d\mathbf{x}}{d\tau}\cdot\frac{d\mathbf{x}}{d\tau} = 0 \tag{3}$$

$$\frac{d^2x^i}{d\tau^2} + 2H\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0$$

where $i = 1, 2, 3$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $d\mathbf{x} = (dx, dy, dz)$, and $H = \dot{a}/a = \frac{1}{a}\frac{da}{dt}$ is the Hubble parameter.

As in special relativity, we define the four-momentum of the particle as

$$P^\mu = (E, \mathbf{P}) = m\frac{dx^\mu}{d\tau}.$$

Note that

$$E = m\frac{dt}{d\tau},$$

so

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \frac{E}{m}\frac{d}{dt}.$$

f) Show that equation (3) can be rewritten as

$$E\frac{dE}{dt} + a\dot{a}\mathbf{P}\cdot\mathbf{P} = 0. \tag{4}$$

g) Like in special relativity, the four-momentum satisfies

$$g_{\mu\nu}P^\mu P^\nu = P^\mu P_\mu = m^2.$$

Let $E = \sqrt{p^2 + m^2}$, where p is the physical momentum, and show that

$$a^2\mathbf{P}\cdot\mathbf{P} = E^2 - m^2 = p^2.$$

This means that $\mathbf{P}$ is the co-moving momentum.

h) Show that equation (4) now can be written as

$$E\frac{dE}{dt} + \frac{1}{a}\frac{da}{dt}(E^2 - m^2) = 0.$$

Solve this differential equation and show that it gives

$$p \propto \frac{1}{a}.$$

The last result implies that we can write

$$p(t) = \frac{a(t_f)}{a(t)}p_f,$$

where p_f is the physical momentum at some reference time t_f.

i) Start from

$$d\tau = \frac{m}{E} dt,$$

Multiply by $H(\tau) = \dot{a}/a$ on both sides (we can consider the Hubble parameter to be a function of τ since $t = t(\tau)$ along the geodesic), and show that

$$\int_{\tau_i}^{\tau_f} H(\tau) d\tau = \int_{a_i}^{a_f} \frac{da}{\sqrt{a^2 + \frac{p_f^2 a_f^2}{m^2}}}, \tag{5}$$

where a_i is the value of the scale factor at some initial time t_i, $\tau_i = \tau(t_i)$, and $\tau_f = \tau(t_f)$ (Hint: The relation $\dot{a} = \frac{da}{dt}$ is trivial, but very useful).

j) Evaluate the integral on the right-hand side in equation (5) and show that

$$\int_{\tau_i}^{\tau_f} H(\tau) d\tau \leq \ln\left(\frac{m + E_f}{p_f}\right), \tag{6}$$

where $E_f = \sqrt{p_f^2 + m^2}$. (Hint: Use the substitution $a = \frac{a_f p_f}{m} \sinh x$. You will also need the relation $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$)

k) We define the time-averaged Hubble parameter as

$$H_{\text{ave}} \equiv \frac{1}{\tau_f - \tau_i} \int_{\tau_i}^{\tau_f} H(\tau) d\tau.$$

Equation (6) implies

$$H_{\text{ave}} \leq \frac{1}{\tau_f - \tau_i} \ln\left(\frac{m + E_f}{p_f}\right).$$

Explain why this means that $\tau_f - \tau_i$ must be finite if $H_{\text{ave}} > 0$.

You have now shown that if the Universe has been on average expanding, there will be geodesics that cannot be extended infinitely far back in proper time, so they must end at some point in the past. Since the de Sitter solution is of this type ($H = H_0 = \text{constant} > 0$), this means that it cannot be truly eternal, even though for observers following the expansion, cosmic time can be extended back to $t = -\infty$. For comparison: In a spacetime with a black hole, there are geodesics that don't end up at the singularity (just choose the initial conditions so that they stay away from the event horizon), but the existence of geodesics that *do* end up at the singularity is enough to ensure that this spacetime is singular. So the existence of some geodesics that cannot be extended infinitely far back in time is enough to show that the de Sitter universe is not eternal, or is *past-incomplete* to use more fancy language.

The argument you have now worked your way through is taken from A. Borde, A. H. Guth, and A. Vilenkin: 'Inflationary spacetimes are not past-complete', Phys.Rev.Lett. 90 (2003) 151301, which you can also find at https://arxiv.org/abs/gr-qc/0110012. They also give a much more general proof, valid for spacetimes that do not obey the Cosmological Principle.