Problem 1: The QCD phase transition

In the lectures I talked briefly about the epoch when quarks become confined inside hadrons like protons and neutrons. Quantum Chromodynamics (QCD) is the quantum field theory describing how quarks interact through the exchange of gluons. When the quarks have very high energies, scattering processes etc. can be calculated perturbatively. At low energies, however, the interactions between quarks become so strong that they cannot be treated as small perturbations. This is related to the property of *confinement*: at low energies, quarks (and gluons) are trapped inside hadrons (protons, neutrons, π mesons,...) and do not exist as free particles.

Physicists are interested in when and how this transition from a state with individual, essentially free quarks, to a state where they are confined within hadrons takes place. To investigate this, one must treat QCD non-perturbatively, and this is usually done in large numerical simulations, in an approximation known as lattice QCD.

However, simpler, phenomenlogical models are also used in order to gain physical understanding. One of the simplest models of quark confinement is the MIT Bag Model. In this model, the hadrons are treated like bags. Inside a bag, the quarks can move freely, but they cannot get out of the bag. The confinement is modelled by a so-called bag constant, B, in the region of free quarks, simulating a pressure exerted by the vacuum.

Let us model a hadron as a sphere of radius R. The contribution from the bag pressure to the energy of the hadron is then $BV = 4\pi R^3 B/3$.

a) Explain why a free, relativistic quark in the bag has energy $\propto 1/R$.

Let us therefore write the energy of a hadron as

$$E_H = \frac{4\pi}{3} R^3 B + \frac{C}{R},$$

where C is a constant.

b) Make a qualitative sketch of E_H as a function of R and convince yourself that it has a minimum. Use this to show that

$$C = 4\pi B R^4,$$

and that the minimum energy is

$$E_{H,\text{min}} = \frac{16\pi}{3} R^3 B.$$

c) Equate this to the rest-mass energy of a nucleon, $\sim 10^3$ MeV, and find a value for B if $R = 1$ fm $= 10^{-15}$ m

d) Particle physicists like to use so-called *natural units*, where $\hbar = c = 1$, and in applications of the bag model they also often use the value

$$B^{1/4} = 200 \text{ MeV}.$$

Translate this value for B to "normal" units, and find out which value of R this corresponds to. (The combination $\hbar c = 197.327$ MeV $\cdot$ fm is useful here.)

We will use the value for B from d) in the following.

To model the phase transition from free quarks (often called a quark-gluon plasma) to hadrons, consider the confined and deconfined phases of quarks separately. Assume only u and d quarks are present (the 2nd and 3rd generations of quarks are much heavier, so they have already become non-relativistic and annihilated). Assume that these quarks are relativistic and essentially massless. Near the phase transition, the lightest hadrons dominate, and they are the three spin-0 π mesons: $\pi^{\pm}$ and π^0. Finally, assume that all particles have zero chemical potential.

e) Explain why the bag pressure is

$$P_B = -\frac{\partial}{\partial V}(BV) = -B,$$

and show that in the hadronic phase the pressure is

$$P_H = \frac{\pi^2}{30}\frac{(k_{\mathrm{B}}T)^4}{(\hbar c)^3},$$

while in the deconfined quark-gluon plasma phase the pressure is

$$P_{\mathrm{QGP}} = \frac{37\pi^2}{90}\frac{(k_{\mathrm{B}}T)^4}{(\hbar c)^3} - B.$$

(Hint: Count the degrees of freedom in each phase, remember the difference between fermions and bosons, and don't forget the gluons!)

f) Equate the two pressures and show that the critical temperature is

$$k_{\rm B}T_c \approx 140 \text{ MeV}.$$

(Note: This means that we treat the transition as a 1st order phase transition. Lattice QCD simulations show that the transition is more likely to be of 2nd order, or a crossover).

g) Estimate how long after the Big Bang this transition took place.

Problem 2: Cosmological horizons and entropy

This problem is based on a paper by P. C. W. Davies in Classical and Quantum Gravity, volume 5, page 1349 (1988).

We saw in the lectures that we can associate an entropy with a black hole, and that the entropy is proportional to the area of the event horizon. The background leading up to the notion of black hole entropy was a series of results obtained in the late 1960s by Stephen Hawking and others, establishing that in all processes the area of the event horizon of a black hole never decreases.

We have seen that there are horizons in cosmology as well, so one could ask whether an entropy can be associated with those as well. The situation is more complicated in the cosmological case than in the black hole case, but you will now prove that the area of the event horizon, if it exists, never decreases in an expanding universe provided $p \geq -\rho c^2$ and $a \to \infty$ as $t \to \infty$. We will restrict ourselves to the spatially flat case, but the result is also valid when there is spatial curvature.

a) Define the variable η by

$$\eta = -\int_t^\infty \frac{cdt'}{a(t')}.$$

Show that the proper distance to the event horizon is given by $d_{\rm EH}(t) = -a(t)\eta$, that $\eta \leq 0$, and $\frac{d\eta}{dt} > 0$.

b) Argue that in order to show that the area of the event horizon is non-decreasing, it suffices to show that $\frac{d(d_{\rm EH})}{d\eta} \geq 0$.

c) Use the first Friedmann equation and the continuity equation to show that

$$\dot{H} = -4\pi G\left(\rho + \frac{p}{c^2}\right)$$

d) Introduce $K = \frac{1}{a}\frac{da}{d\eta} = \frac{a'}{a}$ (so $'$ will denote derivatives with respect to η) and show that the result in c) implies

$$K' - K^2 = -4\pi G a^2(\rho c^2 + p),$$

and therefore

$$K' - K^2 \leq 0$$

e) Show that $d'_{\text{EH}} \geq 0$ implies $a'\eta + a \leq 0$, and since $\eta < 0$, this in turn gives

$$K \geq -\frac{1}{\eta}$$

f) Integrate $K' - K^2 \leq 0$ from $\eta < 0$ to 0 and show that

$$\frac{1}{K_0} - \frac{1}{K} \geq \eta$$

where $K_0 = K(\eta = 0) = K(t = \infty)$.

g) If we can show that $K > 0$ and $K_0 = \infty$. The first follows from the definition of K and the assumption of an expanding universe. To show the latter, show that

$$\int_t^\infty \frac{cdt'}{a(t')} = \int_a^\infty \frac{da}{aK}$$

and therefore it follows from the assumption that the event horizon exists that the last integral is finite. Now assume $1/|K|$ is bounded from below so that

$$\frac{1}{[K|} > \epsilon > 0$$

for $a \to \infty$, $t \to \infty$. Show that this implies

$$\int_a^\infty \frac{da}{aK} = \infty,$$

contradicting the existence of an event horizon. Hence $\epsilon = 0$, and $K_0 = \infty$, and the result that the area of the event horizon is non-decreasing follows.

h) Show by direct calculation that this general results holds in the special case of a spatially flat model with $a(t) = a_0(t/t_0)^2$.