## Problem 1

Estimate: At what redshift was the energy density of the Universe equal to that of a typical ice cream cone? Here I am thinking about the energy of the ice cream solely in terms of the how many calories it adds to your diet, and I do not include things like rest-mass energy.

## Problem 2

Observations and experiments have firmly established that neutrinos oscillate between the three different flavours. The data require that at least two states have a non-zero mass, and most likely all three of them are massive. You will in the following look at one species of neutrino with mass $m$, estimate the redshift when it became non-relativistic, and find its speed after that point in time.

a) While it is relativistic the momentum of a single neutrino is $p = E/c$. One way of defining the average momentum of the neutrino background is then to take

$$\langle p \rangle = \frac{1}{c} \frac{\rho c^2}{n}$$

where $\rho c^2$ is the energy density and $n$ is the number density. Use results from the lecture notes to show that

$$p = 3.15 \frac{k_B T_\nu}{c}$$

b) We will take $\langle p \rangle = mc$ as the point where the neutrinos go from being relativistic to being non-relativistic. Let $z_{\rm nr}$ be the redshift when this happens, and show that

$$1 + z_{\rm nr} = \frac{mc^2}{0.54 \text{ meV}}$$

(1 meV $= 10^{-3}$ eV).

c) Find the speed of this type of neutrino as a function of redshift for $z < z_{\rm nr}$.

d) Oscillation experiments can only measure the differences between the squares of the neutrino masses, not the masses themselves. If we call the three mass states $m_1$, $m_2$, and $m_3$, the experiments find (roughly) that $m_2^2 - m_1^2 = 75$ meV$^2$ and $|m_3^2 - m_1^2| = 2500$ meV$^2$. Assume $m_3 > m_2 > m_1 = 0$, and find the speed of the mass states $m_1$ and $m_2$ today.

e) The KATRIN experiment uses $\beta$ decay of to constrain the mass of the electron neutrino (which is a mix of the mass eigenstates). So far, they have found an upper bound of 1.1 eV. Given the mass-squared differences measured in oscillation experiments, what would it mean for $m_1$, $m_2$, and $m_3$ if the mass of the electron neutrino is equal to or near this upper limit? How fast would the cosmic neutrinos be moving today if this were the case?

## Problem 3

This problem is based on an article you can find at https://arxiv.org/abs/1710.01785.

We have seen that the neutrinos started out as a contribution to the radiation energy density, but later became non-relativistic. In general a component like this has to be treated numerically, but we will study a toy model which can be solved analytically. In this model there is just one type of neutrino, with mass $m$. Rather than following the Fermi-Dirac distribution, though, at any given time these neutrinos all have the same momentum $p$.

We will use units where $c = 1$, and assume that the spatial curvature $k = 0$, so we can choose $a = a_0 = 1$ today.

a) Explain why the energy density is given by

$$\rho = n\sqrt{p^2 + m^2},$$

where $n$ is the number density of neutrinos.

The pressure can be shown to be given by

$$P = \frac{1}{3}\frac{np}{\sqrt{1 + \left(\frac{m}{p}\right)^2}}.$$

b) Find an expression for the equation of state parameter $w$.

As the universe expands, the momentum of the neutrinos scales as $p = p_0/a$.

c) Show that

$$w(a) = \frac{1}{3}\frac{1}{1+\frac{a^2}{f_0^2}}$$

where $f_0 = p_0/m$.

d) Solve the continuity equation and show that

$$\rho = \rho_0 a^{-3}\exp[-3I_w(a)],$$

where

$$I_w(a) = \int_1^a \frac{da'}{a'}w(a').$$

e) Show that

$$\rho = \rho_0 g^{1/2}(a)a^{-3},$$

where

$$g(a) = \frac{1+f_0^2 a^{-2}}{1+f_0^2}.$$

Hint:

$$\frac{1}{a(f_0^2+a^2)} = \frac{1}{f_0^2}\left(\frac{1}{a} - \frac{a}{f_0^2+a^2}\right).$$

f) Use the first Friedmann equation to show that

$$H_0 t = \int_0^a \frac{(a')^{1/2}da'}{g^{1/4}(a')},$$

and carry out the integral to obtain

$$H_0 t = \frac{2}{3}(1+f_0^2)^{1/4}\left[(f_0^2+a^2)^{3/4} - f_0^{3/2}\right].$$

g) What can you say about the behaviour of this type of neutrino in the limits $f_0 \to 0$ and $f_0 \to \infty$? Show that we regain familiar results for $a(t)$ in these limits.

h) Show that in the general case

$$a(t) = \left[\left(\frac{t}{t_{\text{fid}}} + f_0^{3/2}\right)^{4/3} - f_0^2\right]^{1/2},$$

where

$$t_{\text{fid}} = \frac{2(1 + f_0^2)^{1/4}}{3H_0}.$$

i) What is the age of the Universe in this model?