Sample solutions for Project 2, AST3220, 2022

a) Lets first compute

$$\frac{dY_n}{d(\ln T)} = \frac{dY_n}{dt}\frac{dt}{dT}\frac{dT}{d(\ln T)}$$

where

$$\frac{dT}{d(\ln T)} = \left(\frac{d(\ln T)}{dT}\right)^{-1} = \left(\frac{1}{T}\right)^{-1} = T,$$

$$\frac{dt}{dT} = \left(\frac{dT}{dt}\right)^{-1} = \left(\frac{d(T_0/a)}{dt}\right)^{-1}$$

$$= -T_0 a^{-2}\frac{da}{dt} = -\frac{T_0}{a}H = -TH,$$

and

$$\frac{dY_n}{dt} = \frac{d(n_n/n_b)}{dt} = \frac{1}{n_b}\frac{dn_n}{dt} - \frac{n_n}{n_b^2}\frac{dn_b}{dt}$$

since $n_b = n_{b0} a^{-3}$,

$$\frac{dn_b}{dt} = \frac{d(n_{b0} a^{-3})}{dt} = -3 n_{b0} a^{-4} \frac{da}{dt}$$

$$= -3 H n_b$$

$$\Rightarrow \frac{dX_n}{dt} = \frac{1}{n_b} \frac{dn_n}{dt} + \frac{3 H n_n}{n_b}$$

Inserting now the eq. for $\frac{dn_n}{dt}$

$$\Rightarrow \frac{dX_n}{dt} = \frac{-3 H n_n}{n_b} + \frac{n_p}{n_b} \Gamma_{p\to n} - \frac{n_n}{n_b} \Gamma_{n \to p}$$

$$+ \frac{3 H n_n}{n_b}$$

$$= Y_p \Gamma_{p \to n} - X_n \Gamma_{n \to p}$$

Combining everything gives

$$\frac{dX_n}{d(\ln T)} = -\frac{1}{H} \left[Y_p \Gamma_{p \to n} - X_n \Gamma_{n \to p} \right]$$

The eq. for X_p is entirely equivalent, only with $n \rightleftarrows p$.

Bonus Q) The baryon density at $T \sim 10^9 K$ is roughly

$$\rho_b = \rho_{b0} a^{-3} = 0.05 \rho_{c0} \left(\frac{T}{T_0}\right)^3$$
$$\sim \underline{\underline{10^{-2} \, kg/m^3}}$$

The mean density of the Sun is $\sim 10^3 \, kg/m^3$, so the baryon density during BBN is quite a bit smaller than the density of the Sun.

The ratio ρ_b/ρ_r at $T \sim 10^9 K$ is

$$\rho_b/\rho_r = \frac{\Omega_{b0} \rho_{c0} a^{-3}}{\Omega_{r0} \rho_{c0} a^{-4}} = \frac{\Omega_{b0}}{\Omega_{r0}} a$$
$$= \frac{\Omega_{b0}}{\Omega_{r0}} \frac{T_0}{T} \sim \underline{\underline{10^{-6}}}$$

b) We can show this by considering conservation of entropy before and after $e^- - e^+$ annihilation.

When $e^- - e^+$ annihilates, energy is released via the production of energetic photons, which causes the cosmic plasma to heat up slightly. Neutrinos do not get any of this heating since they are decoupled from the other particles.

The entropy is dominated by relativistic particles, hence we need only count the relativistic degrees of freedom g_{*s}

$$s = \frac{2\pi^2}{45} k_B g_{*s} \left(\frac{k_B T}{\hbar c} \right)^3$$

where

$$g_{*s} = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3$$

Conservation of entropy means

$$g_{*s} S a^3 = g_{*s} (T a)^3 = \text{constant}$$

Before $e^- - e^+$ annihilation, there were photons, electrons, and positrons

$$g_{*s}(\text{before}) = 2 + \frac{7}{8} \times 2 \times 2$$

$$= 2 + \frac{7}{2} = \frac{11}{2}$$

After there are only photons

$$g_{*s}(\text{after}) = 2$$

Hence

$$g_{*s} (aT)^3 \Big|_{\text{before}} = g_{*s} (aT)^3 \Big|_{\text{after}}$$

$$\Rightarrow \frac{11}{2}(aT)^3\Big|_{\text{before}} = 2(aT)^3\Big|_{\text{after}}$$

$$\Rightarrow \quad aT\Big|_{\text{before}} = \left(\frac{4}{11}\right)^{1/3} aT\Big|_{\text{after}}$$

The temperature of neutrinos, which are decoupled, simply decrease as $T_\nu \propto 1/a$, hence

$$aT_\nu\Big|_{\text{before}} = aT_\nu\Big|_{\text{after}}$$

After e^--e^+ annihilation we therefore have

$$\underline{\underline{T_\nu = \left(\frac{4}{11}\right)^{1/3} T}}$$

We assume that e^--e^+ annihilation is complete by the time of BBN by using the above relation throughout our treatment of BBN.

c) The total radiation energy is due to photons and neutrinos, which have the energy densities

$$\rho_\gamma c^2 = \frac{\pi^2}{30} g_\gamma \frac{(k_B T)^4}{(\hbar c)^3}$$

$$\rho_\nu c^2 = \frac{7}{8} \times \frac{\pi^2}{30} g_\nu \frac{(k_B T_\nu)^4}{(\hbar c)^3}$$

Photons have $g_\gamma = 2$, while N_{eff} number of neutrinos have $g_\nu = 2 N_{eff}$.

The factor 2 for g_ν is due to anti-neutrinos.

The total radiation is

$$\rho_r = \rho_\gamma + \rho_\nu = \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^5} \left[2T^4 + \frac{7}{8} \times 2 N_{eff} T_\nu^4 \right]$$

Inserting for $T_\nu = \left(\frac{4}{11}\right)^{1/3} T$

$$\rho_r = \frac{\pi^2}{15} \frac{(k_B T)^4}{\hbar^3 c^5} \left[1 + N_{eff} \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \right]$$

Dividing $\rho_{r0} = \rho_r(T = T_0)$ by

$\rho_{c0} = \dfrac{3H_0^2}{8\pi G}$ gives Ω_{r0};

$$\Omega_{r0} = \frac{\rho_{r0}}{\rho_{c0}} = \frac{8\pi^3}{45} \frac{G}{H_0^2} \frac{(k_B T_0)^4}{\hbar^3 c^5}$$

$$\times \left[1 + N_{eff} \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \right]$$

d) The Friedmann eq is

$$H = \frac{1}{a}\frac{da}{dt} = H_0\sqrt{\Omega_{r0}}\,a^{-2}$$

$$\Rightarrow a\,da = H_0\sqrt{\Omega_{r0}}\,dt$$

$$\Rightarrow \int_0^a a\,da = \int_0^t H_0\sqrt{\Omega_{r0}}\,dt$$

$$\Rightarrow \frac{1}{2}a^2 = H_0\sqrt{\Omega_{r0}}\,t$$

$$\Rightarrow \underline{\underline{a(t) = \sqrt{2H_0\sqrt{\Omega_{r0}}\,t}}}$$

Since $T = T_0/a \Rightarrow a = T_0/T$

$$\Rightarrow \frac{1}{2}\left(\frac{T_0}{T}\right)^2 = H_0\sqrt{\Omega_{r0}}\,t$$

$$\Rightarrow t(T) = \frac{1}{2H_0\sqrt{\Omega_{r0}}}\left(\frac{T_0}{T}\right)^2$$

This gives

$t(T = 10^{10}K) \simeq 3.6s$

$t(T = 10^{9}K) \simeq 360\,s = 6\,min$

$t(T = 10^{8}K) \simeq 3.6\times10^{4}s \simeq 10h$

e) It is assumed all of ρ_b is in protons and neutrons

$$\rho_b = \rho_p + \rho_n$$

$$\Rightarrow \frac{\rho_b}{m_p} = n_b = \frac{\rho_p}{m_p} + \frac{\rho_n}{m_p} \simeq n_p + n_n$$

Dividing by n_b gives

$$1 = \frac{n_p}{n_b} + \frac{n_n}{n_b} = Y_p + Y_n$$

Eq. (4) in the problem text with $\frac{m_p}{m_n} \simeq 1$ gives

$$\frac{n_n}{n_p} = \frac{Y_n}{Y_p} = e^{-(m_n - m_p)c^2/k_B T}$$

Inserting $1 = Y_p + Y_n$

$$\Rightarrow \frac{Y_n}{1 - Y_n} = e^{-(m_n - m_p)c^2/k_B T}$$

$$\Rightarrow \quad Y_n = (1 - Y_n)\, e^{-(m_n - m_p)c^2/k_B T}$$

$$\Rightarrow \quad Y_n = \frac{1}{1 + \exp\left[(m_n - m_p)c^2/k_B T\right]}$$

f) See project document

g) The terms $\frac{dY_i}{d(\ln T)}$ and

$$\sum_{j\neq i}\left[Y_j \Gamma_{j\to i} - Y_i \Gamma_{i\to j}\right]$$ are

essentially the same as in task a).

The remaining term

$$\sum_{jkl}\left[n_k n_l \gamma_{kl\to ij} - n_i n_j \gamma_{ij\to kl}\right]$$

is multiplied by $-\frac{1}{H n_b}$, and we define $\Gamma_{ij\to kl} = n_b \gamma_{ij\to kl}$

$$\Leftrightarrow -\frac{1}{H}\sum_{jkl}\left[\frac{n_k n_l}{n_b^2}\Gamma_{kl\to ij} - \frac{n_i n_j}{n_b^2}\Gamma_{ij\to kl}\right]$$

$$= -\frac{1}{H}\sum_{jkl}\left[Y_k Y_l \Gamma_{kl\to ij} - Y_i Y_j \Gamma_{ij\to kl}\right]$$

Combining everything gives eq. (19).

h) See project document for plot.

Initially, there is only n and p in thermal equilibrium, and as the universe expands and cools, the system increasingly prefers to have p since it is ligther than n.

At some point, however, the interaction rate between n and p decreases to below the rate of expansion, which suppresses their interactions. The number densities of n and p therefore "freeze" out.
Decay of $n \to p$ causes n to decrease further despite this freeze out.

Eventually, at around $T \sim 10^9 K$, the universe becomes sufficiently cool for the deuterium that form from p and n to not be immediately disintigrated by energetic photons. Up to this point this has been the so-called "deuterium bottleneck" in BBN, but once the are no longer enough photons to destroy all deuterium, its production takes place at a very quick pace.

i) See project document.

j) Since $\Omega_{b0} \simeq 0.05 < \Omega_{m0} \simeq 0.3$, not all matter in the universe can be baryonic. Instead it must be some other particle(s) that does not interact much with the standard model particles we know.

k) $N_{eff} \simeq 3$ agrees with what we know from the standard model of particle physics, i.e. that there are three types of neutrinos.