Part 2 of project 3 in AST3220, 2024

Please write your solutions of the following problems in LaTeX, and hand them in as a pdf file. As usual, you are only allowed to use real intelligence, no AI.

Miscellaneous problems

Problem 1 (10 points)

In this problem you can assume that the Universe is described by the Einstein-de Sitter model.

a) Write down the expression for the scale factor as a function of time. Use this expression to show that the age of the Universe at redshift z is given by

$$t(z) = \frac{t_0}{(1+z)^{3/2}},$$

where t_0 is the present age of the Universe.

b) Assume that we at the present epoch observe two objects, one with redshift $z_1 = 3$ and another with redshift $z_2 = 8$. When, in units of t_0, was the light we observe today emitted by these objects?

c) Determine the comoving radial coordinates of the two objects.

d) The light we receive now from the object with $z = z_2 = 8$ was emitted at a time we can call t_e. Determine the comoving radial coordinate of the light ray heading towards us at an arbitrary later time t.

e) Imagine that there was an observer situated at the object with $z = z_1 = 3$. What redshift did she measure for the light from the object we observe today with redshift $z_2 = 8$?

Problem 2 (10 points)

a) Use the Friedmann equations to explain why there must be a time in the past when the scale factor vanished in models where the total density and pressure satisfy $\rho + 3p/c^2 > 0$, the density decreases faster with a than $1/a^2$, and $H_0 > 0$.

b) We seem to be living in a universe where the expansion is presently accelerating, $\Omega_{\rm m0} = 0.3$, and $\Omega_{\Lambda 0} = 0.7$. Was there a time in the past when $a = 0$ in this model? Why/why not?

Problem 3 (10 points)

The proper distance to the particle horizon at time t is given by

$$d_{\rm P,PH}(t) = a(t) \int_0^t \frac{c dt'}{a(t')}$$

a) Show that the proper distance to the particle horizon at redshift z is

$$d_{\rm P,PH}(z) = \frac{c}{1+z} \int_z^\infty \frac{dz'}{H(z')}$$

b) Calculate $d_{\rm P,PH}(z)$ for a matter-dominated universe with

$$H(z) = H_0 \sqrt{\Omega_{\rm m0}} (1+z)^{3/2},$$

and for a radiation-dominated universe with

$$H(z) = H_0 \sqrt{\Omega_{\rm r0}} (1+z)^2$$

Show that in both cases,

$$d_{\rm P,PH}(z) \sim \frac{c}{H(z)}$$

A typical neutron star has a radius of around 10 kilometers, and a mass around 1.5 solar masses. Take the radius of the observable universe at any given time to be equal to the proper distance to the particle horizon at that time.

c) At what redshift was the radius of the observable universe equal to that of a typical neutron star? What was the mass density of the universe at that redshift? What was the radiation density (use units of kilograms per cubic meter)? Assume $\Omega_{\rm m0} = 0.3$ and $\Omega_{\rm r0} = 10^{-4}$. Compare with the average density of a typical neutron star. Assume that the dimensionless Hubble constant $h = 0.7$.

d) What was the CMB temperature at that time?

e) Show that the age of the Universe at redshift z is given by

$$t(z) = \int_z^\infty \frac{dz'}{(1+z')H(z')},$$

and that for the radiation-dominated universe

$$t(z) = \frac{1}{2H(z)}$$

How old was the Universe at the redshift found in c)?

On inflation

The following problems are all about inflation, and we will use units where $\hbar = 1$ and $c = 1$.

Problem 4 (5 points)

Give brief descriptions of the horizon problem and the flatness problem.

Problem 5 (5 points)

Assume inflation is driven by a scalar field with the potential

$$V(\phi) = \lambda \phi^p,$$

where λ is a positive constant and $p \geq 2$. Show that the total number of e-foldings during inflation is guaranteed to be large if the slow-roll conditions are fulfilled.

Problem 6 (5 points)

a) Is inflation possible if $V(\phi) = 0$ for all ϕ?

b) Is inflation possible if the dynamics of the scalar field is such that we always have $\dot{\phi}^2 = 2V(\phi)$?

Problem 7 (10 points)

We will look at inflation driven by a scalar field with the potential $V(\phi) = V_0 e^{-\lambda\phi}$, where V_0 and λ are positive constants. In addition to taking $\hbar = 1 = c$, we will also simplify the equations by introducing the so-called reduced Planck mass $M_\text{P} = 1/\sqrt{8\pi G}$.

a) Write down the equations for ϕ and H in the slow-roll approximation with this potential.

b) Solve the equations from a) and find $\phi(t)$ and $a(t)$.

c) The full equations, without the slow-roll approximation, are

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= 0 \quad (1) \\ H^2 &= \frac{1}{3M_\text{P}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \quad (2) \end{aligned}$$

and in the case of the exponential potential we consider in this problem it turns out that they have an exact analytical solution. The solution you found in b) suggests the ansatz

$$\begin{aligned} a(t) &= Ct^\alpha \\ \phi(t) &= \frac{2}{\lambda}\ln(Bt) \end{aligned}$$

Determine the constants α and B that make this ansatz an exact solution, and show that you regain the slow-roll solution in the appropriate limit.

d) What is the main problem with this potential as a model of inflation?

Problem 8 (20 points)

The slow-roll approximation turns the second-order differential equation for the scalar field into a first-order equation. This can only work if the dynamics of the field makes the precise initial conditions on the field (apart from the fact that the field must start out somewhere on the potential where the slow-roll conditions are satisfied) redundant. In this problem you are going to set up an argument for why this will usually be the case. Without loss of generality, you can assume that $\dot{\phi} > 0$ during inflation.

a) Use equations (1) and (2) from problem 7 to show that

$$\dot{\phi} = -2M_{\rm P}^2 H'(\phi),$$

where $H'(\phi) = dH/d\phi$.

b) Use the result from a) to show that the first Friedmann equation can be written as

$$[H'(\phi)]^2 - \frac{3}{2M_{\rm P}^2} H^2(\phi) = -\frac{1}{2M_{\rm P}^4} V(\phi) \tag{3}$$

The scalar field ϕ is in itself not directly observable, the important physical quantity is the Hubble parameter. If we can show that we end up on the same curve in the ϕ-H plane regardless of the initial conditions on the scalar field, we have the result we want.

Consider a linear perturbation around a solution of equation (3):

$$H(\phi) = H_0(\phi) + \delta H(\phi)$$

where H_0 is a solution of equation (3) (and *not* the Hubble constant!)

c) Assume that H is also a solution of equation (3) and show that to first order in the perturbation we have

$$H_0' \delta H' = \frac{3}{2M_{\rm P}^2} H_0 \delta H,$$

where ’ again denotes derivatives with respect to ϕ.

d) Show that the equation in c) has the general solution

$$\delta H(\phi) = \delta H(\phi_{\rm i}) \exp\left[\frac{3}{2M_{\rm P}^2} \int_{\phi_{\rm i}}^{\phi} \frac{H_0(\varphi)}{H_0'(\varphi)} d\varphi\right]$$

where $\phi_{\rm i}$ is the initial value of the scalar field, and φ is just a dummy integration variable. Use this result to explain why the perturbation δH quickly dies out.