AST3220, spring 2024: Project 1 [1]

Read this before you start

This project consists of a set of problems, some analytical, some numerical. There is no need to structure your answers as a report with an introduction, methods, results, discussion and conclusion, you can just answer the questions, one by one. **In fact, we prefer it that way.** It is important that you explain how you think, just writing down a bunch of equations with no explanations will not give you a maximum score. You should write your report/answers using LaTeX. Posting handwritten lecture notes and solutions to problems is a privilege that belongs to the lecturer alone. Your figures should have a clear layout with proper axis labels and units, and with a caption explaining what the figure shows. The figures should be referenced in the main text. You are also required to hand in your source code in a separate file that can be easily compiled/executed and tested. Results and equations taken from the lecture notes need not be referenced, but other sources that you use must be cited. You can work together to solve the problems, but you must write your own code and report.

Using AI like ChatGPT to generate text, code and mathematics is strictly forbidden

Please don't write your name or other facts which may disclose your identity anywhere!

NOTE: To simplify the notation, we will use units where $\hbar = c = 1$, unless otherwise stated

Quintessence: Scalar fields as an alternative to the cosmological constant

We have (I hope) learnt that the expansion of the Universe is accelerating, and that the simplest explanation for that is the cosmological constant. But introducing the cosmological constant in our model creates a problem: The

[1] This project is a slightly modified and extended version of a project originally developed by Mattia Mina

value that fits the observations is many, many orders of magnitude smaller than the theoretical prediction. If you have read the bonus material in the lecture notes you will know that this way of stating the problem is inaccurate, but nevertheless, the fact remains that the value of the cosmological constant is one of the greatest unsolved problems in theoretical physics. But can we really be sure that the cosmological constant really is the best explanation for the accelerated expansion? Not until we have tested alternative explanations. In this project we will investigate one of the most popular alternatives.

The class of models we will look at was introduced by Reuter and Wetterich (Physics Letters B 188, 38-43 (1987), and independently by Peebles and Ratra (Astrophysical Journal 325, L17-20 (1988)). This was in fact ten years before the accelerating expansion was discovered, but already as the 1990s approached hints that something new was needed in our cosmological models were appearing. The idea was to introduce a scalar field, eventually named 'quintessence'. A scalar field is just a function ϕ that assigns a (in our case real) number to each point in spacetime. We will assume the cosmological principle, and in this case ϕ must at any given time assign the same number to every point in space, so ϕ can only be a function of cosmic time, t: $\phi = \phi(t)$. In particle physics, fields are associated with particles, and the most famous example of a fundamental scalar field (and the only one known to exist) is the Higgs field. Unfortunately, the Higgs field cannot do the job we want our scalar field to do, so we must simply postulate that this field exists and work out what observable consequences it would have.

It can be shown that a scalar field has an energy density (remember that we set $c = 1$, so mass density, energy density and pressure now all have the same units)

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

where $V(\phi)$ is referred to as 'the potential' (but must have the units of an energy density). We will not care about how this potential is derived, but we will later consider two specific guesses. The equation of state (EoS) parameter for the field is

$$w_\phi = \frac{p_\phi}{\rho_\phi} \tag{3}$$

Since ϕ depends on the time, we see that so will in general w_ϕ. In this respect it differs from the contributions to energy density and pressure we have considered so far: non-relativistic matter ($w = 0$), radiation ($w = \frac{1}{3}$), and the cosmological constant ($w = -1$). However, from equations (1), (2), and (3) we see that if $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, then $w_\phi \approx -1$. So if ϕ varies slowly enough with time, it will act in the same way as the cosmological constant, and can therefore explain the accelerating expansion.

You can assume that the quintessence field behaves as a fluid and therefore follows the same continuity equation as the other components:

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi) = -3H(1 + w_\phi)\rho_\phi, \tag{4}$$

but now you have to bear in mind that w_ϕ is a function of the time t.

Problem 1 (5 points) Show that the general solution of the continuity equation for the quintessence field is given by

$$\rho_\phi(z) = \rho_{\phi 0} \exp\left\{\int_0^z dz' \frac{3[1 + w_\phi(z')]}{(1 + z')}\right\} \tag{5}$$

where $\rho_{\phi 0}$ is the present value of the energy density of the quintessence field.

Problem 2 (2 points) By using equations (1) and (2), show that the continuity equation for the scalar field also gives

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{6}$$

where $V'(\phi) = \frac{dV}{d\phi}$.

We will in the following consider spatially flat ($k = 0$) models of the Universe containing non-relativistic matter, radiation, and the quintessence field. Using equation (5) and results from the lectures we can write the Hubble parameter for such models as

$$H^2 = H_0^2 \left[\Omega_{\mathrm{m}0}(1 + z)^3 + \Omega_{\mathrm{r}0}(1 + z)^4 + \Omega_{\phi 0} \exp\left\{\int_0^z dz' \frac{3[1 + w_\phi(z')]}{(1 + z')}\right\}\right] \tag{7}$$

where for a component i $\Omega_{i0} = \rho_{i0}/\rho_{c0} = \frac{8\pi G}{3H_0^2}\rho_{i0}$ and the subscript 0 refers to present-day values.

Problem 3 (5 points) By using the first and second Friedmann equations together with equations (1) and (2), show that the time derivative of the Hubble parameter can be written as

$$\dot{H} = -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1 + w_r) + \dot{\phi}^2\right], \tag{8}$$

where we have introduced the shorthand notation $\kappa^2 = 8\pi G$.

Although we have most of the time in this course used present-day values of the various density parameters, we can define them at any time as

$$\Omega_i = \frac{\rho_i(t)}{\rho_c(t)}, \tag{9}$$

where

$$\rho_c(t) = \frac{3H^2}{8\pi G} = \frac{3H^2}{\kappa^2} \tag{10}$$

is the critical density (the total density that gives $k = 0$) at time t.

One of the main goals in the following is to solve numerically for the time evolution of the densities of the different components. To this end, it turns out to be advantageous to introduce the following dimensionless variables:

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \tag{11}$$

$$x_2 = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \tag{12}$$

$$x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \tag{13}$$

Problem 4 (5 points) Show that we have

$$\Omega_\phi = x_1^2 + x_2^2,\ \Omega_r = x_3^2,\ \Omega_m = 1 - x_1^2 - x_2^2 - x_3^2 \tag{14}$$

We want to track the evolution of the densities over almost the entire history of the Universe, so the cosmic time t will vary by many orders of magnitude and is therefore a bit awkward to use in the numerics. We therefore introduce a new time variable

$$N = \ln\left(\frac{a}{a_0}\right), \tag{15}$$

where a_0 is the present value of the scale factor. From this definition, we get

$$\dot{N} = \frac{dN}{dt} = \frac{d}{dt}\left[\ln\left(\frac{a}{a_0}\right)\right] = \frac{1}{a}\frac{da}{dt} = H$$

Using this result, we can rewrite the time derivative of any quantity f as

$$\frac{df}{dt} = \frac{dN}{dt}\frac{df}{dN} = H\frac{df}{dN}$$

Problem 5 (5 points) By using equation (8) and the definitions of the dimensionless variables, derive the following expression:

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}(3 + 3x_1^2 - 3x_2^2 + x_3^2) \tag{16}$$

You will soon derive the equations of motion for the dimensionless variables, but first we introduce the quantites

$$\lambda = -\frac{V'}{\kappa V} \tag{17}$$

$$\Gamma = \frac{VV''}{(V')^2} \tag{18}$$

where $V'' = \frac{d^2V}{d\phi^2}$.

Problem 6 (10 points) Show that the equations of motion for the dimensionless variables are

$$\frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1(3 + 3x_1^2 - 3x_2^2 + x_3^2) \tag{19}$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1^2 - 3x_2^2 + x_3^2) \tag{20}$$

$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3(3 + 3x_1^2 - 3x_2^2 + x_3^2) \tag{21}$$

where λ is given by equation (17).

Problem 7 (2 points) What must V look like if λ is a constant? What is the corresponding value of Γ?

Problem 8 (3 points) For the case when λ is not a constant, show that

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x_1 \tag{22}$$

We will now study the numerical solutions of equations (19)-(21) (+(22) when needed). To do so, we must choose the potential $V(\phi)$. We will look at two possibilities: An inverse power-law potential

$$V(\phi) = M^{4+\alpha}\phi^{-\alpha} \tag{23}$$

where M is a mass scale and we choose $\alpha = 1$, and an exponential potential

$$V(\phi) = V_0 e^{-\kappa\zeta\phi} \tag{24}$$

where we will set $\zeta = \frac{3}{2}$. You are going to integrate the equations of motion for $0 \leq z \leq 2 \times 10^7$ with initial conditions at $z = 2 \times 10^7$ as follows:

- For the power-law potential: $x_1 = 5 \times 10^{-5}$, $x_2 = 10^{-8}$, $x_3 = 0.9999$, $\lambda = 10^9$
- For the exponential potential: $x_1 = 0$, $x_2 = 5 \times 10^{-13}$, $x_3 = 0.9999$

Problem 9 (40 points) Write a code that integrates the equations of motion for x_1, x_2, x_3, and λ for both the inverse power-law potential and the exponential potential. Use the results to plot the following quantities as functions of the redshift:

1. The density parameters for matter, radiation and the quintessence field
2. The EoS parameter w_ϕ

Problem 10 (10 points) Calculate and plot the Hubble parameter for the two models using the expression in equation (7), and plot it as a function of z. In the same figure, also plot the Hubble parameter for the spatially flat ΛCDM model with $\Omega_{\mathrm{m}0} = 0.3$. (Hint: For the two quintessence models, it may be a good idea to integrate over N rather than z in equation (8)).

Problem 11 (10 points) Calculate the dimensionless age $H_0 t_0$ of the Universe for the two quintessence models and the ΛCDM model. The formula you need can be found in the lecture notes. Which model gives the oldest universe (for a given H_0)?

Problem 12 (10 points) Calculate and plot the dimensionless luminosity distance $H_0 d_{\mathrm{L}}(z)/c$ for the two models for $0 \leq z \leq 2$.

Along with this text you will receive a table of measured luminosity distances with associated errors. The table is in the format (redshift, luminosity distance, error). The distances and the errors are given in units of Gpc (1 Gpc = 10^9 pc). You will now check how well our models fit these measurements. Let us, as an example, look at the case when we want to test the spatially flat ΛCDM model for a given $\Omega_{\rm m0}$. Let us call the expression for the luminosity distance based on this parameter for 'the model'. Given values for $\Omega_{\rm m0}$ and $\Omega_{\Lambda 0}$, we want to know the probability of the model, given the data, $P(\text{model}|\text{data})$. There is no ready recipe for calculating this probability, but a result known as Bayes' theorem says that

$$P(\text{model}|\text{data}) = \frac{P(\text{data}|\text{model})P(\text{model})}{P(\text{data})} \tag{25}$$

The second factor in the numerator is the probability we would assign to the model before obtaining the data, and it is called the *prior*. The factor in the denominator is known as the *evidence*. We will, as is quite common, consider both of these factors to be constants, and we then have the result

$$P(\text{model}|\text{data}) \propto P(\text{data}|\text{model}). \tag{26}$$

The probability on the right-hand side is known as the *likelihood*, and the point is that it is possible to work out how to calculate it. For example, we will assume that the observations are drawn from a Gaussian distribution. This means that we assume that if we measure the luminosity distance to the ith redshift z_i to be $d_{\rm L}^i$ with measurement error σ_i, then the probability distribution for the true luminosity distance $d_{\rm L}(z_i)$ is

$$P(d_{\rm L}(z_i)) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{(d_{\rm L}(z_i) - d_L^i)^2}{2\sigma_i^2}\right].$$

If we also assume that the measurements are uncorrelated, it can be shown that the likelihood is given by

$$P(\text{data}|\text{model}) = \frac{1}{(2\pi\prod_{i=1}^N \sigma_i^2)^{1/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^N \frac{(d_L(z_i;\vec{p}) - d_L^i)^2}{\sigma_i^2}\right],$$

where $d_L(z_i, \vec{p})$ is the model prediction for the luminosity distance to redshift z_i for given parameter vector $\vec{p}$, and N is the number of observations. The quantity

$$\chi^2(\vec{p}) = \sum_{i=1}^N \frac{(d_L(z_i;\vec{p}) - d_L^i)^2}{\sigma_i^2}.$$

therefore determines the likelihood of the model: Smaller values of χ^2 means a higher likelihood and a better fit to the data.

Problem 13 (5 points) Calculate χ^2 for the inverse power-law potential and the exponential potential. You can use $h = 0.7$ when calculating the luminosity distances in units of Gpc. Which of the two models provides a better fit to the data?

Problem 14 (10 points) Determine the value of Ω_{m0} which provides the best fit for the spatially flat ΛCDM model to the data. Is this a better or worse fit than for the two quintessence models we considered? Is this a fair comparison?