1986A&A...163..135M

Astron. Astrophys. 163, 135–139 (1986)

ASTRONOMY AND ASTROPHYSICS

Contribution functions and the depths of formation of spectral lines

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Received February 12, accepted February 27, 1986

Summary. We derive a rigorous expression for the contribution function to the spectral line depression in a stellar atmosphere, giving the contribution of the different atmospheric layers to the formation of the line depression. This contribution function is the solution of the transfer equation for the line depression. It is the only appropriate distribution function for the computation of the depths of formation of spectral lines. It indicates, in contrast with some earlier contribution functions, that a faint spectral line is not necessarily formed in the same layers as the continuum. The response function of the line depression to a given perturbation is also briefly discussed.

Key words: lines: formation – stars: atmospheres of – Sun: photosphere of

1. Introduction

The knowledge of the average depth of formation of spectral lines is essential for the study of depth-dependent phenomena in stellar atmospheres, as long as that study is based on the analysis of spectral lines. Among such phenomena, we may mention the variation with depth of turbulence, magnetic field, solar oscillations or solar rotation (an example of this last case may be found in Solonsky, 1971).

The determination of the depth of formation of spectral lines is generally carried out through the use of a *contribution function* (CF), which should give the relative contribution of the different atmospheric layers to the observed quantity, or of a *response function* (RF), which is designed to measure the response of some observed quantity to a given perturbation.

As was first pointed out by De Jager (see Gurtovenko et al., 1974), it is extremely important, in studying the formation of spectral lines, to distinguish between the region of origin of the *emergent radiation* and the region where the *line depression* is formed. This point will be further discussed in Sect. 7.

However, if everybody agrees on the form of the CF to the emergent intensity, several forms have been proposed for the CF to the line depression, leading sometimes to very different (or even contradictory) results. In our opinion, this is due to the lack of a proper definition for *the* CF to a given quantity. In this paper, we propose a coherent definition, by analogy with the CF for the emergent intensity. Using this definition, we determine *the* CF to the line depression and compare it with previously proposed "CF's". We also show that it gives results in agreement with physical intuition, contrary to some other CF's. Finally, we show how to derive a response function for the line depression.

2. Earlier contribution functions

2.1. Contribution function to the specific intensity

If I is the specific intensity at frequency ν in the direction s, the transfer equation may be written (Gray, 1976):

$$\frac{\mu}{\varrho}\frac{dI}{dz} = -\kappa(I-S), \tag{1}$$

where $\mu=\cos\theta$, the cosine of the angle between the s direction and the normal to the surface, ϱ is the density of the gas, z the geometrical depth, κ the absorption coefficient and S the source function. Introducing, as usual, the optical depth τ by the relation:

$$d\tau = \kappa\varrho\, dz \tag{2}$$

the solution of (1) for $\tau=0$ [that is the emergent intensity $I(0)$ in the direction considered] may be written:

$$I(0) = \int_0^{+\infty} S e^{-\tau/\mu} d\tau/\mu\,. \tag{3}$$

The integrand of (3) represents thus the fraction of the emergent intensity originating from depth τ. It is usually referred to as the CF to I, which we shall note $\mathscr{C}_I(\tau)$. In stellar atmospheres, however, it is more convenient to work in the $\log\tau_0$ scale, τ_0 being the optical depth at a reference wavelength λ_0. One of the reasons for this is that $\log\tau_0$ is approximately proportional to the geometrical depth. Writing x for $\log\tau_0$, one has:

$$\mathscr{C}_I(x) = \mathscr{C}_I(\tau)\cdot\frac{d\tau}{dx} \tag{4}$$

so that:

$$\mathscr{C}_I(x) = \mu^{-1}\ln 10\,\tau_0\frac{\kappa}{\kappa_0}S e^{-\tau/\mu}, \tag{5}$$

κ_0 being the absorption coefficient at wavelength λ_0.

2.2. Contribution functions to the line depression

The relative depression at some frequency in the line is defined by:

$$R(0)=[I_c(0)-I_l(0)]/I_c(0)\,, \tag{6}$$

where $I_l(0)$ is the emergent intensity at the frequency considered and $I_c(0)$ the continuous intensity at the same frequency (i.e. the intensity that we would observe if the line was absent). Replacing the quantities in the numerator of (6) by their expressions of the form (3) and combining the two integrals into one, we obtain, in LTE (putting all source functions equal to the Planck function B):

$$R(0)=I_c(0)^{-1}\int_0^{+\infty} B\left[\frac{\kappa_c}{\kappa_0}\,e^{-\tau_c/\mu}-\frac{\kappa_c+\kappa_l}{\kappa_0}\,e^{-(\tau_c+\tau_l)/\mu}\right]d\tau_0/\mu\,, \tag{7}$$

where κ_c and κ_l (resp. τ_c and τ_l) are the continuous and line absorptions (resp. optical depths). One can thus define (?) a CF (see Cowley, 1970):

$$\mathscr{C}_R^{(1)}(x)=\mu^{-1}\ln 10\,\tau_0\,\frac{\kappa_c}{\kappa_0}\,\frac{B}{I_c(0)}\left[1-\left(1+\frac{\kappa_l}{\kappa_c}\right)e^{-\tau_l/\mu}\right]e^{-\tau_c/\mu}\,. \tag{8}$$

By mathematical transformations of the integral (7), including integrations by parts, one obtains the "weight-saturation" (Pecker, 1952) or "Planchian-gradient" (Mugglestone, 1958) CF's (noted $\mathscr{C}_R^{(2)}$ and $\mathscr{C}_R^{(3)}$, respectively):

$$\mathscr{C}_R^{(2)}(x)=\mu^{-1}\ln 10\,\tau_0\,\frac{\kappa_l}{\kappa_0}\,\frac{I_c}{I_c(0)}\left(1-\frac{B}{I_c}\right)e^{-(\tau_c+\tau_l)/\mu} \tag{9}$$

and

$$\mathscr{C}_R^{(3)}(x)=\mu^{-1}\ln 10\,\tau_0\,\frac{\kappa_c}{\kappa_0}\,\frac{1}{I_c(0)}\,\frac{dB}{d\tau_c}\,(1-e^{-\tau_l/\mu})\,e^{-\tau_c/\mu}\,. \tag{10}$$

These four CF's, namely $\mathscr{C}_I$, $\mathscr{C}_R^{(1)}$, $\mathscr{C}_R^{(2)}$, and $\mathscr{C}_R^{(3)}$ have been used by different authors to compute average depths of formation of spectral lines, leading sometimes to strongly different results. Some authors, including Edmonds (1969), Ruhm (1969), Gurtovenko et al. (1974), Babii and Rikalyuk (1981) have compared these CF's, presenting arguments in favor of one or the other of these CF's, with contradictory conclusions. In the following section, we derive a new CF, based on a rigorous treatment, and which we argue is *the* CF to the line depression.

3. The contribution function to the line depression

Let us recall the procedure followed to obtain the CF to the emergent intensity. To obtain $\mathscr{C}_I$, we have:

- written the transfer equation for I in the form (1);
- written its formal solution in the form (3);
- identified the integrand of (3) as $\mathscr{C}_I$.

It may be seen, from (1) and (3) that, when it is written in the τ_0 scale, the CF to I is just the source term in the right-hand side of Eq. (1), times the exponential factor which accounts for the fraction of light emitted at depth τ_0 which is absorbed in the layers between τ_0 and the stellar surface. It represents thus just what we mean by a CF, that is the contribution of the layer situated at depth τ_0 to the emergent intensity.

So, if we want to obtain the CF to the line depression, we must follow the same procedure, that is:

- write the transfer equation for R in the form (1);
- write its formal solution in the form (3);
- identify the integrand of (3) as $\mathscr{C}_R$.

This is the only valid procedure if we want $\mathscr{C}_R(\tau_0)$ to be interpreted as the contribution of the layer situated at depth τ_0 to the observed line depression. So, the basic problem is to write a transfer equation for R, which we thus define as the relative line depression at any depth in the atmosphere:

$$R=(I_c-I_l)/I_c\,. \tag{11}$$

The transfer equations for I_c and I_l are the following:

$$\frac{\mu}{\varrho}\frac{dI_c}{dz}=-\kappa_c(I_c-S_c)\,, \tag{12}$$

$$\frac{\mu}{\varrho}\frac{dI_l}{dz}=-(\kappa_c+\kappa_l)\,(I_l-S_t)\,, \tag{13}$$

where S_c is the source function in the continuum and S_t the source function in the line (ratio of total emission coefficient to total absorption coefficient). It is related to the line source function S_l (ratio of line emission coefficient to line absorption coefficient) by:

$$S_t=\left(S_c+\frac{\kappa_l}{\kappa_c}S_l\right)\bigg/\left(1+\frac{\kappa_l}{\kappa_c}\right) \tag{14}$$

see, e.g., Gray (1976). From (11), we get:

$$\frac{dR}{dz}=\frac{I_l}{I_c^2}\frac{dI_c}{dz}-\frac{1}{I_c}\frac{dI_l}{dz}\,. \tag{15}$$

Introducing (12) and (13) in (15) and rearranging the terms, taking (11) into account, we obtain the transfer equation for R:

$$\frac{\mu}{\varrho}\frac{dR}{dz}=-\kappa_R(R-S_R)\,, \tag{16}$$

where the "effective absorption coefficient" κ_R and the "effective source function" S_R are given by:

$$\kappa_R=\kappa_l+\kappa_c\frac{S_c}{I_c} \tag{17}$$

and

$$S_R=\left(1-\frac{S_l}{I_c}\right)\bigg/\left(1+\frac{\kappa_c}{\kappa_l}\frac{S_c}{I_c}\right). \tag{18}$$

The emergent line depression is thus:

$$R(0)=\int_0^{+\infty} S_R e^{-\tau_R/\mu}d\tau_R/\mu\,, \tag{19}$$

where

$$d\tau_R=\kappa_R\varrho\,dz\,. \tag{20}$$

And the CF to R may be written:

$$\mathscr{C}_R(x)=\mu^{-1}\ln 10\,\tau_0\,\frac{\kappa_l}{\kappa_0}\left(1-\frac{S_l}{I_c}\right)e^{-\tau_R/\mu}\,. \tag{21}$$

This is *the* CF to R in the $\log\tau_0$ scale.

A nice feature of the transfer equation (16) and of the CF (21) is that they allow a straightforward interpretation in terms of line formation. From (21), we see that $\mathscr{C}_R$ is non zero only if:

$\kappa_l\neq 0$, so that some absorber must be present in order for a line to be formed, and

$S_l\neq I_c$, so that the re-emitted light must not be equal to the absorbed light. The line will appear in absorption if $S_l<I_c$ and in emission if $S_l>I_c$.

4. The average depth of formation of a spectral line

Using the CF, we can calculate the average depth of formation of the line depression at a given frequency by the formula:

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\,\mathscr{C}_R(x)\,dx \Big/ \int_{-\infty}^{+\infty} \mathscr{C}_R(x)\,dx\,. \tag{22}$$

In a similar way, we can compute the mean of other quantities, such as τ_0, the geometrical depth z, the temperature $T, \ldots$. This is valid only if $\mathscr{C}_R(x)$ does not change sign, which is generally true in the case of an absorption line. If some layers contribute to line emission, they may be treated separately, calculating a mean depth for the absorption and a mean depth for the emission. Note, in passing, that $\langle \log \tau_0 \rangle \neq \log \langle \tau_0 \rangle$. They would be equal in the limit of an extremely sharp-peaked CF (ideally, a δ-function).

5. Comparison of the different contribution functions

For the sake of simplicity, we consider the case of LTE and the center of the stellar (solar) disk ($\mu = 1$). This does not affect any of our conclusions. Let us denote by α the common factor:

$$\alpha = \ln 10 \, \frac{\tau_0}{\kappa_0} = \frac{1}{\kappa} \frac{d\tau}{dx}. \tag{23}$$

We then have:

$$\mathscr{C}_I(x) = \alpha \kappa_c B \left(1 + \frac{\kappa_l}{\kappa_c}\right) e^{-(\tau_c + \tau_l)}, \tag{24}$$

$$\mathscr{C}_R(x) = \alpha \kappa_l \left(1 - \frac{B}{I_c}\right) e^{-\tau_R}, \tag{25}$$

$$\mathscr{C}_R^{(1)}(x) = \alpha \kappa_c \frac{B}{I_c(0)} \left[1 - \left(1 + \frac{\kappa_l}{\kappa_c}\right) e^{-\tau_l}\right] e^{-\tau_c}, \tag{26}$$

$$\mathscr{C}_R^{(2)}(x) = \alpha \kappa_l \frac{I_c}{I_c(0)} \left(1 - \frac{B}{I_c}\right) e^{-(\tau_c + \tau_l)}, \tag{27}$$

$$\mathscr{C}_R^{(3)}(x) = \alpha \kappa_c \frac{1}{I_c(0)} \frac{dB}{d\tau_c} (1 - e - \tau_l)\, e^{-\tau_c}. \tag{28}$$

A first striking point is the similarity of $\mathscr{C}_R$ and $\mathscr{C}_R^{(2)}$. They behave in roughly the same way since $I_c \simeq I_c(0)$ in the layers where the line is formed and τ_R is generally not very different from $\tau_c + \tau_l$. In fact, they coincide in the case of large τ_l. This similarity gives a posteriori support to those earlier computations of the average depths of formation which were based on $\mathscr{C}_R^{(2)}$ (e.g., Gurtovenko et al., 1974). It also means that the arguments of these authors supporting $\mathscr{C}_R^{(2)}$ equally support $\mathscr{C}_R$.

When comparing these five CF's, it is very instructive to consider the case of a layer where no absorber is present ($\kappa_l = 0$). In this case, only $\mathscr{C}_R$ and $\mathscr{C}_R^{(2)}$ cancel automatically, while the other CF's are non zero if $\tau_l \neq 0$, that is if some absorber is present between the layer considered and the stellar surface. These CF's thus indicate some contribution to the line depression from layers situated below those where the absorber is present. This is clearly unphysical and would be sufficient to reject these CF's as indicating the layers where the line depression is formed. This is hardly surprising in the case of $\mathscr{C}_I$ since it is the CF to the emergent intensity and not to the line depression. Unfortunately, it is sometimes used to determine the depth of formation of spectral lines. At this point, it may be added that the methods of determination of the depth of formation from the equality of the emergent intensity to the source function at the depth of formation also give some indication of where the emergent intensity originates, but not where the line depression is formed.

The comparison of the different CF's is illustrated in Figs. 1 and 2 which give the CF's at the center of an Fe I line at 5000 Å in a solar model. The excitation potential of the line is 3 eV and the calculations are carried out in LTE and at the center of the solar disk ($\mu = 1$). Figure 1 corresponds to a faint line and Fig. 2 to a rather strong line. $\mathscr{C}_I$ has been normalised in order to have the same integral over x as the $\mathscr{C}_R$'s, so that the comparison is easier. Note that all these $\mathscr{C}_R$'s have the same integral, which is the emergent line depression. In the case of the faint line, all CF's, except $\mathscr{C}_R$ and $\mathscr{C}_R^{(2)}$, which is here indistinguishable from $\mathscr{C}_R$, indicate that the line is formed near $\tau_0 = 1$, that is where the continuous intensity is formed. For the stronger line, $\mathscr{C}_I$ indicates roughly the same depth of formation as $\mathscr{C}_R$, while $\mathscr{C}_R^{(1)}$ and $\mathscr{C}_R^{(3)}$ remain confined to the deeper layers. Figure 3 shows the variation of the average depth of formation with the equivalent width for lines of 0 and 3 eV excitation potential as indicated by the different CF's. These average depths $e^{-\tau_l}$ of formation correspond to the line center.

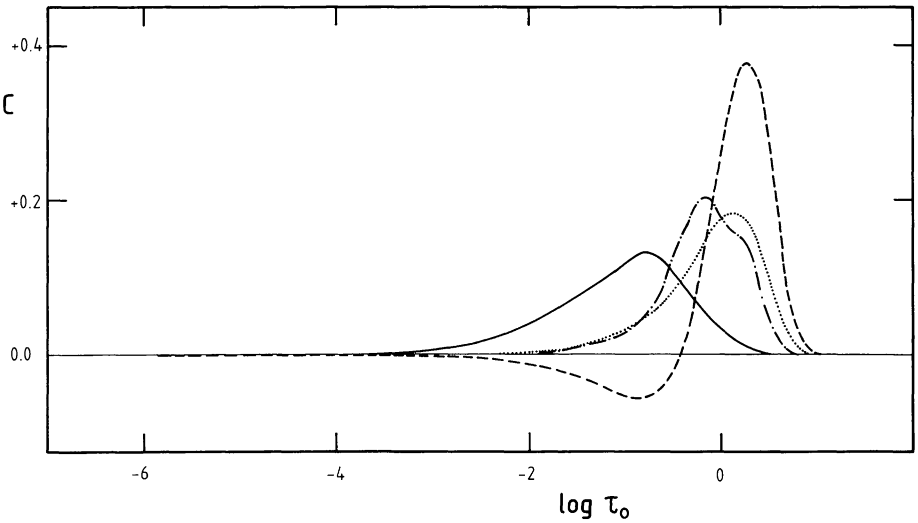

Fig. 1. Contribution functions to the line depression at the center of an Fe I line of 3 eV excitation potential and 10 mÅ equivalent width at a wavelength of 5000 Å. These CF's are computed for the center of the solar disk with the solar model of Gray (1976). These functions are, respectively, $\mathscr{C}_R$ (continuous line), $\mathscr{C}_R^{(1)}$ (dashed line), $\mathscr{C}_R^{(3)}$ (dash-dotted line) and $\mathscr{C}_I$ (dotted line). $\mathscr{C}_R^{(2)}$ is indistinguishable from $\mathscr{C}_R$

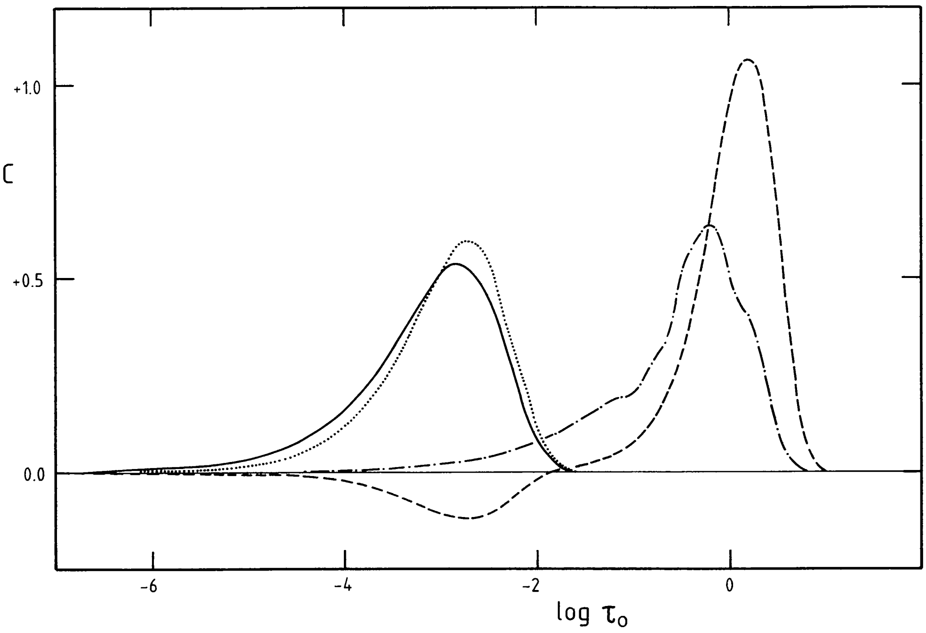

Fig. 2. Same as Fig. 1 for a line of 100 mÅ equivalent width. The irregular appearance of $\mathscr{C}_R^{(3)}$ is due to numerical inaccuracies in the computation of the derivative of the Planck function

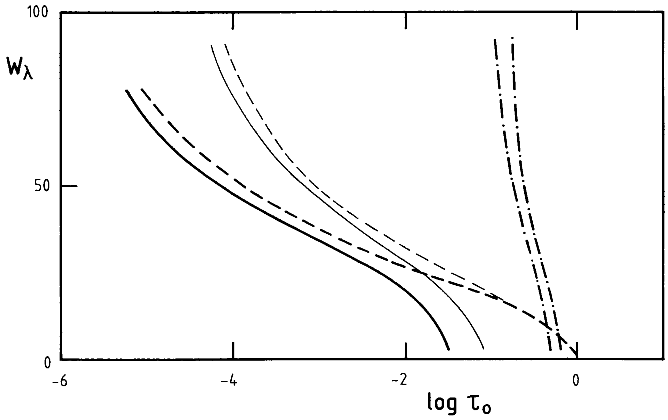

Fig. 3. The average depth of formation of the center of an Fe I line at 5000 Å for the center of the solar disk, as a function of the line equivalent width W_λ. The line excitation potential is 0 eV (thick line) and 3 eV (thin line). The depths of formation are computed from $\mathscr{C}_R$ (continuous line), $\mathscr{C}_I$ (dashed line) and $\mathscr{C}_R^{(3)}$ (dash-dotted line)

6. Response functions

Let us consider a physical quantity $\beta(x)$ eventually varying with depth in a stellar atmosphere (β may be, for example, the temperature or the microturbulent velocity). If the quantity considered is perturbed from $\beta(x)$ to $\beta(x)+\delta\beta(x)$, the change in the observed line depression may be written, to first order:

$$\delta R(0) = \int_{-\infty}^{+\infty} R_{R,\beta}(x)\,\delta\beta(x)\,dx\,, \tag{29}$$

where $R_{R,\beta}(x)$ is the RF of the line depression $R(0)$ to a perturbation in the quantity β. It may be obtained following the lines of Caccin et al. (1977), that is by perturbing the transfer equation for R. We thus obtain:

$$R_{R,\beta}(x) = \mu^{-1} \ln 10\,\tau_0\,\frac{\kappa_R}{\kappa_0}\left[\frac{dS_R}{d\beta} - \frac{1}{\kappa_R}(R-S_R)\frac{d\kappa_R}{d\beta}\right] e^{-\tau_R/\mu}. \tag{30}$$

The use of the RF or of the CF depends on what is analysed. For example, the variation of the line depression due to a change in turbulent velocity should be studied through the corresponding RF, but that function does not tell us where the line is formed: the latter information should be derived from the CF. It may be noted that, in many cases, the RF to the microturbulent velocity has a behaviour similar to the CF, while this is not the case for the RF to the temperature. The main reason for this is that the continuous intensity is sensitive to changes in temperature and insensitive to changes in microturbulent velocity, the latter affecting only the line opacity.

7. Discussion

Some of the basic ideas for deriving the CF to the line depression were already present in the work of Caccin et al. (1977). Apart from the fact that they consider the absolute line depression

$$D = I_c - I_l \tag{31}$$

instead of the relative line depression R (11), they correctly recognize that, to obtain the CF, one "must write a differential equation for D and identify the source and sink terms in it". However, they argue that "the right hand member of this equation can be written in two different ways, separating differently the term in D and the known term". These two ways are, in our notation (and in LTE):

$$\frac{\mu}{\varrho}\frac{dD}{dz} = \kappa_c\left[D - \frac{\kappa_l}{\kappa_c}(I_l - B)\right], \tag{32}$$

and

$$\frac{\mu}{\varrho}\frac{dD}{dz} = \kappa_c\left[\left(1+\frac{\kappa_l}{\kappa_c}\right)D - \frac{\kappa_l}{\kappa_c}(I_c - B)\right]. \tag{33}$$

Arguing that (32) has a clearer interpretation than (33), they consider it as "the equation of transfer for D" and conclude that "the integrand of its formal solution can be properly called the CF for the line depression".

The wrong point in that development is that (32) and (33) are not two equivalent ways of separating the term in D and the known term. In fact, I_l cannot be regarded as known as long as D is unknown. These two quantities are related by Eq. (31), where I_c may be regarded as known since it is independent of the line absorption. So, either D or I_l may be taken as the unknown, but the transfer equation may not be written in terms of these two quantities, considering one as known and the other as unknown. It would be impossible to solve (32) without taking (31) into account, in which case (32) reduces to (33). On the other hand, if both I_c and I_l were known, it would not be necessary to solve a transfer equation for D, it would be simply given by (31). So the only valid transfer equation for D, separating correctly the known and unknown quantities, is (33). A similar discussion applies if D is replaced by R. In fact, it is easily shown that S_R and κ_R are unique, as far as they do not contain any explicit dependence on R (or I_l).

To conclude on the unicity of the CF to the line depression, let us come back to the different CF's arising from mathematical transformations of the integral (7). Let us consider, as an example, a distribution of particles as a function of some quantity y, given by the distribution function

$$f(y) = \alpha e^{-y} \tag{34}$$

for y varying from 0 to $+\infty$. The total number of particles is given by

$$N = \int_0^{+\infty} \alpha e^{-y} dy . \tag{35}$$

Integrating (34) by parts, we see that the total number of particles is also given by

$$N = \int_0^{+\infty} \alpha y e^{-y} dy . \tag{36}$$

Of course, nobody will ever argue that the distribution function of the particles is equally given by $\alpha y e^{-y}$. This function has no physical meaning, except that its integral over y gives the total number of particles. It can be used for nothing else. Only the integrals are equivalent, but not the integrands. This is certainly trivial, but, strange enough, it does not seem so as soon as y stands for $\log \tau_0$, $f(y)$ for the CF and N for the line depression. Showing that the line depression may be computed from various integrals does not allow to conclude that the integrands have any physical meaning. Unfortunately, this has been done many times in the past.

Rather recently, Babii and Rikalyuk (1981) have discussed the various CF's and have listed a number of requirements which, in their opinion, must be satisfied by a CF in order to be physically and mathematically sound. In particular, they note that "with movement into the line wing, i.e. into the region of ever decreasing absorption in the line, the optical depths of formation of absorption lines, [...], must approach the same optical depth of formation of the continuous spectrum at the given λ, although in somewhat different ways".

This is a misunderstanding which appears from time to time in the literature. As already stressed by De Jager (see Gurtovenko et al., 1974), it is essential to distinguish between the region of origin of the emergent radiation and the region where the line depression is formed. If it is true that, as the line opacity tends to zero, the layers from which the intensity in the line originates tend to coincide with the continuum-forming layers, this is not true for the line depression. This is particularly clear in the following extreme example. Consider a telluric water vapor line appearing in the solar spectrum as observed from the ground. It is clear that, even if the line becomes very faint, or if we move to the line wing, it does not shift to the sun. It is always formed in the earth atmosphere. In this case, $\mathscr{C}_R^{(1)}$ and $\mathscr{C}_R^{(3)}$ would predict that most of the line depression originates from the solar photosphere, while $\mathscr{C}_R$ and $\mathscr{C}_R^{(2)}$ would correctly place its layers of formation in the earth atmosphere. So, as correctly indicated by $\mathscr{C}_R$, a faint absorption line is not necessarily formed in the same layers as the continuum.

In conclusion, from the corresponding transfer equation, we have defined the contribution function to the line depression. This function is the only one which gives the contribution of the different layers to the observed line depression. We have also shown that it is in perfect agreement with physical intuition, contrary to what had sometimes been argued in the past about $\mathscr{C}_R^{(2)}$.

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