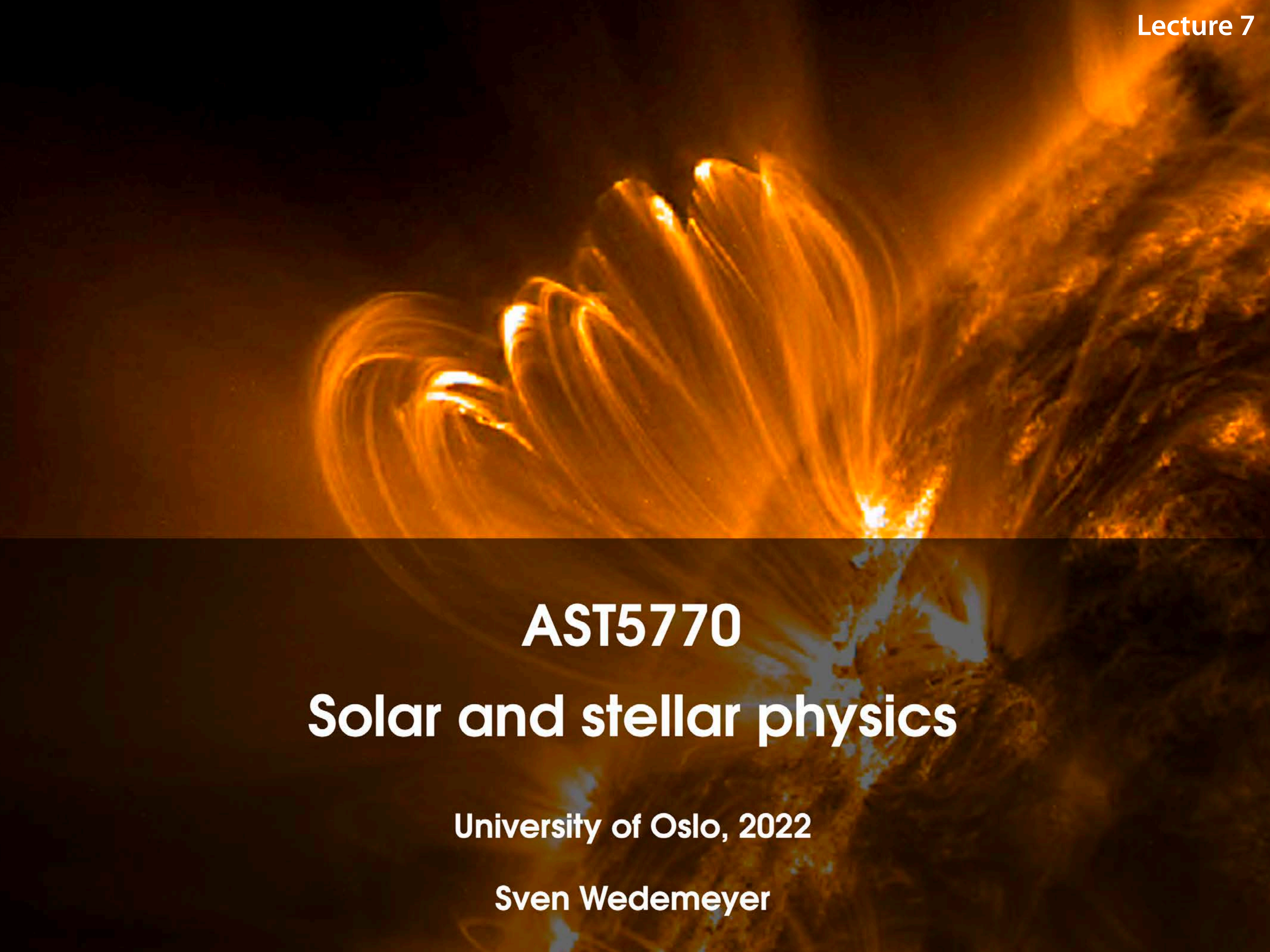


Lecture 7

AST5770

Solar and stellar physics

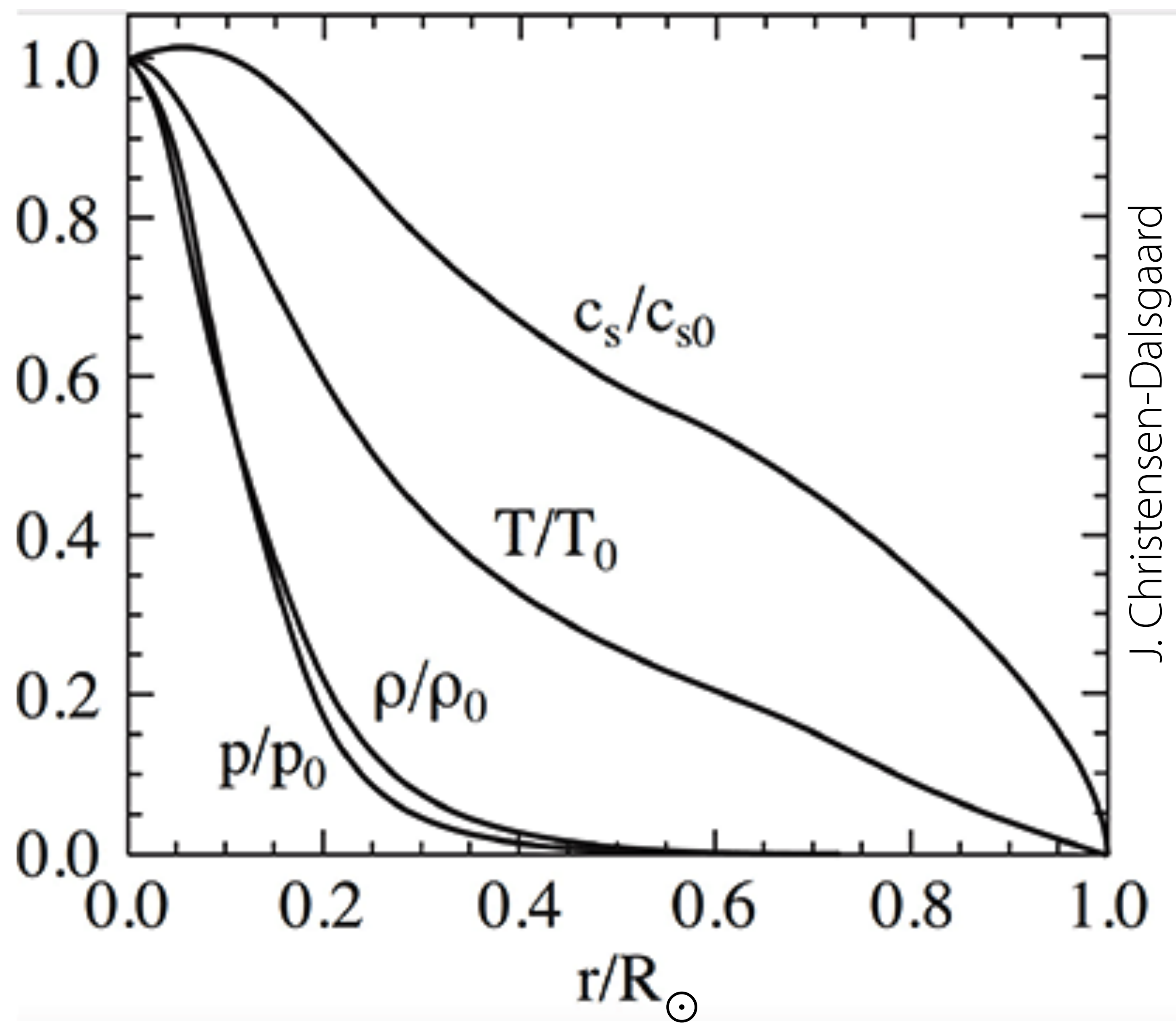
University of Oslo, 2022

Sven Wedemeyer

Stellar interior

Standard model of the solar interior

- Variation of (average) quantities as function of radius in the solar interior ($r/R_\odot$)
- Scaled to value at solar centre
- Temperature T_0=1.57 10^7 K
- Mass density ρ_0=1.54 10^5 kg m^{-3}
- Pressure p_0=2.35 10^{16} Nm^{-2}
- Sound speed $c_{s,0}$=5.05 10^5 ms^{-1}



Energy transport

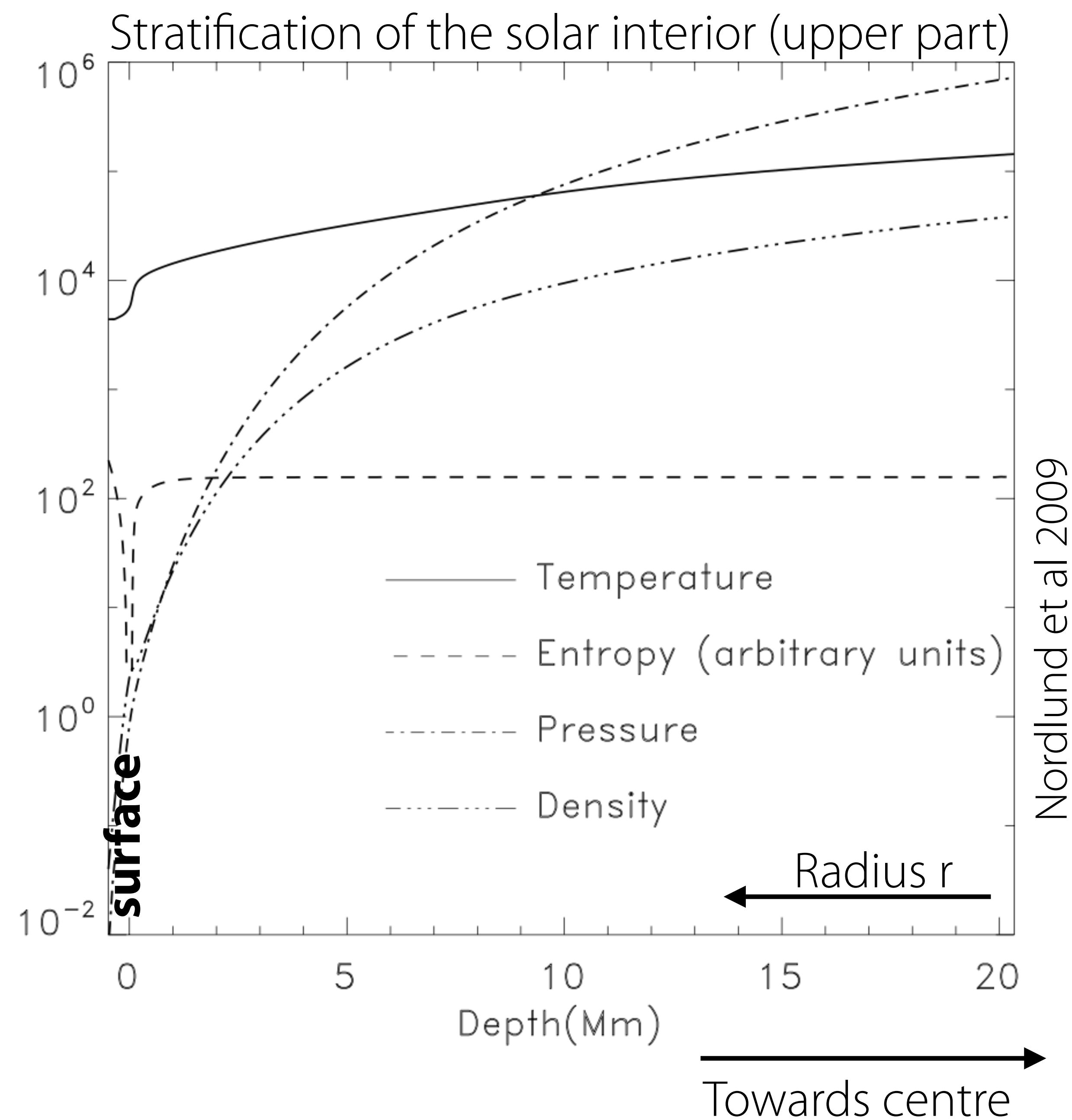
Convection — Stability criterion

- **Plasma/gas inside a star not a perfectly stratified but small perturbations occur**

➡ Is a layer stable against small perturbations?

➡ Or can initially small perturbations grow and result in significant deviations?

- **In the stratified stellar interior:** Pressure and density decreases with radius r towards surface

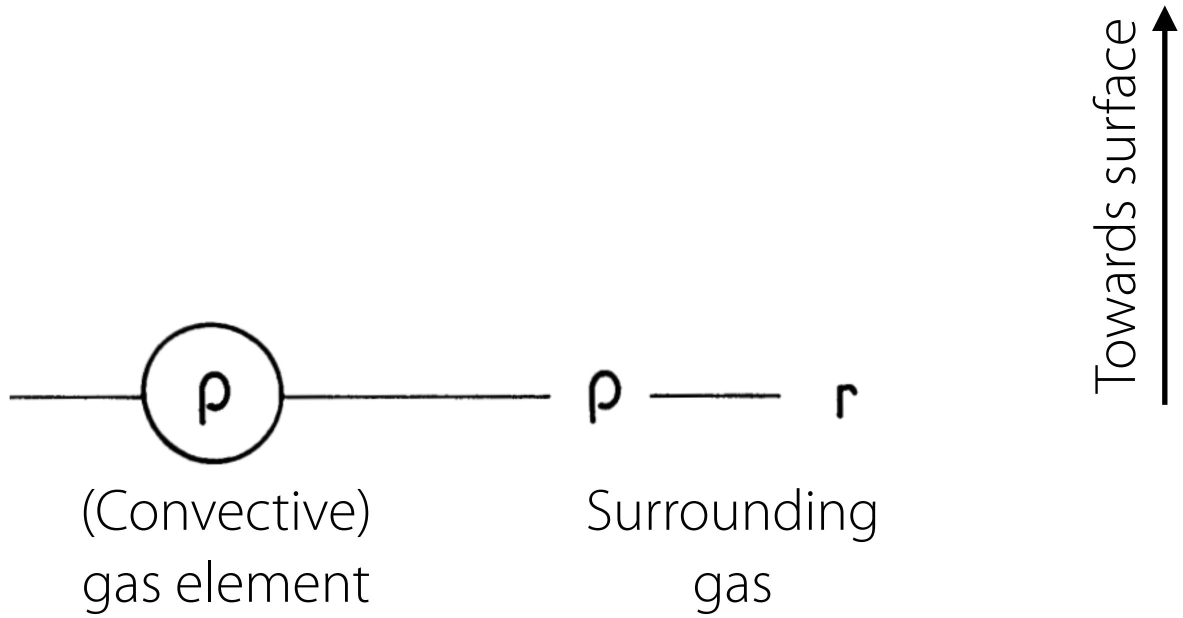


Energy transport

Convection — Stability criterion

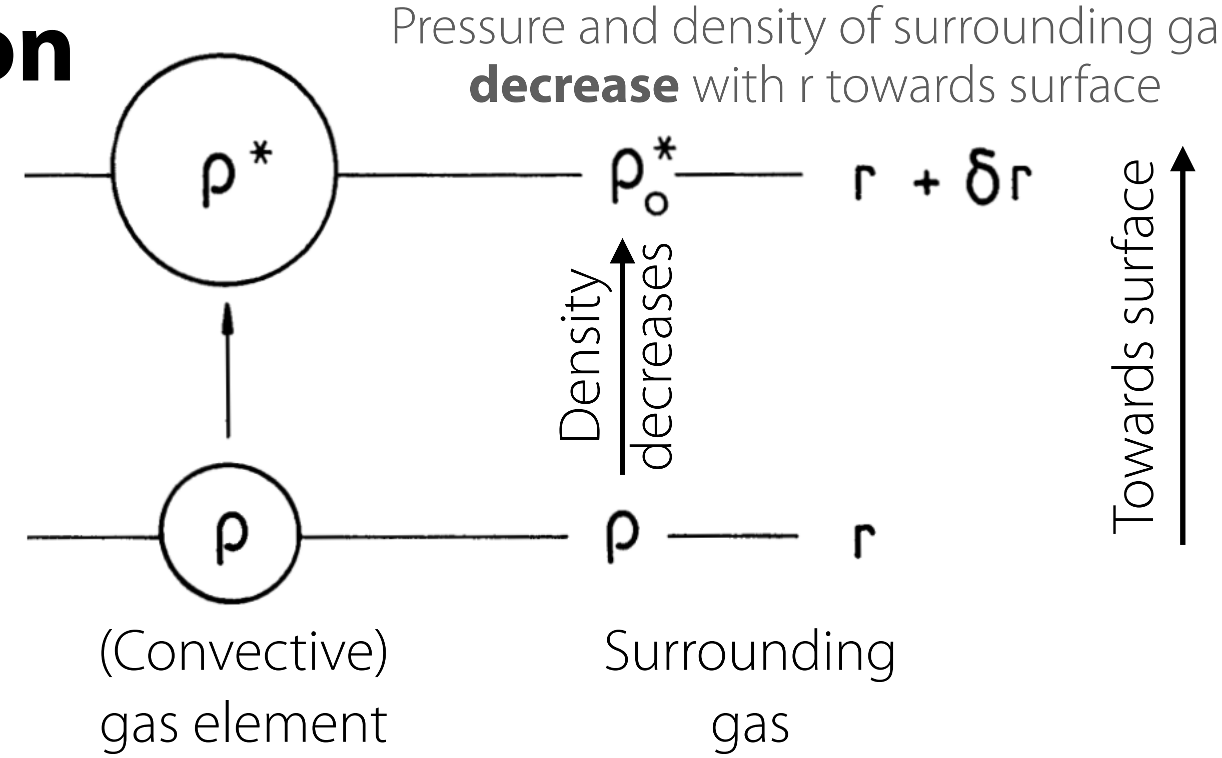
- **Gas element** at distance r from the centre of the star
- Initially: Element in **equilibrium** with its surroundings at r:

➡ Pressure P and density ρ are the same as in its surroundings.



Energy transport

Convection — Stability criterion



- Gas element at distance r from the centre of the star
- Initially: Element in equilibrium with its surroundings at r:

➡ Pressure P and density ρ are the same as in its surroundings.

- Now — perturbation: **element** displaced (**rises**) a vertical distance δr **adiabatically** (no heat exchanged with environment) but slow enough that pressure is adjusted to new balance with outside pressure
  - Occurs when <u>time scale</u> **of heat exchange** is long compared to **time scale of expansion** of the element (the latter = local dynamical time scale, set by local sound speed); happens in the optically thick solar interior

➡ Element expands to restore pressure equilibrium with surrounding

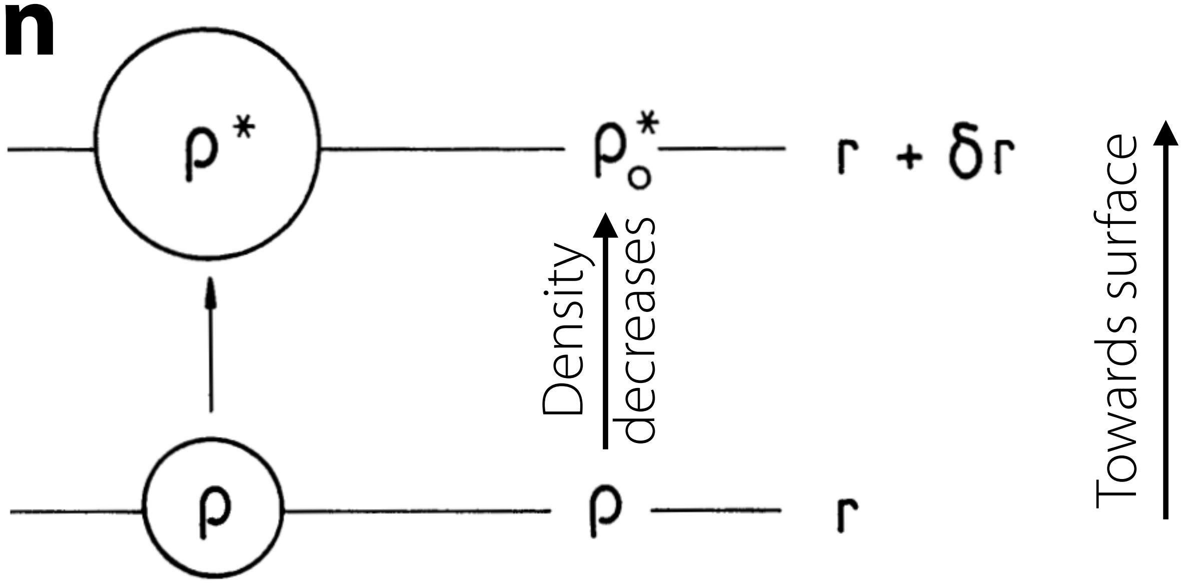
➡ Pressure in the element reduce as it rises

➡ Now at r+δr: Compare density of the element ρ^* with density of (new) surrounding ρ_0^*

Energy transport

Convection — Stability criterion

- Gas element at distance r from the centre of the star rises …
- Initially in equilibrium with its surroundings at r



- $\varrho^* > \varrho_0^*$: Element will fall back to initial height — stratification is **convectively stable**
- $\varrho^* < \varrho_0^*$: Element will keep rising up (net buoyancy!) — stratification is **convectively unstable**

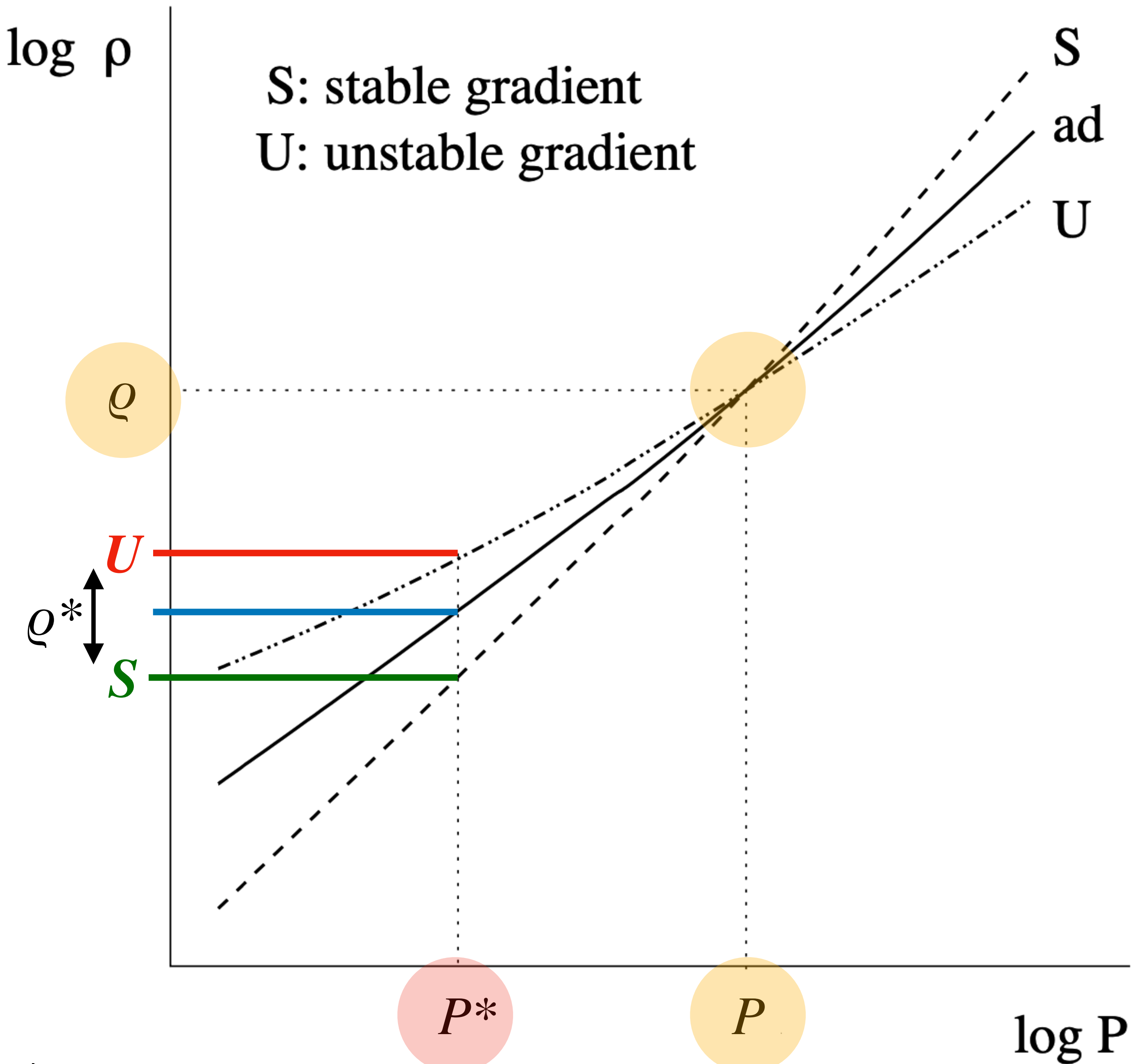
- At $r+\delta r$: Density difference ϱ^* - ϱ_0^* — gradients!
- For adiabatic expansion: $$\frac{\delta P^*}{P^*} = \gamma_{\mathrm{ad}} \frac{\delta \varrho^*}{\varrho^*}$$

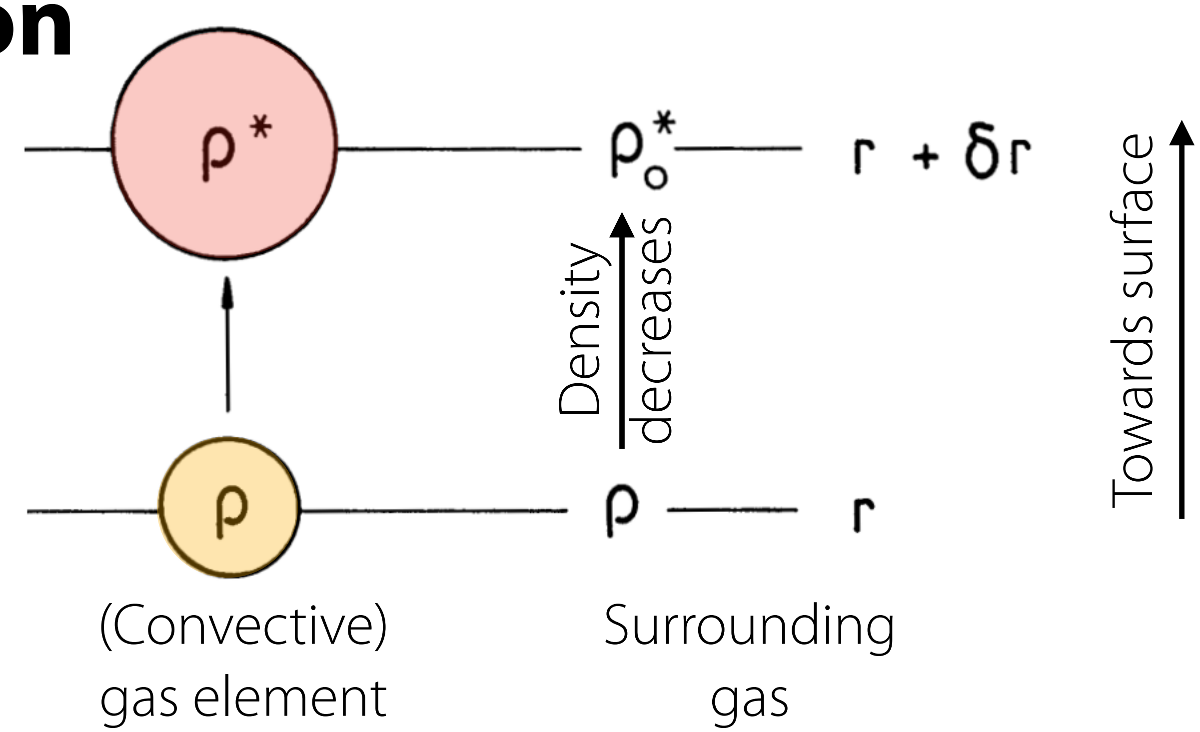
➡ For the situation above, we derive the following criterion for the gas remaining convectively stable:

$$\frac{\mathrm{d}\log\rho}{\mathrm{d}\log P} > \frac{1}{\gamma_{\mathrm{ad}}}$$

Energy transport

Convection — Stability criterion





$$\frac{\mathrm{d}\log\rho}{\mathrm{d}\log P} > \frac{1}{\gamma_{\mathrm{ad}}}$$

Pols

Energy transport

Convection — Stability criterion

- Compare gradient $\nabla_{\rm rad}$ for convectively stable stratification with adiabatic temperature gradient $\nabla_{\rm ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)$

∇_{rad}: spatial gradient of temperature

∇_{μ}: spatial gradient of mean molecular weight

∇_{ad}: adiabatic temperature variation in a gas element undergoing a change in pressure.

- **Ledoux criterion** of stability against convection

$$\nabla_{\rm rad} < \nabla_{\rm ad} - \frac{\chi_\mu}{\chi_T} \nabla_\mu$$

$$\chi_T = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rho, X_i} \quad \chi_\rho = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{T, X_i}$$

: Indices: quantities held constant

- For chemically homogeneous gas: $\nabla_\mu = 0$:

➡ **Schwarzschild criterion** of stability against convection

$$\nabla_{\rm rad} < \nabla_{\rm ad}$$

- Note: In presence of fusion reactions: $\nabla_\mu \geq 0$
- Stabilising effect! (An upwards displaced element is heavier due to higher μ)

Energy transport

Convection — Stability criterion

- Compare gradient $\nabla_{\rm rad}$ for convectively stable stratification with adiabatic temperature gradient $\nabla_{\rm ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)$

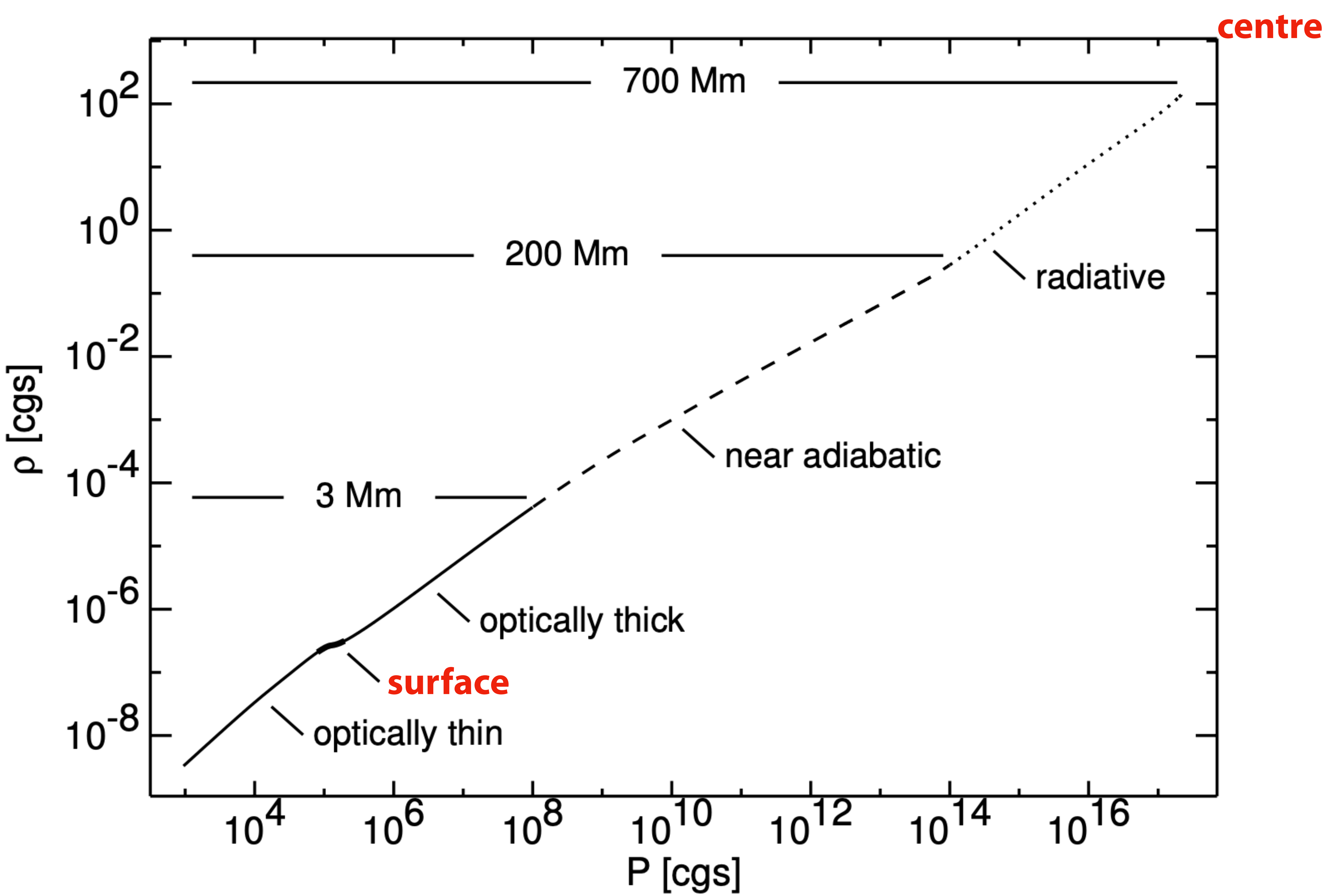
- **Ledoux criterion** of stability against convection

$$\nabla_{\rm rad} < \nabla_{\rm ad} - \frac{\chi_\mu}{\chi_T} \nabla_\mu$$

- Stratification **convectively stable: Energy transport by radiation**
- Stratification **convectively unstable: Energy transport by convection**

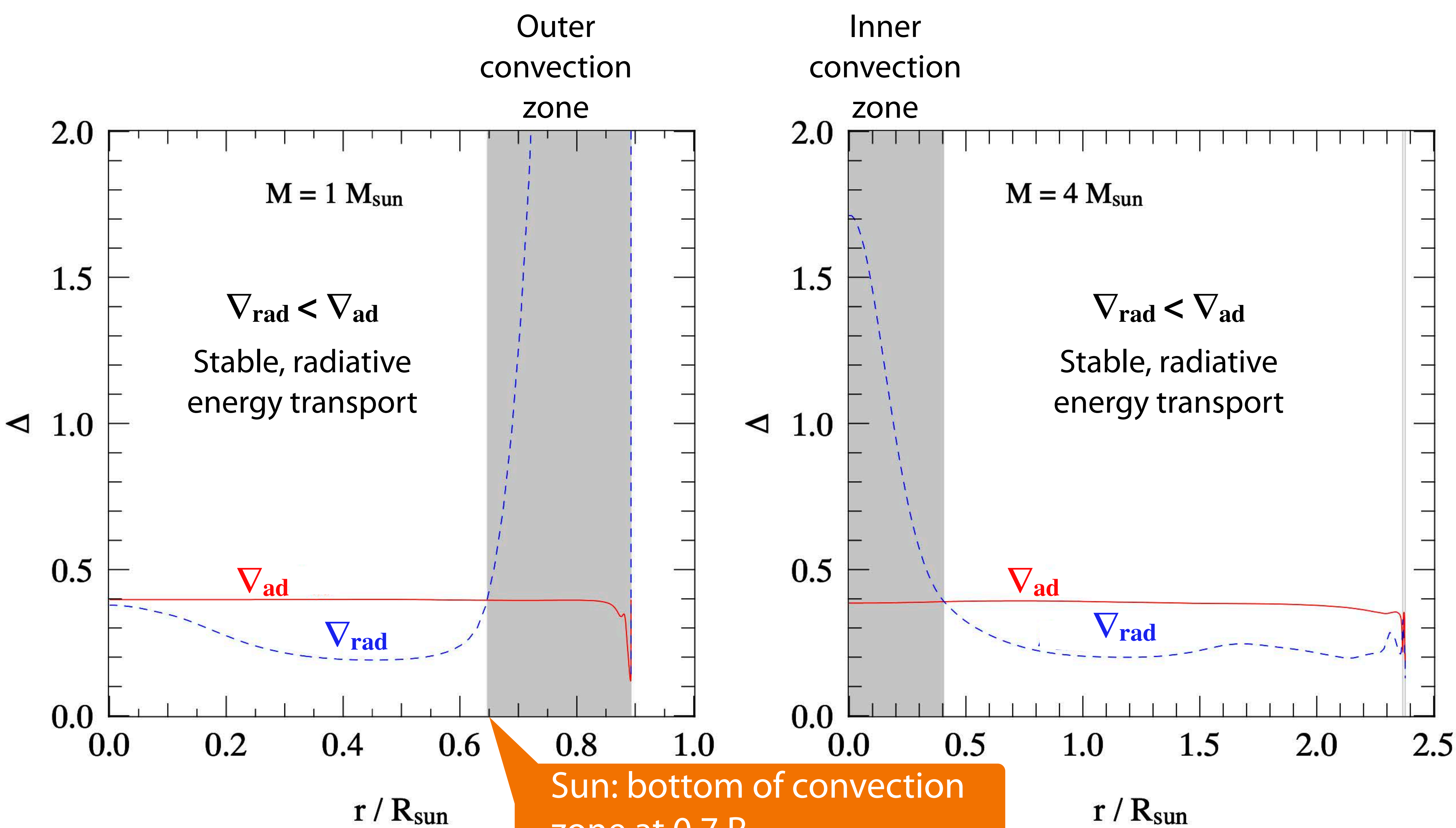
Energy transport

Convection — Stability criterion in the solar interior



Energy transport

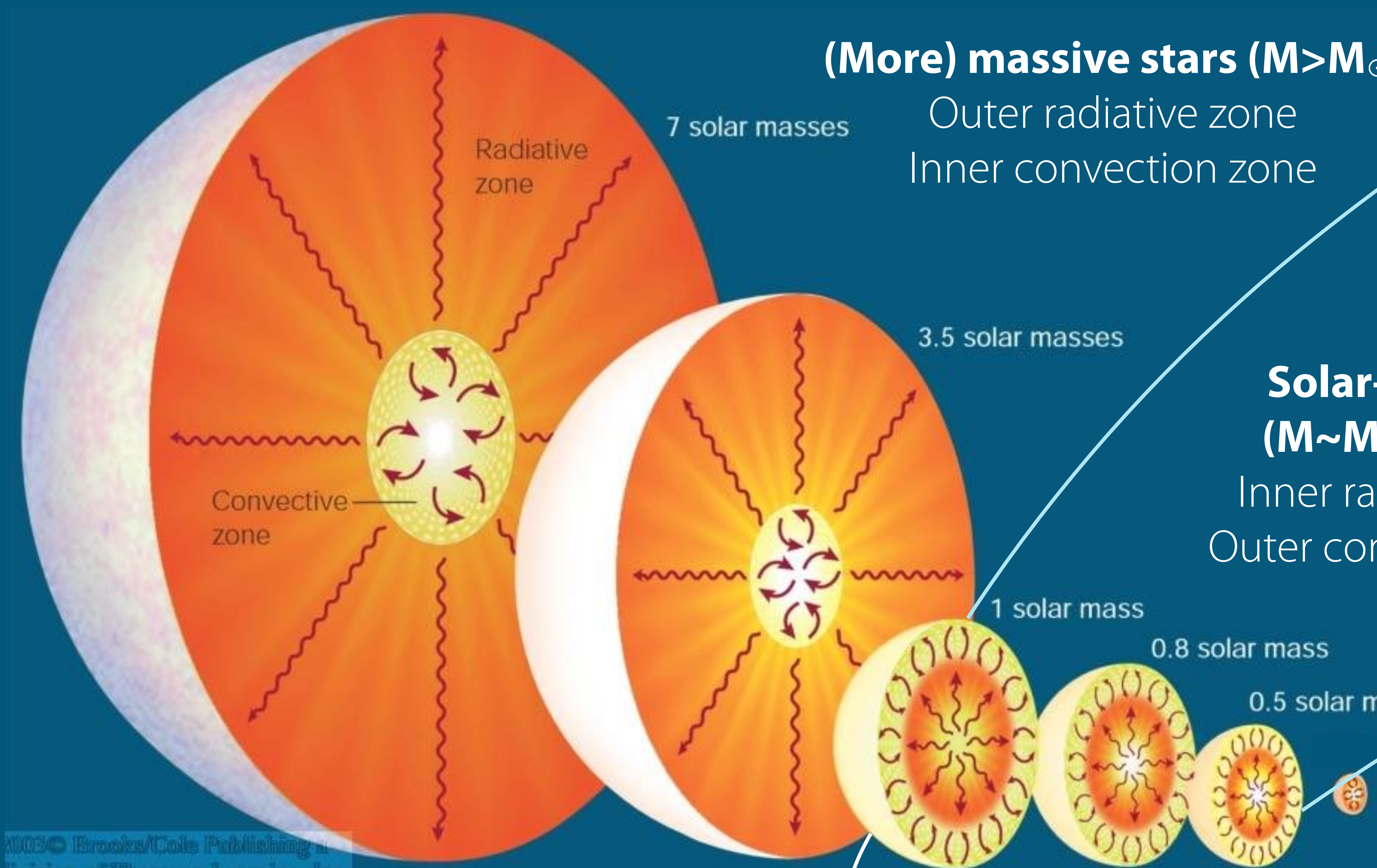
Convection — Stability criterion



Sun: bottom of convection zone at 0.7 $R_\odot$

Energy transport

Differences along the main sequence



(More) massive stars ($M>M_\odot$)
Outer radiative zone
Inner convection zone

Solar-like stars ($M \sim M_\odot ... 0.4 M_\odot$)
Inner radiative zone
Outer convection zone

Low-mass stars ($M< 0.3\text{-}0.4 M_\odot$)
Fully convective

Dominant fusion process	**CNO cycle**	**pp chain**

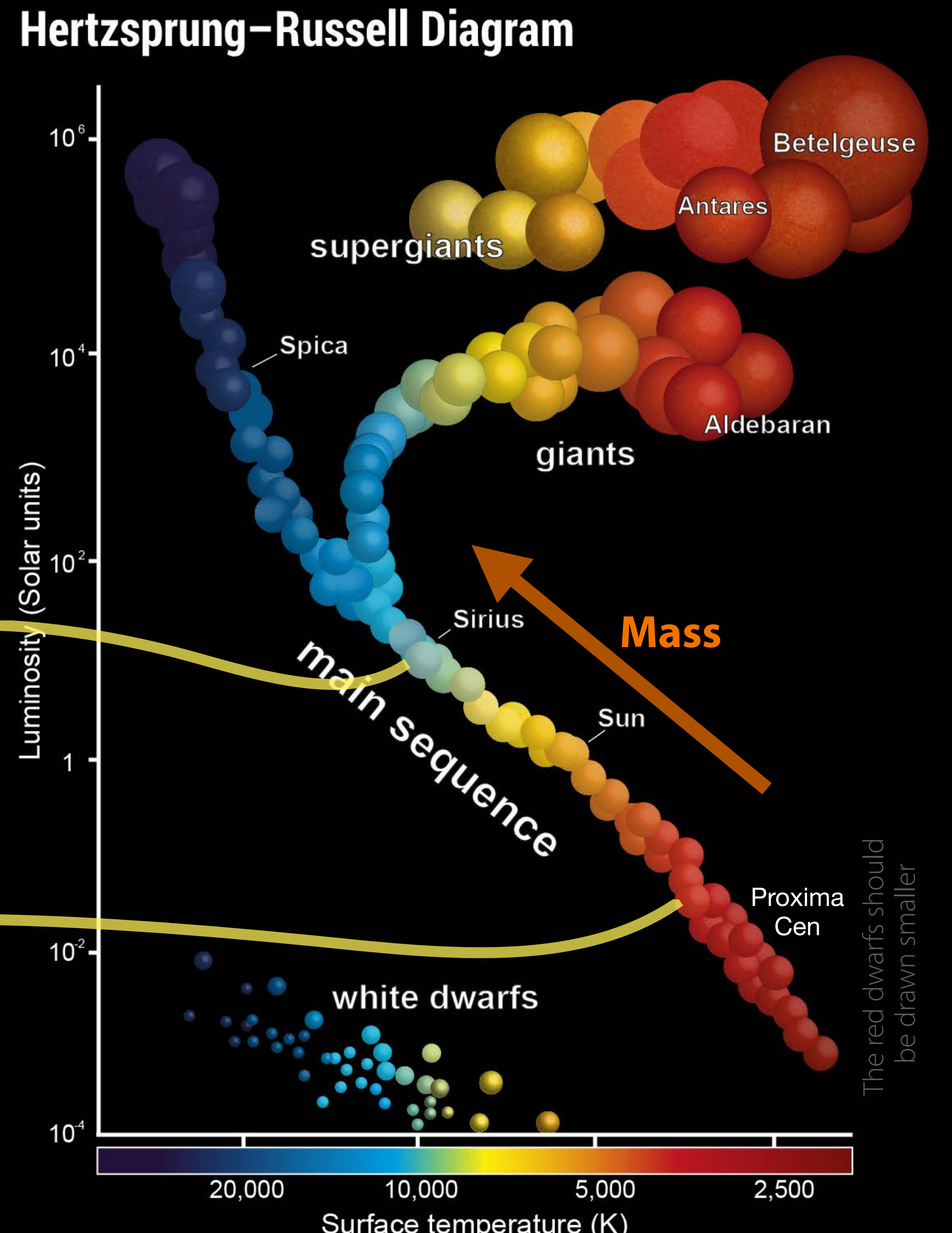
Energy transport

Differences along the main sequence

(More) massive stars (M>M⊙)
Outer radiative zone
Inner convection zone

Solar-like stars (M~M⊙…0.4M⊙)
Inner radiative zone
Outer convection zone

Low-mass stars (M< 0.3-0.4M⊙)
Fully convective



Energy transport

Mixing length theory

- In the context of stellar (interior) structure and evolution, we want to know about convection zones:
  - **How much energy** can be transported by convection?
  - What is the **temperature gradient** ?

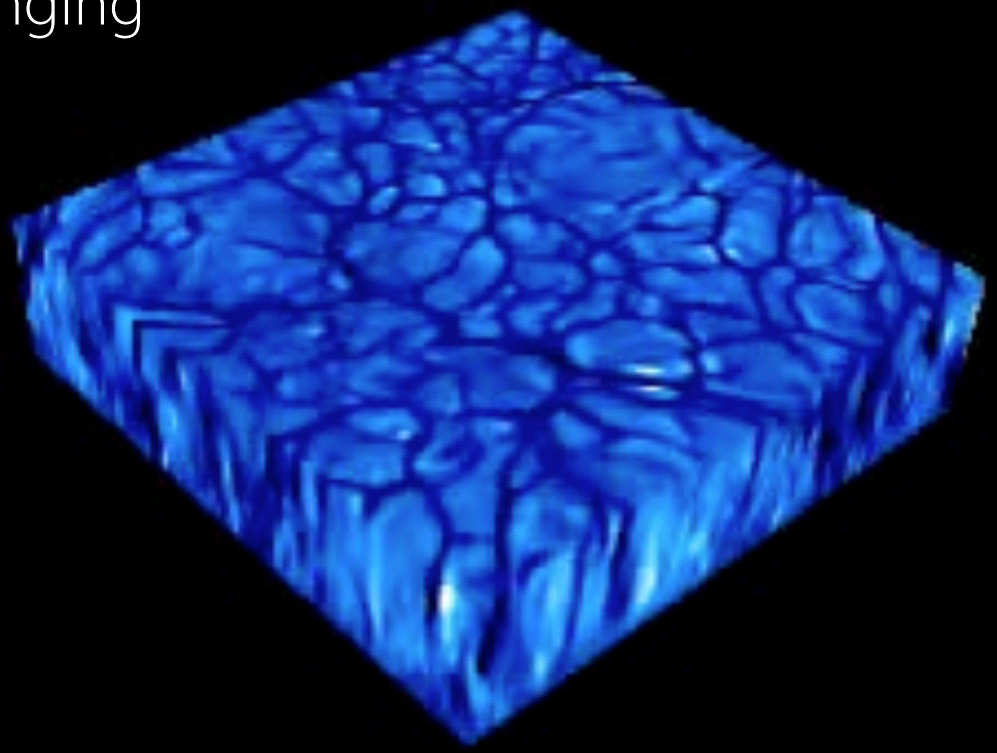
- But: Detailed theory of convection and its practical application still challenging today

- Addressed with numerical simulations but very challenging and computationally expensive

  ➡ Prohibitive to use as part of stellar evolution calculations.

  ➡ Simpler approach needed

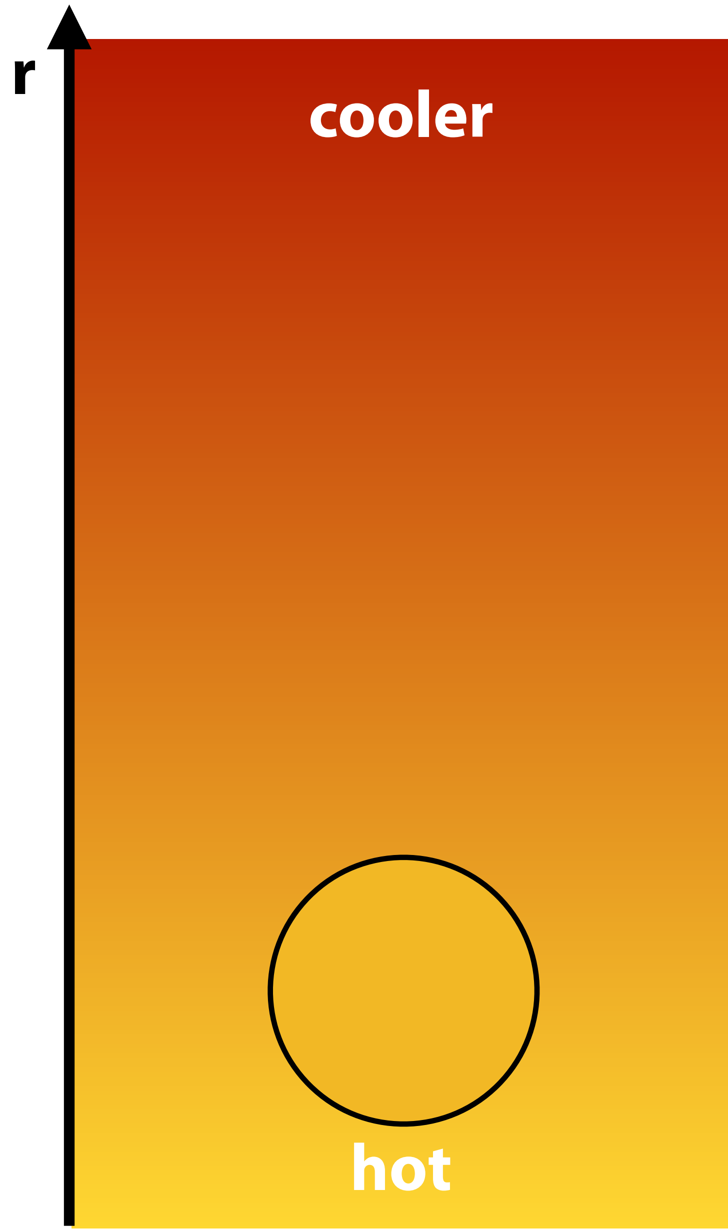
  ➡ **Mixing Length Theory (MLT)**



Energy transport

Mixing length theory

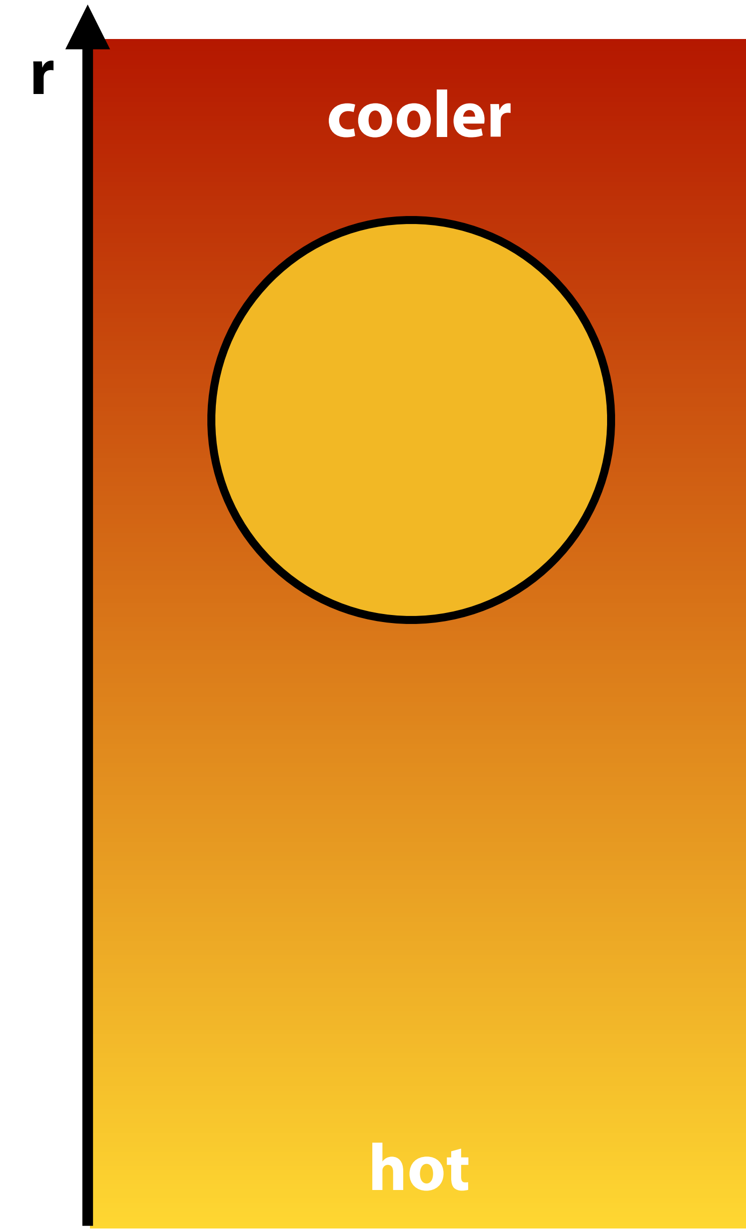
- Simplified picture of convective energy transport:
  1. Gas element rises (or sinks) over a radial distance
  2. Gas element dissolves (becomes part of the new environment) and releases excess heat
  - Mixing length l_m
  - If gas element sinks, it absorbs deficit energy from environment
  - Mixing length l_m is an unknown free parameter!
  - Assumption: l_m on the order of local pressure scale height H_P



Energy transport

Mixing length theory

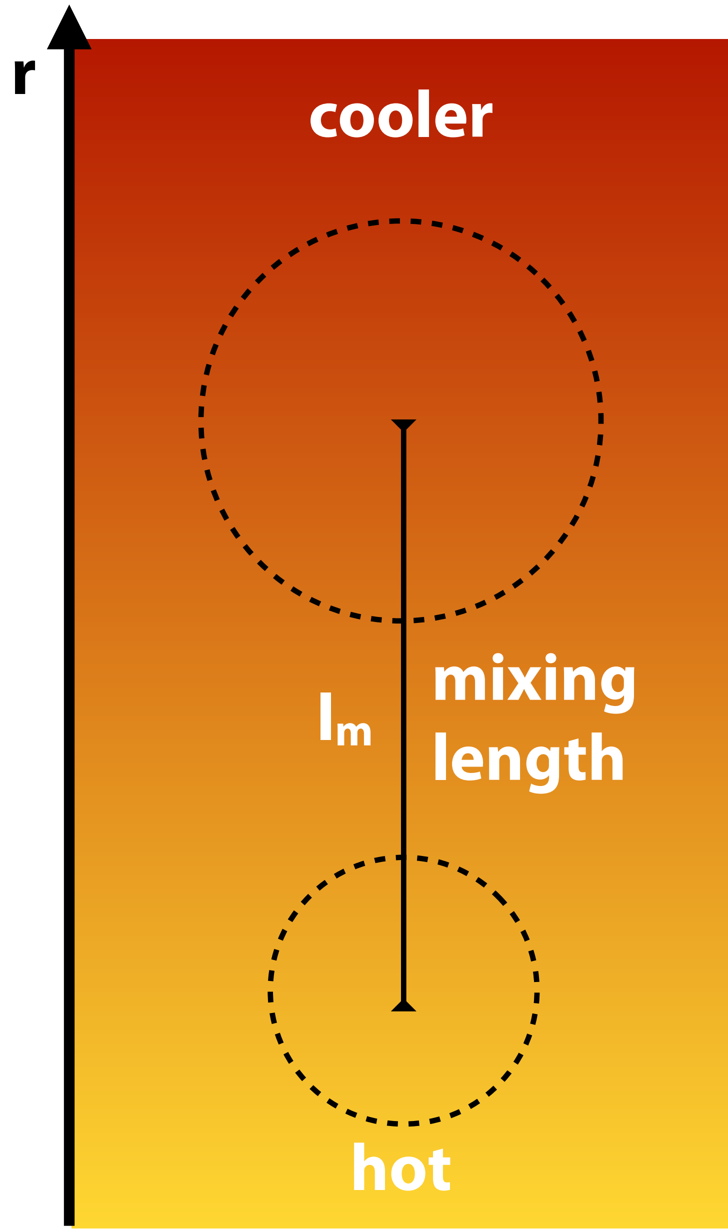
- Simplified picture of convective energy transport:
  1. Gas element rises (or sinks) over a radial distance
  2. Gas element dissolves (becomes part of the new environment) and releases excess heat
  - Mixing length l_m
  - If gas element sinks, it absorbs deficit energy from environment
  - Mixing length l_m is an unknown free parameter!
  - Assumption: l_m on the order of local pressure scale height H_P



Energy transport

Mixing length theory

- Simplified picture of convective energy transport:
    1. Gas element rises (or sinks) over a radial distance
    2. Gas element dissolves (becomes part of the new environment) and releases excess heat
  - Mixing length l_m
  - If gas element sinks, it absorbs deficit energy from environment
  - Mixing length l_m is an unknown free parameter!
  - Assumption: l_m on the order of local pressure scale height H_P



- **Pressure scale height:** (in a stratified medium) the radial distance over which the pressure changes by a factor 1/e

$$H_P = \left|\frac{\mathrm{d}r}{\mathrm{d}\ln P}\right| = \frac{P}{\rho g}.$$

Energy transport

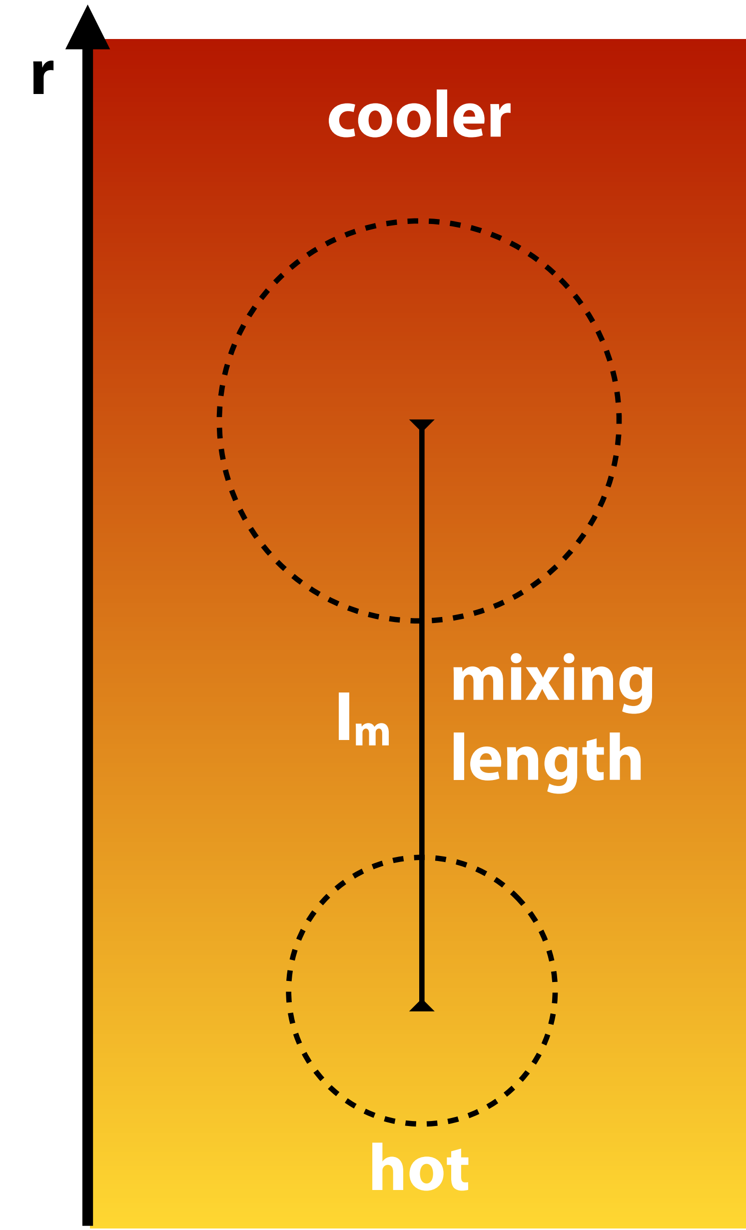
Mixing length theory

- Assumption: $l_m = \alpha\, H_P$
  - Valid in hydrostatic equilibrium.
  - Reasonable as gas element expands while rising

- Assume spherical surface inside the convection zone
  - 1/2 covered by rising blobs
  - 1/2 covered by sinking blobs

  ➡ Expanding rising blobs would cover most of the area after rising 1-2 pressure scale heights.

  ➡ **Net energy transport upwards** (down the gradient)



- **Pressure scale height:** (in a stratified medium) the radial distance over which the pressure changes by a factor 1/e

$$P(r) = P_0 e^{-(r/H_P)} \implies H_P = \left|\frac{\mathrm{d}r}{\mathrm{d}\ln P}\right| = \frac{P}{\rho g}.$$

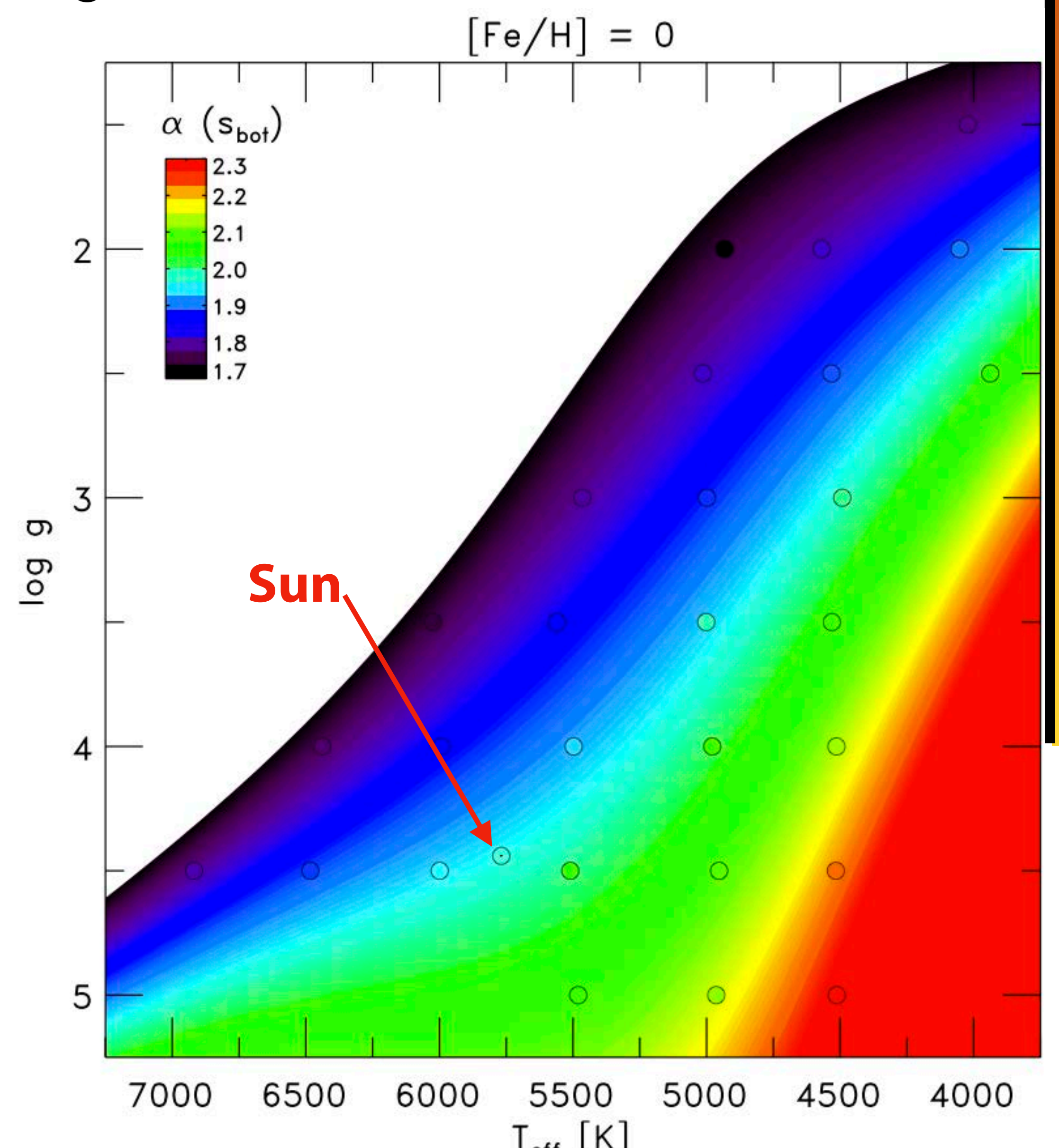
Energy transport

Mixing length theory

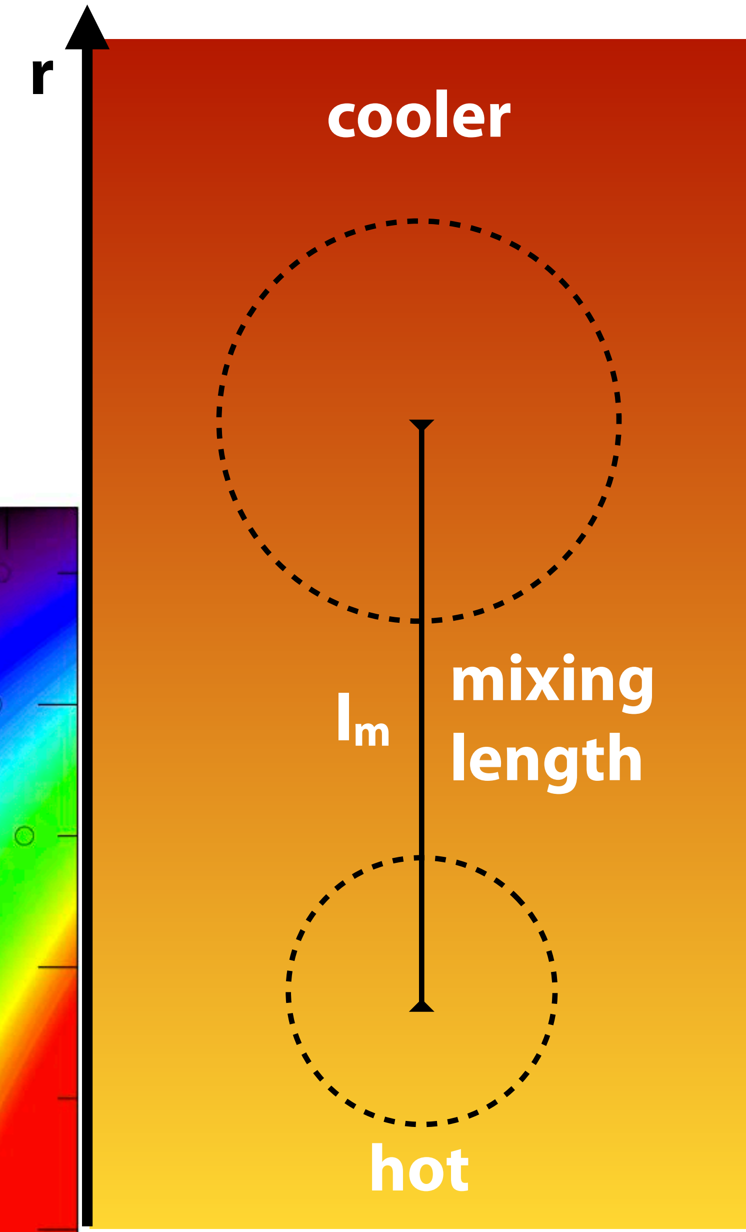
- Detailed numerical model calculations
  (in comparison to observation)
  to derive/calibrate corresponding mixing length

- Mixing length (via parameter depends on
  - Effective temperature $T_{\rm eff}$
  - Grav. Acceleration $log\ g$
  - Metallicity $[Fe/M]$

- Values for α typically ~2



Magic et al. 2015a



Energy transport

The convective energy flux

- Gas element after rising a distance $\delta r = l_m$:

➡ Mean excess temperature ΔT between element and surrounding:

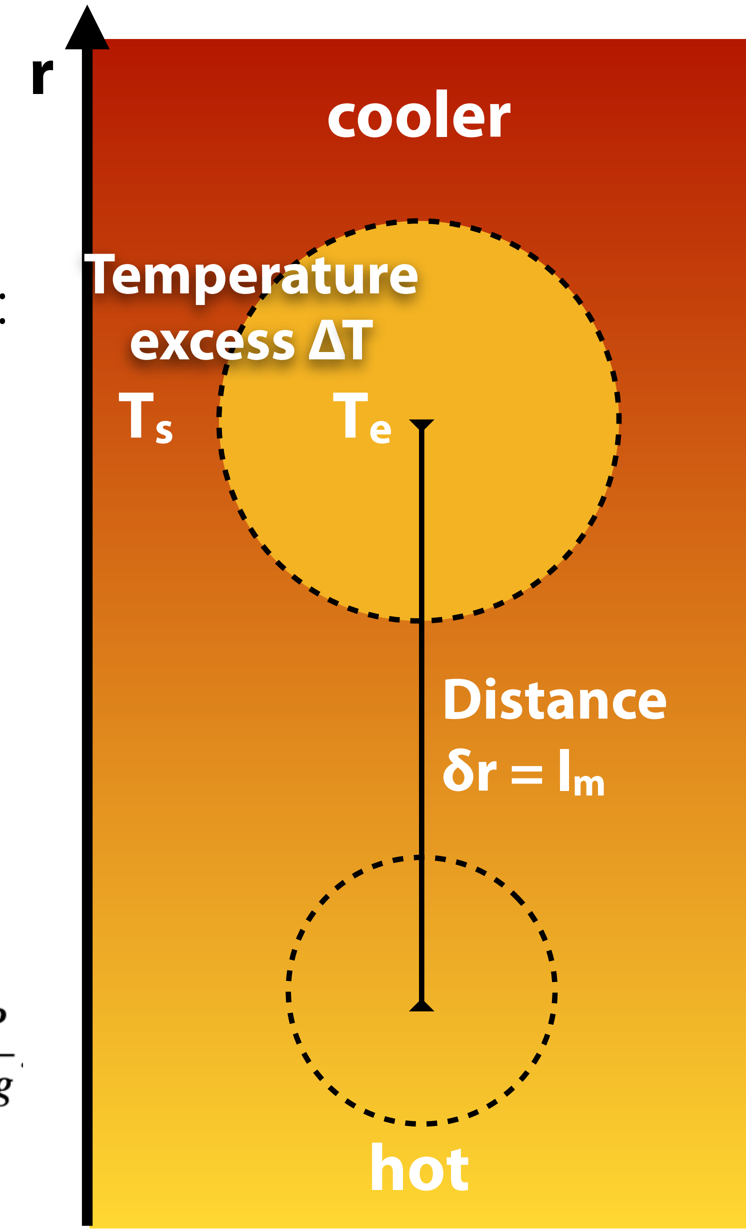
$$\Delta T = T_\mathrm{e} - T_\mathrm{s} = \left[\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_\mathrm{e} - \frac{\mathrm{d}T}{\mathrm{d}r}\right]\ell_\mathrm{m} = \Delta\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)\ell_\mathrm{m}$$

  - dT/dr : temperature gradient in surrounding
  - $(dT/dr)_e$: variation of temperature with radius r for the gas element while rising and expanding adiabatically
  - $\Delta(dT/dr)$: difference between the two gradients.

- Rewrite the equation above with gradients $\nabla \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)$ and $H_P = \left|\frac{\mathrm{d}r}{\mathrm{d}\ln P}\right| = \frac{P}{\rho g}$, deriving and using the following equation

$$\frac{\mathrm{d}T}{\mathrm{d}r} = T\frac{\mathrm{d}\ln T}{\mathrm{d}r} = T\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\frac{\mathrm{d}\ln P}{\mathrm{d}r} = -\frac{T}{H_P}\nabla \quad \text{and} \quad \left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_e = -\frac{T}{H_P}\nabla_\mathrm{ad}$$

➡ $$\Delta T = T\,\frac{\ell_\mathrm{m}}{H_P}(\nabla - \nabla_\mathrm{ad})$$



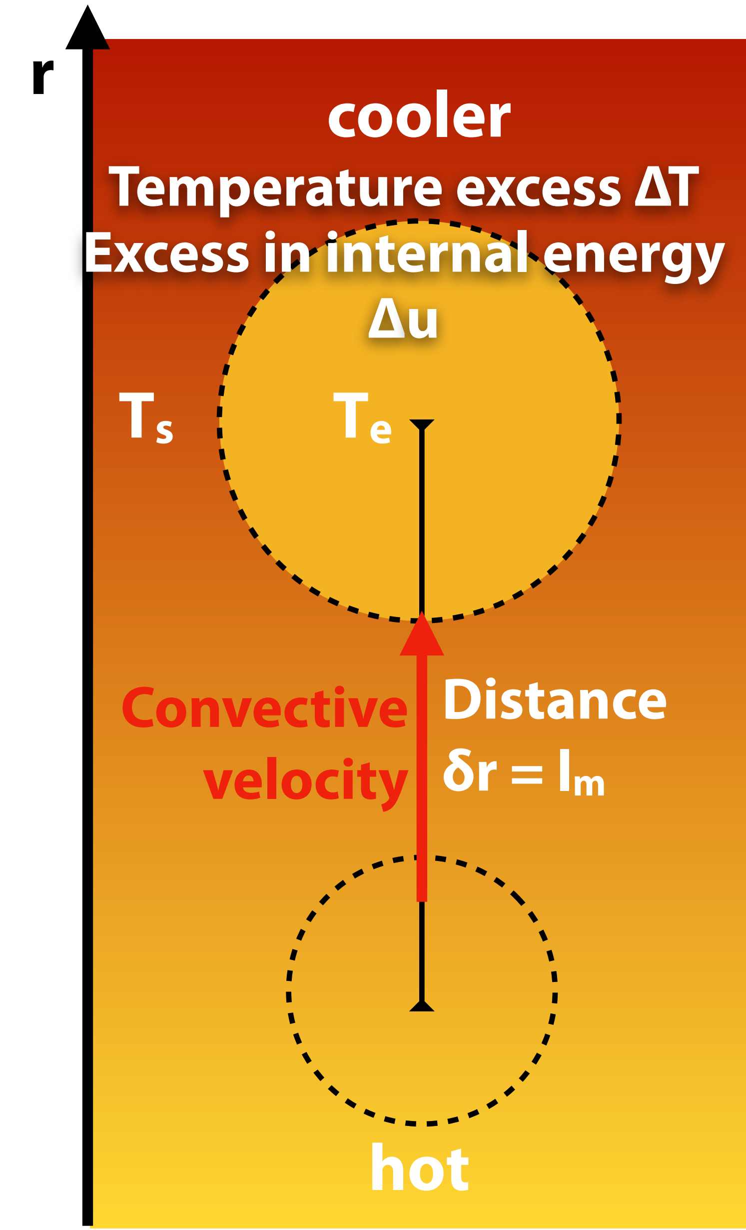
Energy transport

The convective energy flux

- Mean temperature excess ΔT is related to an excess in internal energy between the gas element and the surrounding
  $\Delta u = c_P \, \Delta T \, \varrho$

- Energy flux carried by gas elements at (average) velocity v_c (the convective velocity)

  $F_{\mathrm{conv}} = v_c \, \varrho \, \Delta u = v_c \, \varrho \, c_P \, \Delta T$



➡ What is the **convective velocity**?

➡ Gas element moves over distance l_m in the time t starting from resting position at constant acceleration: $l_m = {}^1\!/_2 \, a \, t^2$

➡ Average velocity $v_c \approx l_m / t = \sqrt{\frac{1}{2}\ell_{\mathrm{m}} a}$

➡ Buoyancy force provides the acceleration $a = -g \, \frac{\Delta\rho}{\rho} \approx g \, \frac{\Delta T}{T}$

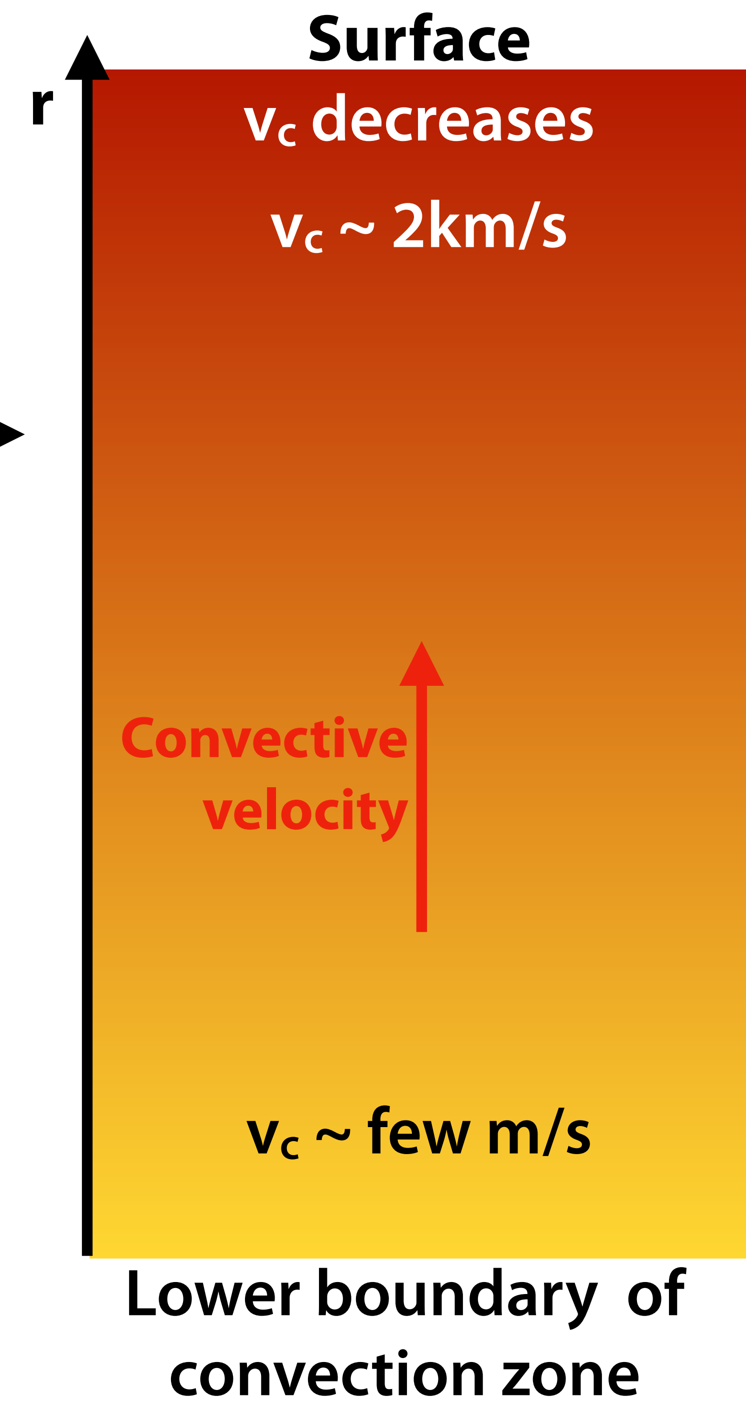
$$\Delta T = T \, \frac{\ell_{\mathrm{m}}}{H_P} (\nabla - \nabla_{\mathrm{ad}})$$

Energy transport

The convective energy flux

➡ Convective velocity $v_c \approx \sqrt{\frac{1}{2}\ell_m g \frac{\Delta T}{T}} \approx \sqrt{\frac{{\ell_m}^2 g}{2H_P}(\nabla - \nabla_{ad})}$

- Increases with radius
- For the Sun up to ~2 km/s (average velocity)



➡ **Convective energy flux** $F_{conv} = \rho c_P T \left(\frac{\ell_m}{H_P}\right)^2 \sqrt{\frac{1}{2} g H_P}\,(\nabla - \nabla_{ad})^{3/2}$

- **Superadiabaticity** $\nabla - \nabla_{ad}$: degree to which the actual temperature gradient ∇ exceeds the adiabatic value ∇_{ad}.
- **What $\nabla - \nabla_{ad}$ is needed to carry the whole energy flux by convection?**
  - With typical values for the whole star (using the virial theorem):

➡ $F_{conv} \sim \frac{M}{R^3}\left(\frac{GM}{R}\right)^{3/2}(\nabla - \nabla_{ad})^{3/2}$

Energy transport

The convective energy flux and temperature gradient

- Combine $F_{\text{conv}} = \rho c_P T \left(\frac{\ell_\text{m}}{H_P}\right)^2 \sqrt{\tfrac{1}{2} g H_P} \, (\nabla - \nabla_\text{ad})^{3/2}$ and $F_{\text{conv}} \sim \frac{M}{R^3}\left(\frac{GM}{R}\right)^{3/2} (\nabla - \nabla_\text{ad})^{3/2}$

➡ **Superadiabaticity** $\nabla - \nabla_\text{ad} \sim \left(\frac{LR}{M}\right)^{2/3} \frac{R}{GM}$

- Typical values in the interior of the Sun $\nabla - \nabla_\text{ad} \sim 10^{-5} - 10^{-7}$

➡ Only very small superadiabaticity needed!

➡ Convective energy transport is very efficient with $F_\text{conv} \gg F_\text{rad}$

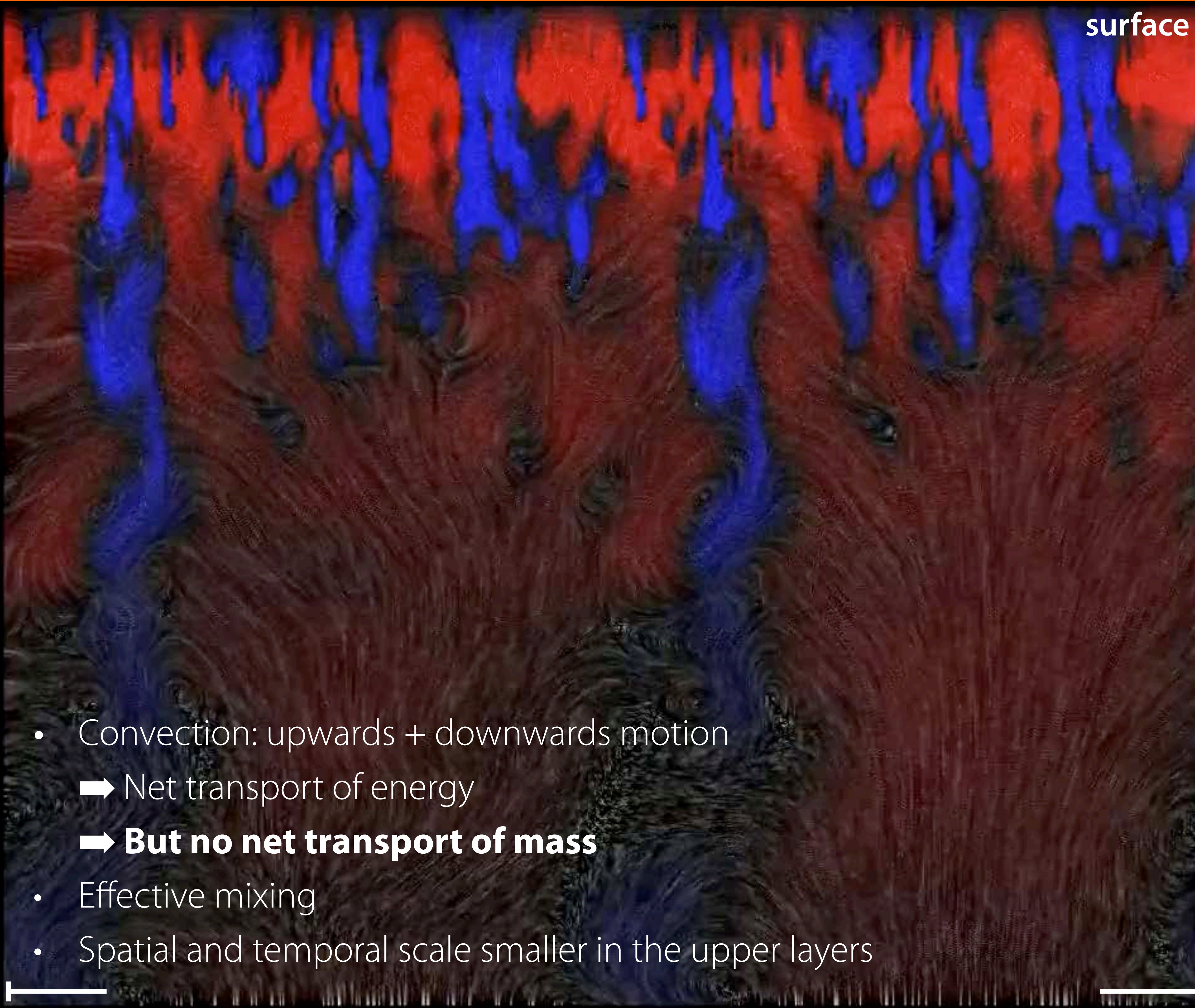
- **Temperature gradient** in a convective region can be derived by simply using $\nabla \approx \nabla_\text{ad}$

➡ $$\frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \frac{T}{P} \nabla$$

Convection near the surface

- Density and temperature much smaller
- Superadiabaticity much larger
- Temperature gradient depends on detailed properties of the convective motions

Convection



- Convection: upwards + downwards motion
  ➡ Net transport of energy
  ➡ **But no net transport of mass**
- Effective mixing
- Spatial and temporal scale smaller in the upper layers

- Simulated vertical velocity in a vertical cross-section trough the Sun's upper convection zone, played fast forward
- **red: upward**
- **blue: downward**
- streamlines

(C. Henze, NASA Advanced Supercomputing Division, Ames Research Center).

Convection

surface

- Convection zone: Mass density changes by orders of magnitudes
  - Density scale height for a stratified medium: $\varrho\,(r) = \varrho_0\, e^{-(z/H)}$
- Gas element rising up (or down) by a density scale height

➡ Expands (or contracts) by a factor e.

➡ Set dominant spatial scale of convective motions

➡ Convection cell size ≈ few local scale height

Rising gas elements
cannot carry mass higher but diverge.

- Most elements turning over within a density scale height (statistically!)
- Upward flows diverge, smoothed out

➡Upflows occupy ~2/3 of the area.*

Sinking gas elements
get compressed and fluctuations increased, becoming turbulent.

- Elements can shoot down as turbulent plumes

➡Downflows occupy ~1/3 of the area. *

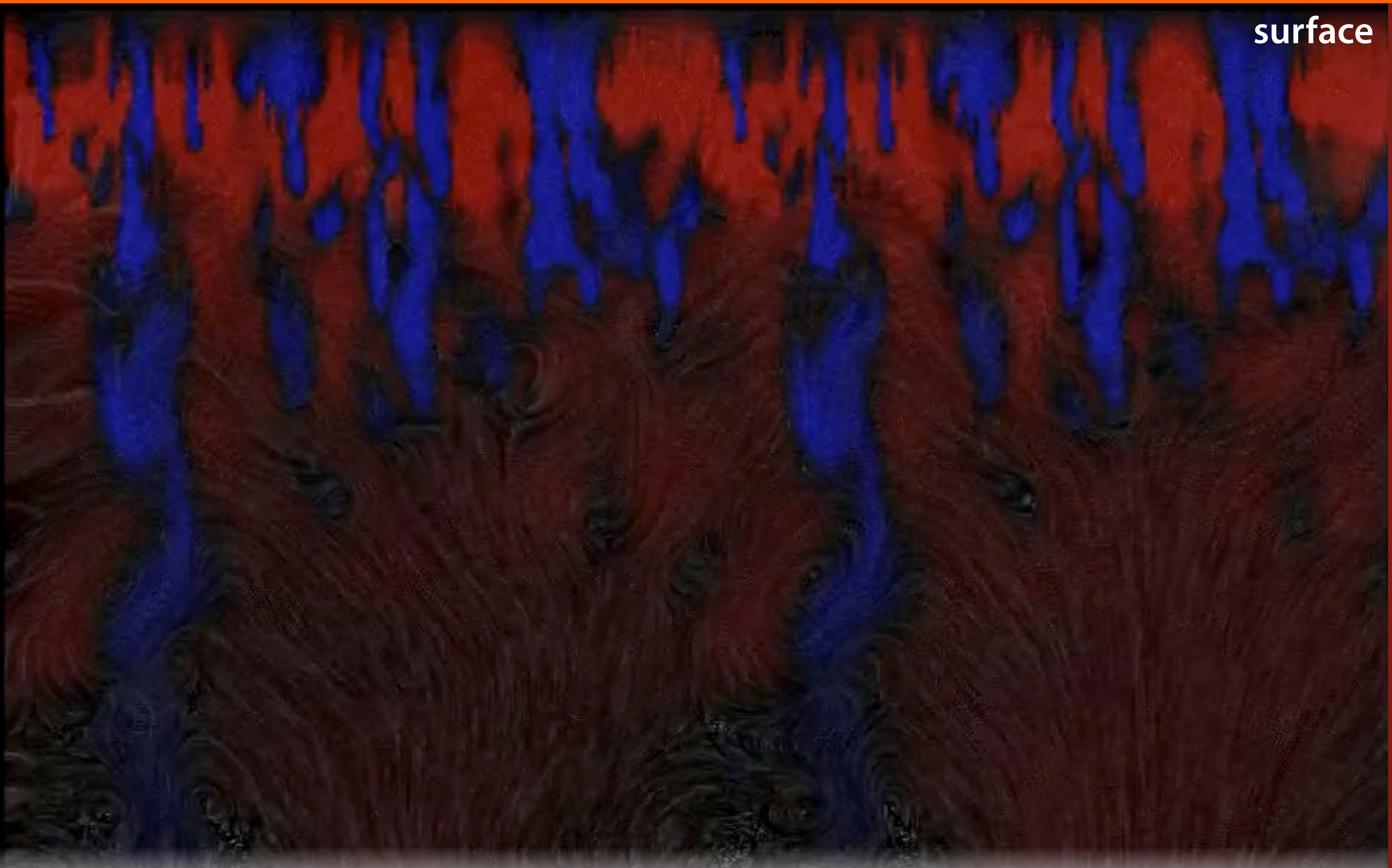
* *Inside the convection zone! The uppermost layer (~100km in the Sun) is special as it is at surface where the plasma becomes transparent.*

- **Asymmetry** up/downflows but **conservation of mass** must be obeyed!
- Temperature and density increase with depth

➡ Scale height $H_P = P/(\varrho g)$ increases with depth

➡ Typical size of convective cells set by mass conservation and local conditions, increase with depth

Convection



- **Asymmetry** up/downflows but **conservation of mass** must be obeyed!
- Temperature and density increase with depth

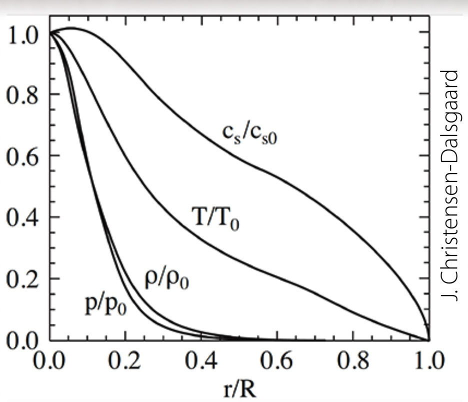
➡ Scale height $H_P = P/(\varrho g)$ increases with depth

➡ Typical size of convective cells set by mass conservation and local conditions, increase with depth

- **Sound speed** in the adiabatic case (no heat exchange with surrounding) increases with depth, too!

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} = \sqrt{\frac{\gamma P_0}{\rho_0}} = \sqrt{\frac{\gamma k_B T_0}{\mu m_H}} = \sqrt{\gamma g H_p}$$

k_B: Boltzmann constant μ: mean molecular weight m_H: mass of hydrogen



Convection

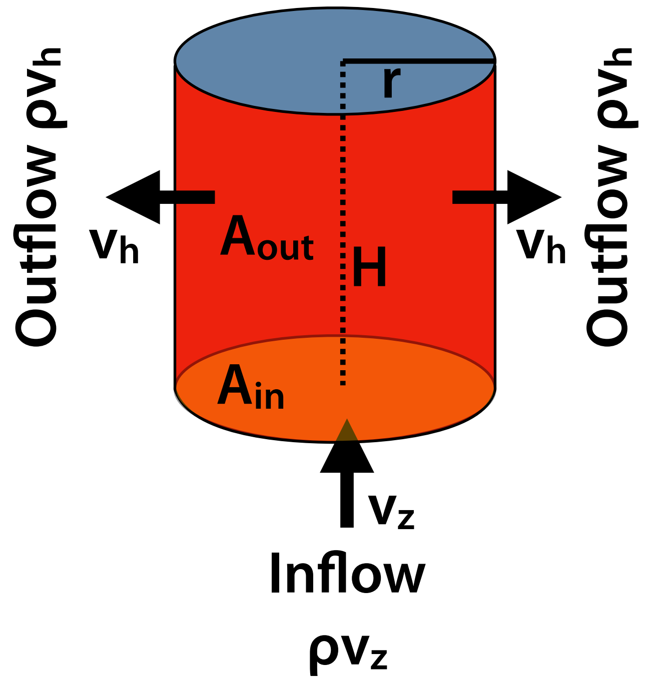
Spatial scales — Convection cell sizes

- Conservation of mass must be obeyed!
- **Rough estimate for typical radius of a convection cell:**
  - Assume a volume in the form of a cylinder with height H and radius r
  - Inflow of matter through bottom with area $\mathrm{A_{in}} = \pi\, \mathrm{r}^2$
  - Flow turns over within one scale height H
  - ➡ Outflow through the sides of the cylinder with area $\mathrm{A_{out}} = 2\, \pi\, \mathrm{r\, H}$
  - Conservation: Outflow = inflow

➡ $A_{\mathrm{in}}\, \varrho v_z = A_{\mathrm{out}}\, \varrho v_h$

➡ $\pi\, \mathrm{r}^2\, \varrho v_z = 2\, \pi\, \mathrm{r\, H}\, \varrho v_h$

➡ $\mathrm{r} = 2\, \mathrm{H}\, v_h / v_z$



v_z: vertical velocity
v_h: horizontal velocity

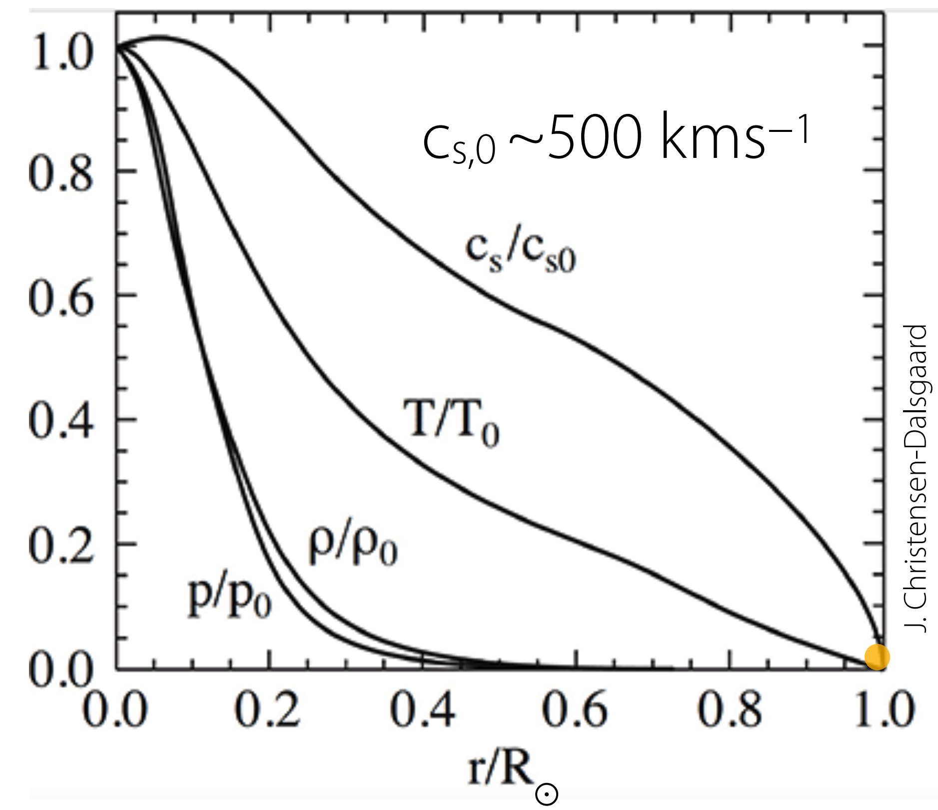
Convection

Spatial scales — Convection cell sizes

- **Typical radius of a convection cell** $r = 2\,H\,v_h/v_z$
- Vertical velocity v_z = convective velocity
- Horizontal velocity? **Upper limit** set by local sound speed

$$v_h < c_s$$

- Convective velocity v_z and sound speed c_s ($>v_h$) depend on the local thermodynamic conditions and are thus functions of radius

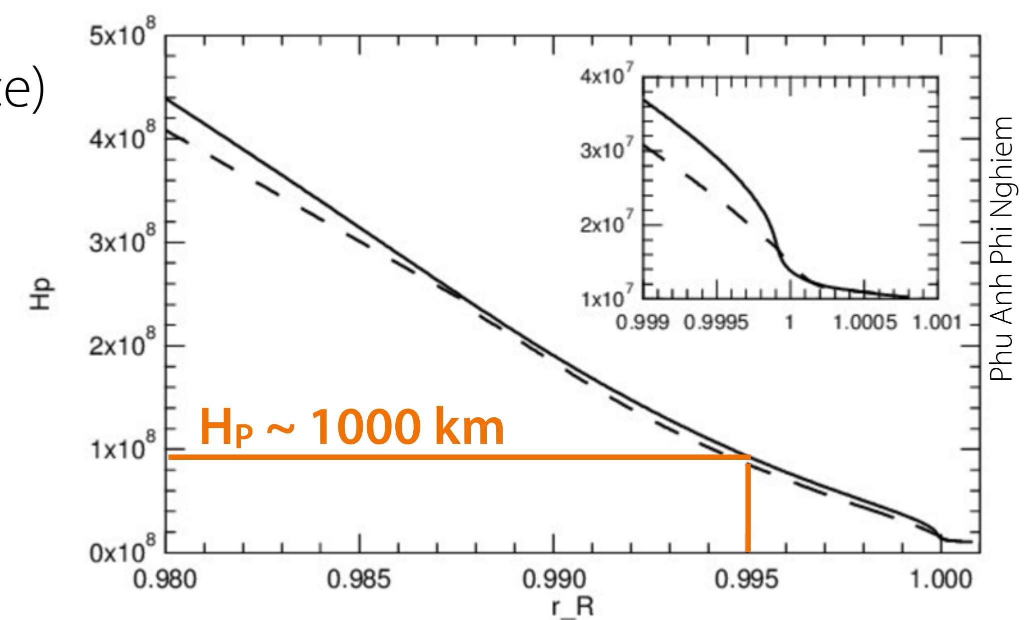


- **Example 1:** Top of the convection zone (well below surface)
  - $v_z \sim 2 km/s$, $H \sim 1000$ km, $c_s = (\gamma\, H_P\, g)^{1/2} \approx 20$ km/s

➡ $r = 2\,H\,v_h/v_z < 2000$ km x $20/2 = 20\,000$ km

➡ Horizontal cell diameters are <u>smaller</u> than that

➡ Will change strongly as function of radius / depth!



3500 km below surface

✳ $\gamma = 5/3$ (ideal monatomic gas). $\log g_\odot = 4.4$ (cgs!)

Convection

Spatial scales — Convection cell sizes

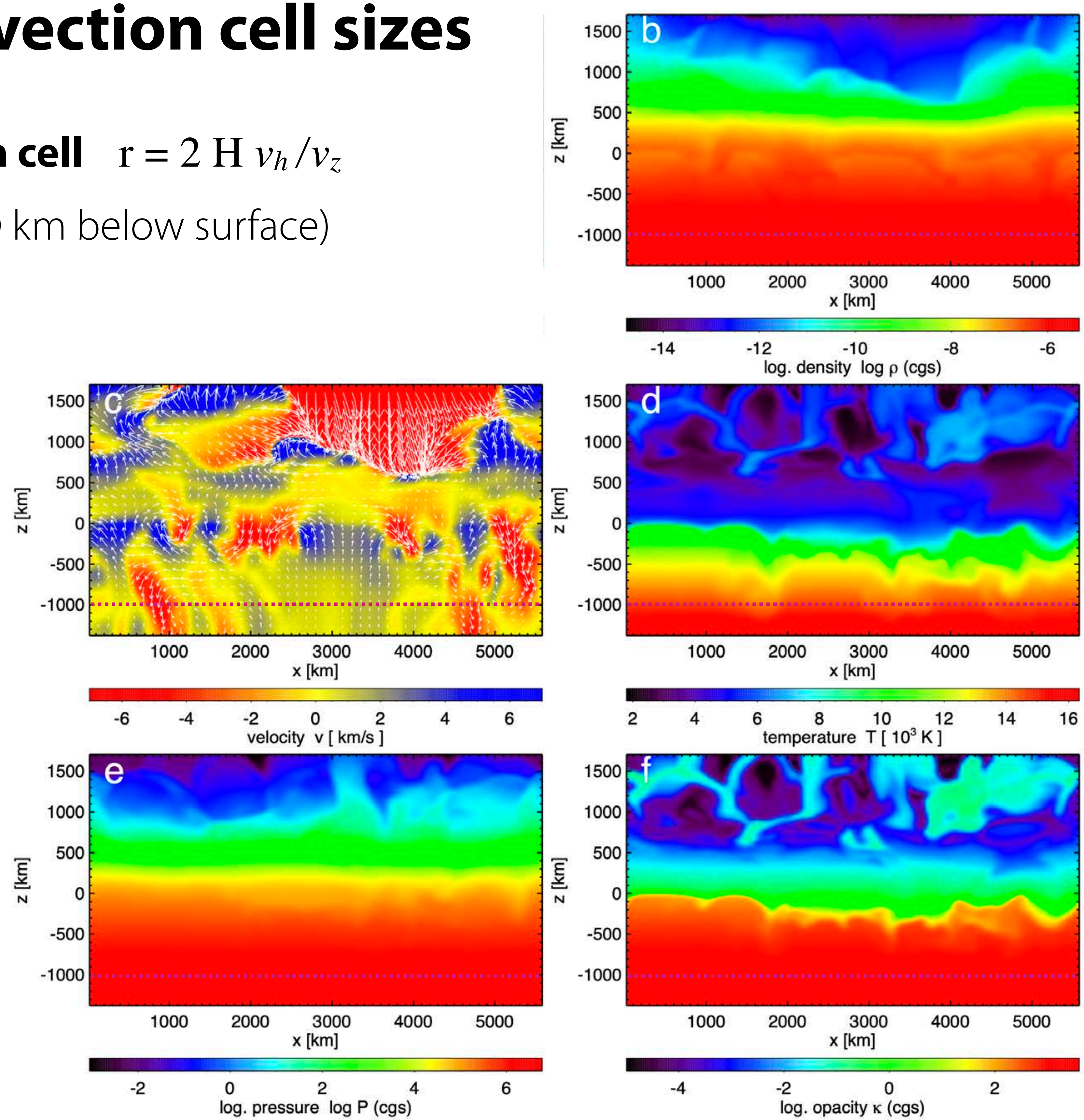
- **Typical radius of a convection cell** $r = 2\,H\,v_h/v_z$
- **Example 2:** 3D simulation, 1000 km below surface)
- All in cgs units!
  - $log\ g = 4.4$
  - $v_z \sim 2$km/s
  - $log\ P \sim 6$
  - $log\ \varrho \sim -6$

➡ $H_P = P/(\varrho g) = 400$ km

➡ $c_s = (\gamma\ H_P\ g)^{1/2} \approx 13$ km/s

➡ $r < 2\ H_P\ c_s/v_z \approx 5000$ km

➡ Convection cell diameter smaller than 10 Mm



Wedemeyer (2003)

Convection

Spatial scales — Convection cell sizes

- **Typical radius of a convection cell** $r = 2\,H\,v_h / v_z$
- **Example 2:** 3D simulation, 1000 km below surface)
- All in cgs units!
  - $log\ g = 4.4$
  - $v_z \sim 2\text{km/s}$
  - $log\ P \sim 6$
  - $log\ \varrho \sim -6$

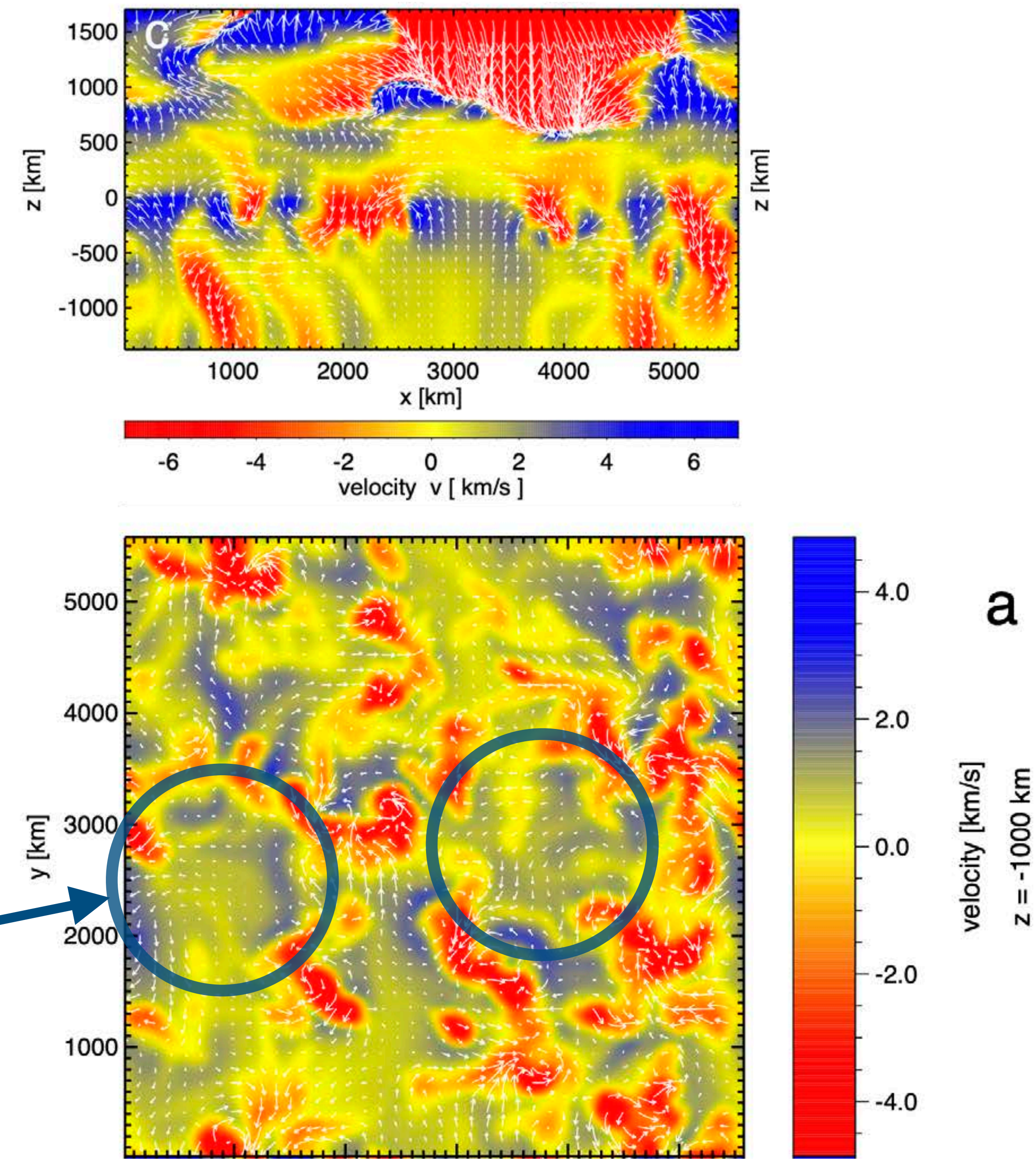
➡ $H_P = P/(\varrho g) \ = 400\ km$

➡ $c_s = (\gamma\ H_P\ g)^{1/2} \approx 13\ \text{km/s}$

➡ $r < 2\ H_P\ c_s / v_z \approx 5000\ km$

- **Very rough upper limit only!**

➡ From simulation diameter of convection cells at z=-1000km on the order of ~2000km

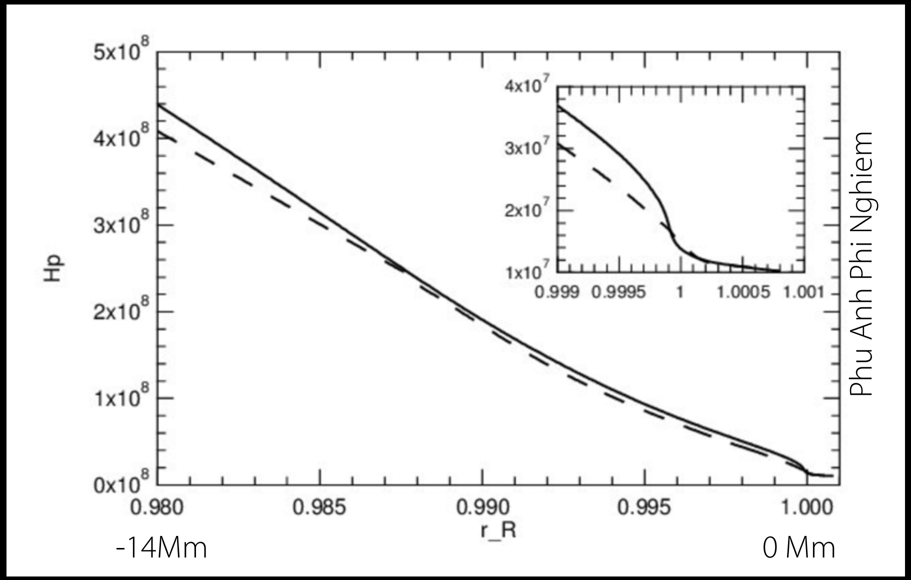


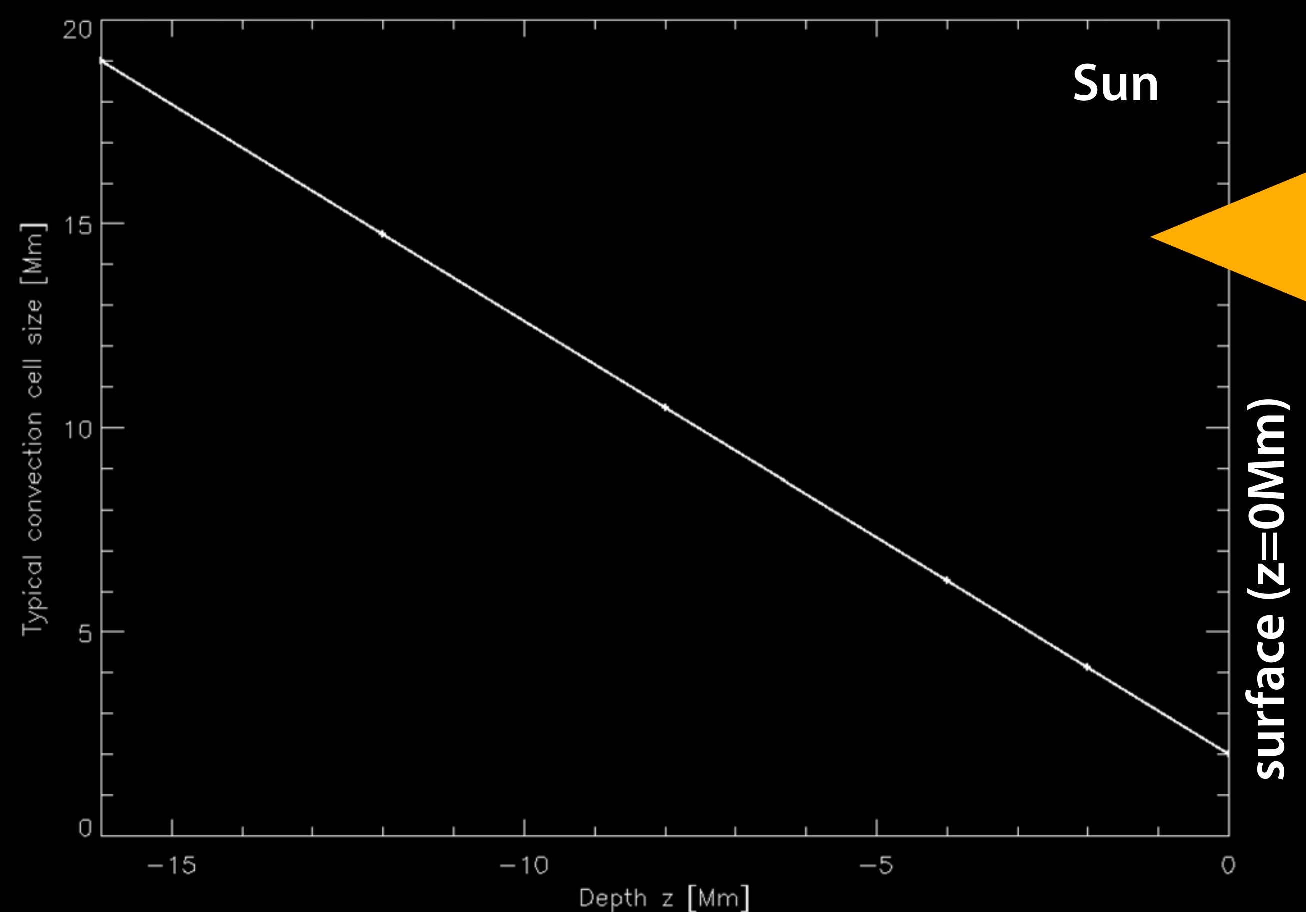
Wedemeyer (2003)

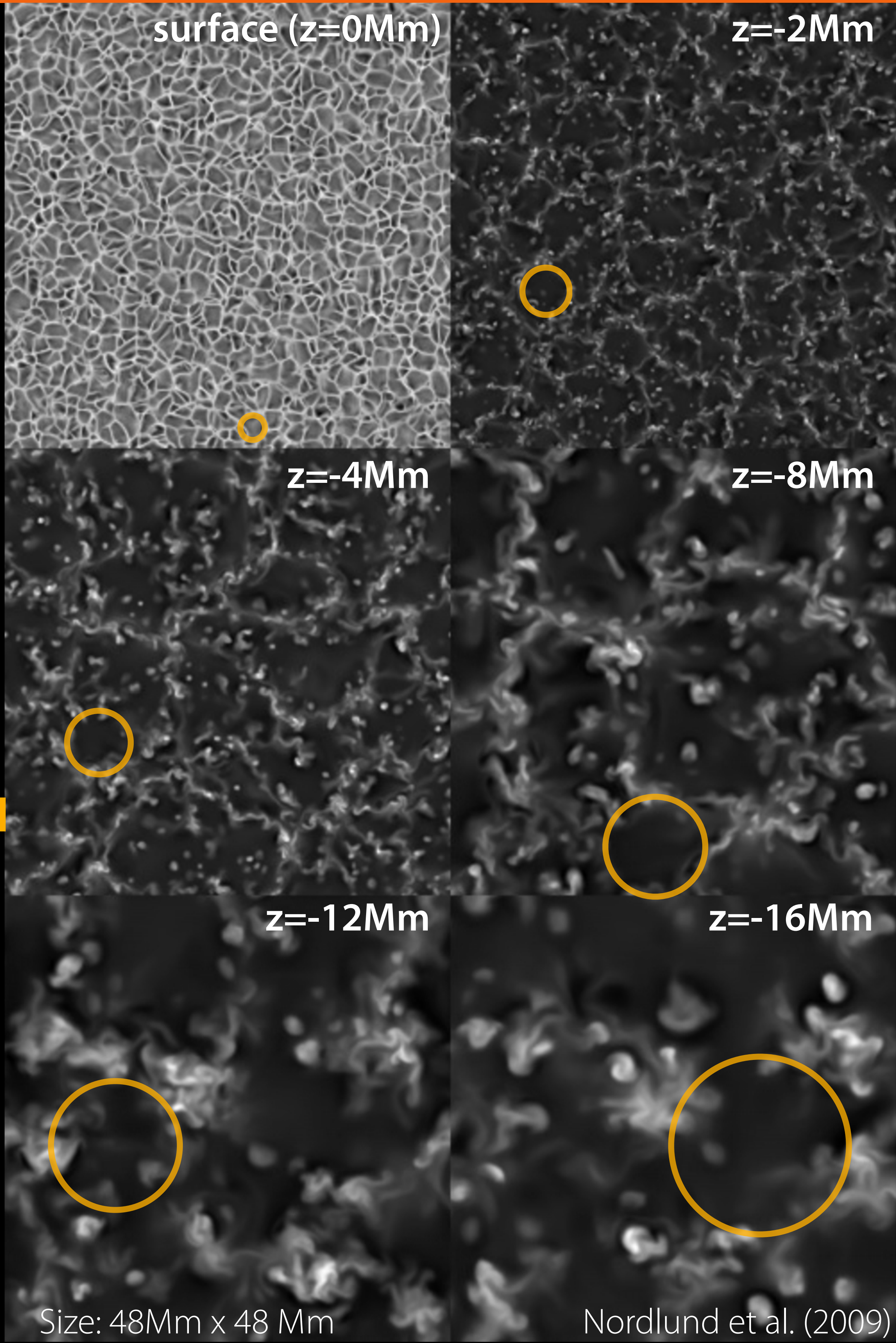
Convection

Spatial scales

- Pressure scale height $H_P = P/(\varrho g)$







Convection

Convective turnover time scale

- Turnover time scale $t_{to} \propto H_P / v_c$

- Solar convection zone highly stratified!

➡ Convective turnover time scale varies by 4 orders of magnitude from surface bottom of the convection zone

- Surface: $t_{to} \approx 200$ s
- Bottom of convection zone: $t_{to} \approx 25$ d

Convection

Spatial scales — Convection cell sizes

- Pressure scale height $H_P = P/(\varrho g)$
- **Upshot — Convection cell size**
  - Scales with pressure scale height
  - Increases with depth due to increasing pressure, density, temperature
  - About 2Mm at surface but many Mm deeper in the interior
  - Rough estimates, more precise numbers require detailed numerical simulations (and/or a more detailed description of convection)
- Note the dependence on g^{-1}

➡ What does it mean when we consider a giant star with $\log g \sim 0$ instead of the Sun with $\log g = 4.4$?

Nordlund & Stein 1997