**The following people have participated in creating these solutions: Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger**
*NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know*

# Partial solutions to problems: part 2B

We have not inserted numbers here, but leave this for the reader.

## Exercise 2B.1

You have to do this one yourself and ask your group teacher if it is correct.

## Exercise 2B.2

A Lorentz transformation is denoted $c_{\mu\nu}$, where $\mu$ and $\nu$ runs through $0-3$. Thus $c_{\mu\nu}$ is a $4 \times 4$ matrix, where the $\mu$ and $\nu$ specifies which element of the matrix one is working with. For instance, $c_{12}$ would correspond to the element located at the 2st row, 3rd column. A Lorentz transformation (matrix) operates on a vector (in 4-dimensional flat Minkowski space-time) as such:

$$c_{\mu\nu} x_\nu = x'_\mu \tag{0.1}$$

Here, Einstein's summation convention was used: $\sum_{\mu=0}^{\mu=3} x_\mu x_\mu \equiv x_\mu x_\mu$. In matrix form, equation 0.1 is nothing but

$$\begin{pmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ c_{10} & c_{11} & c_{12} & c_{13} \\ c_{20} & c_{21} & c_{22} & c_{23} \\ c_{30} & c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

We now define $D_\mu = A_\mu + B_\mu$, where $A$ and $B$ are 4-vectors. To show $D_\mu$ is a four-vector, we must show that it transforms as equation 0.1.

$$c_{\mu\nu} D_\nu = c_{\mu\nu}(A_\nu + B_\nu) = c_{\mu\nu} A_\nu + c_{\mu\nu} B_\nu = A'_\mu + B'_\mu = D'_\mu \tag{0.2}$$

Thus the sum of two 4-vectors is a 4-vector.

## Exercise 2B.3

1. In the rest frame of the neutron, $v = 0$ such that $P_\mu(n) = (m_n, 0)$.

2. In the rest frame of the neutron, $p'_p = \gamma'_p m_p v'_p$ and $E'_p = \gamma'_p m_p$. The 4-vector is then

$$P_\mu(p) = (\gamma'_p m_p, \gamma'_p m_p v'_p) = \gamma'_p m_p(1, v'_p).$$

Here $v'_p$ is the velocity of the proton from the neutron frame and $\gamma'_p = 1/\sqrt{1-(v'_p)^2}$.

3. In the rest frame of the neutron, $p'_e = \gamma'_e m_e v'_e$ and $E'_e = \gamma'_e m_e$, such that $P_\mu(e^-) = \gamma'_e m_e(1, v'_e)$. Here $v'_e$ is the velocity of the electron from the neutron frame and $\gamma'_e = 1/\sqrt{1-(v'_e)^2}$.

4. We use conservation of momentum:

$$P'_\mu(n) = P'_\mu(p) + P'_\mu(e^-)$$

Inserting, we find

$$\begin{bmatrix} m_n \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma'_p m_p \\ \gamma'_p m_p v'_p \end{bmatrix} + \begin{bmatrix} \gamma'_e m_e \\ \gamma'_e m_e v'_e \end{bmatrix}$$

Conservation of energy ($P_0$, first line) then gives

$$m_n = \gamma'_p m_p + \gamma'_e m_e,$$

while the second line gives

$$\gamma'_p m_p v'_p = -\gamma'_e m_e v'_e.$$

Squaring the second line and writing it in terms of $\gamma$-factors:

$$(\gamma'_p)^2 m_p^2 - m_p^2 = (\gamma'_e)^2 m_e^2 - m_e^2$$

Solve for $\gamma'_e$ from the first equation:

$$\gamma'_e m_e = m_n - \gamma'_p m_p$$

Insert in the second equation to obtain

$$\gamma'_p = \frac{m_n^2 + m_p^2 - m_e^2}{2 m_p m_n}$$

From which we easily find that $v'_p = 0.001262$. Going back to the first equation we then find that $v'_e = -0.9183016$ (where did we get the minus sign from?). This is one of two possible solutions, the other solution has the signs on the two velocities switched. It is completely random which of the particles will go to the right and which will go to the left, the randomness in quantum physics will decide. We choose to continue with the solution where the proton goes to the right.

5. We now transform between the lab frame (where nothing is at rest) and the neutron rest frame. We use that $P_\mu(e^-) = c_{\mu\nu}P'_\nu(e^-)$ (note that the prime is now on the right hand side, meaning that we need to use $-v_n$ instead of $v_n$, why?). In matrix form for the electron,

$$P_\mu = c_{\mu\nu}P'_\nu(e^-) = \begin{pmatrix} \gamma_n & v_n\gamma_n \\ v_n\gamma_n & \gamma_n \end{pmatrix} \begin{bmatrix} 1 \\ v'_e \end{bmatrix} \gamma'_e m_e = \begin{bmatrix} \gamma_n + v'_e v_n \gamma_n \\ v_n\gamma_n + v'_e\gamma_n \end{bmatrix} \gamma'_e m_e$$

where $v_n$ is the neutron velocity in the lab frame and $\gamma_n = 1/\sqrt{1-v_n^2}$. Inserting numbers we have $E_e = 1.481 \times 10^{-30}$kg and $p_e = 1.168 \times 10^{-30}$kg. In exactly the same way we find $E_p = 1.187 \times 10^{-26}$kg and $p_p = 1.175 \times 10^{-26}$kg.

6. We use the expression for relativistic energy (using the previous result)

$$E_e = \frac{m_e}{\sqrt{1-v_e^2}}$$

Solving for $v_e$ we obtain $v_e = 0.788922$ Similarly we obtain $v_p = 0.990025$

7. Using the formula for relativistic addition of velocities we have

$$v_e = \frac{v'_e + v_n}{1 + v'_e v_n}$$

using again the $v_{\text{rel}} = -v_n$ as the relative velocity between the systems (check again that you understand why!). Similarly for the proton.

8. I don't like long and ugly calculations.

## Exercise 2B.4

1. We let the electron move in the positive x-direction $v_e = v$ and the positron in the negative x-direction $v_p = -v$ such that

$$v'_p = \frac{v_p - v_e}{1 - v_p v_e} = \frac{-2v}{1+v^2}$$

2. $P_\mu(e) = \gamma m(1, v)$ and $P_\mu(p) = \gamma m(1, -v)$, where $m$ is the electron/positron mass and $\gamma = 1/\sqrt{1-v^2}$.

3. 

$$P'_\mu(e\pm) = c_{\mu\nu}P_\nu(e\pm) = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{bmatrix} 1 \\ \mp v \end{bmatrix} m\gamma = \begin{bmatrix} 1 \pm v^2 \\ -v \mp v \end{bmatrix} m\gamma^2$$

4. In general, $E^2 = p^2 + m^2$. Photons have zero mass, so $E = \pm p$. A four-vector is generally expressed $P_\mu = (E, p, 0, 0)$, such that the four-vector of a photon is always $P_\mu(\gamma) = (E, \pm E, 0, 0)$.

5. Conservation of four-vectors gives (omitting the y-z-directions)

$$P_\mu(e) + P_\mu(p) = P_\mu(\gamma_1) + P_\mu(\gamma_2),$$

   Giving

$$(2m\gamma, 0) = (E_1 + E_2, E_1 - E_2)$$

   Momentum conservation gives $E_1 - E_2 = 0$, so $E_1 = E_2$.

6. The wavelength is given as $E = hc/\lambda$, so $\lambda = hc/E$. From the previous question we have $E = m\gamma$ such that $\lambda = hc/(m\gamma)$

7. A Lorentz boost (omitting y and z directions) is given by

$$c_{\mu\nu} P_\nu = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = \begin{bmatrix} P'_0 \\ P'_1 \end{bmatrix} = P'_\mu$$

   Inserting, one of the equations give

$$E' = \gamma E - \gamma v E = E\gamma(1 - v)$$

8. This is found by insertion of the electron velocity $v$:

$$E' = E\gamma(1 \pm v)$$

   where $E$ is the energy of the photons in the laboratory frame.

9. We start with

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \frac{\lambda'}{\lambda} - 1 = \frac{E}{E'} - 1,$$

   where we used $E = hc/\lambda$. Inserting the expression for energy,

$$\frac{\Delta\lambda}{\lambda} = \frac{E}{E'} - 1 = \frac{1}{\gamma(1-v)} - 1 = \frac{\sqrt{1-v^2}}{1-v} - 1 = \sqrt{\frac{(1-v)(1+v)}{(1-v)^2}} - 1 = \sqrt{\frac{1+v}{1-v}} - 1$$

   which is the relativistic Doppler formula.

10. We Taylor expand the expression $f(v) = \sqrt{(1+v)/(1-v)}$ to first order, as $v$ is very small (and hence $v^2$ even smaller).

$$f(v) \approx f(0) + f'(0) \cdot v$$

where $f(0) = 1$ is trivial. We differentiate f:

$$f'(v) = \frac{d}{dv}\sqrt{\frac{1+v}{1-v}} = \frac{1}{2\sqrt{\frac{1+v}{1-v}}}\left(\frac{1}{(1-v)^2} + \frac{1}{1-v} + \frac{v}{(1-v)^2}\right)$$

letting $v = 0$,we find $f'(0) = 1$, such that we end up with

$$\frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+v}{1-v}} - 1 = f(v) - 1 \approx 1 + v - 1 = v$$

which is the non-relativistic Doppler effect.