

The following people have participated in creating these solutions:  
**Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger**  
*NOTE: There might be errors in the solution. If you find something which  
 doesn't look right, please let me know*

## Partial solutions to problems: Part 2D

### Exercise 2D.1

1. The relativistic effective potential is given as

$$V_{eff}(r) = \sqrt{\left(1 - \frac{2M}{r}\right)\left(1 + \frac{(L/m)^2}{r^2}\right)} = \sqrt{1 - \frac{2M}{r} + \frac{(L/m)^2}{r^2} - \frac{2M(L/m)^2}{r^3}}$$

We find the minimum of the potential by  $dV/dr = 0$ :

$$\frac{dV}{dr} = 0 = \frac{\frac{2M}{r^2} - \frac{2(L/m)^2}{r^3} + 6\frac{M(L/m)^2}{r^4}}{\sqrt{\left(1 - \frac{2M}{r}\right)\left(1 + \frac{(L/m)^2}{r^2}\right)}}$$

When we're interested in when a fraction  $n/m$  is zero, we only need to focus on the nominator  $n$ . Dividing by two, we see that

$$\frac{M}{r^2} - \frac{(L/m)^2}{r^3} + 3\frac{M(L/m)^2}{r^4} = 0$$

Multiplying by  $r^4$  and obtain a second-order equation in  $r$ :

$$Mr^2 - (L/m)^2r + 3M(L/m)^2 = 0$$

Using the standard solution  $r = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$ , we obtain

$$r = \frac{1}{2M} \left( (L/m)^2 \pm \sqrt{(L/m)^4 - 4M \cdot 3M(L/m)^2} \right)$$

and extract the  $(L/m)^2$ :

$$r = \frac{(L/m)^2}{2M} \left( 1 \pm \sqrt{1 - \frac{12M^2}{(L/m)^2}} \right)$$

For  $r$  to stay real and not complex, then  $\frac{12M^2}{(L/m)^2} < 1$ . In this case, the maximum of the potential is given by the **minus** sign (as seen by insertion into  $V(r)$ ). We already knew that the top of the potential is closer to the black hole than the minimum so it was clear that the minus sign gives the position of the top.

2. Here I will give only some hints, not the detailed solution. The angular momentum was obtained in equation (5) in the lecture notes as

$$\frac{L}{m} = r^2 \frac{d\phi}{d\tau}$$

This equation is written in terms of the proper time  $d\tau$  which is the wristwatch time of the moving object. We need to express it in terms of velocities measured by a local shell observer. Thus, one needs to find a way to transform this equation to shell coordinates. Shell observers see Lorentz geometry for events measured within a very short time interval. If you have Lorentz geometry, you already know how to transform from lab-frame time to the proper time of an object, you don't even need general relativity.

This is all we need in order to arrive at the correct expression. Remember that for changing coordinates in a derivative you need to use a rule which you have used many times in this course. The angular coordinate  $\phi$  is the same for the shell observer and the far-away observer.

The local tangential speed is given by

$$v_{\phi,\text{shell}} = v_{\text{shell}} \sin \theta = \frac{d\ell}{dt_{\text{shell}}},$$

where  $d\ell$  is the proper distance (we use  $\ell$  for proper distance instead of  $L$  as before in order not to confuse with the angular momentum) which an object has moved along the tangential direction during the time interval  $dt_{\text{shell}}$  measured on a local shell clock. We know that in the theory of relativity (we used this already several times in special relativity), we find a proper distance by measuring the end points of the interval at the same time such that  $dt = 0$ . Since we have tangential movement, we also have that  $dr = 0$ . Inserting this in the Schwarzschild line elements we have

$$d\ell = ds = r d\phi$$

Combining all this, you find that

$$\frac{L}{m} = r \gamma_{\text{shell}} v_{\text{shell}} \sin \theta$$

3. Again we will only hint on the solution. Starting with the general expression for Energy

$$\frac{E}{m} = \left(1 - 2M/r\right) \frac{dt}{d\tau}$$

all we need to do is to transfer from Schwarzschild coordinates to shell coordinates in order to express this in terms of local shell velocity. In

the lecture notes you have learned to transform from Schwarzschild coordinate intervals  $dr$  and  $dt$  to corresponding intervals measured by shell observers. Combine this with some principles from the previous exercise and you find

$$\frac{E}{m} = \gamma_{shell} \sqrt{1 - 2M/r}$$

4. Inserting numbers, we have that  $L/m \approx 38M$  giving a minimum in the potential at  $r = 1426M$  and a maximum at  $r = 3M$ . The values of the effective potential at these points are  $V_{eff}/m = 0.9997$  and  $V_{eff}/m = 7.3$ .
5. For the rocket to be captured, we need to decide if  $E/m$  is larger or less than  $V(r_{max})$ . Using the expression for the energy per mass above using the numbers for  $r = 20M$  we get  $E/m = 8.03$  which is larger than  $V(r_{max})/m = 7.3$  (as we found in the previous question), the rocket is therefore captured.
6. We need to find a new expression for  $dr/dt$ . We will now call the total energy per mass of the rocket  $E_0/m$  which is a constant. Going back to the previous lecture and repeating the calculations for finding  $dr/dt$  of a freely falling object you now find

$$\frac{dr}{dt} = - \left(1 - \frac{2M}{r}\right) \sqrt{1 - \frac{m^2}{E_0^2} \left(1 - \frac{2M}{r}\right)}$$

Repeating the calculation of the time to reach the singularity made in this lecture using the new total energy per mass  $E_0/m$  you then find that the total time to reach the singularity is given by

$$\tau = - \int_{2M}^0 dr \left( \frac{E_0^2}{m^2} - 1 + \frac{2M}{r} \right)^{-1/2}$$

Using the Integrator you find that the solution is

$$\tau = - \left[ \frac{1}{a} \sqrt{a + \frac{b}{r}} - \frac{b}{2a\sqrt{a}} \ln \left( b + 2ar + 2\sqrt{a} \sqrt{a + \frac{b}{r}} \right) \right]_{2M}^0,$$

where  $a = E_0^2/m^2 - 1$  and  $b = 2M$ . We now insert numbers. Using the expression for the total energy above, inserting  $r = 20M$  and  $v_{shell} = 0.993$  we find  $E_0/m = 8.03$  and therefore  $a = 63.5$  and  $b = 8 \times 10^6 M_\odot = 8 \times 10^6 \times 1482m$ . Before inserting numbers in the previous expression we rewrite it in the following manner (in order to get a

dimensionless number inside the logarithm), check that you understand how to arrive at this expression

$$\tau = \frac{1}{a} \sqrt{a + \frac{b}{2M}} 2M - \frac{b}{2a\sqrt{a}} \ln \left( \frac{b + 2a2M + 2\sqrt{a}\sqrt{a + \frac{b}{2M}} 2M}{2M} \right)$$

which can be written even simpler using  $b = 2M$

$$\tau = \frac{2M}{a} \sqrt{a + 1} - \frac{M}{a\sqrt{a}} \ln(1 + 2a + 2\sqrt{a}\sqrt{a + 1})$$

which gives about 10 seconds.

- The astronaut will be stretched in the radial direction, but will *also* be compressed in the tangential direction: he/she/it is falling towards a point singularity, so the forces don't work in the same direction! The astronaut will have the shape of spaghetti just before being swallowed by the black hole.

## Exercise 2D.2

This program is a possible solution to problem 17.2. It calculates the trajectory of a "spaceshuttle" that falls into a black hole, with given initial conditions. It plots the trajectory of the shuttle and the Schwarzschild radius in the same figure, and prints the angular coordinate when the shuttle approaches the horizon.

For convenience we define  $M = 1$ .

```
#Remember to import the scitools library.
from scitools.all import *
```

```
"""
```

```
EXERCISE 2D.2.1
```

```
***
```

```
Define variables as found in exercise 17.1
```

```
We use that M=c=1
```

```
"""
```

```
r=<...>
```

```
theta=<...>
```

```
vshell=<...>
```

```
gammashell=1/sqrt(1-vshell**2)
```

```
rm=20.0 #=r/M
```

```

phi=0.0

Lm=<...>
Em=<...>

n=1000 #Number of evaluationpoints
dt=0.005#Proper timestep

"""
EXERCISE 2D.2.2
***
Declare arrays to store r and phi for all timesteps dt,
and then initialize the first elements in this array.
"""

#The function zeros() makes an array where all elements equals 0.
#In this case we want an array with n elements of type float.
rvec=zeros(n,float)
phivec=zeros(n,float)

rvec[0]=rm
phivec[0]=phi

"""
EXERCISE 2D.2.3
***
Calculate the orbit of the spaceship
"""

for i in range(1,n):
    #Terminate the loop if rm<2
    if rm<2:
        #Terminates loop at timestep i. Store i as n
        n=i
        break
    else:
        #Calculate the small change of position during a timeinterval dt
        deltar=<...>
        deltaphi=<...>

        #Update the position
        rm+=<...>
        phi+=<...>

```

```

        #Store the new position in the positionarrays
        rvec[i]=<...>
        phivec[i]=<...>

"""
EXERCISE 2D.2.4
***
Plot the orbit of the spaceship.
"""

#Convert from polar coordinates to cartesian coordinates.
x=<...>
y=<...>

#Plot all indexes in the array (from 0 to n-1).
#OBS: x[:n] = array([ x[0], x[1], x[2], ... , x[n-1] ]).
#'. .' means that all points are plotted as dots.
#Be aware that n was changed in the if-test in the for-loop.
plot(x[:n],y[:n],'.')
axis([-5,20,-6,8])

"""
EXERCISE 2D.2.5
***
Plot the Schwarzschild radius in the same plot as the orbit.
"""

#The hold function makes it possible to plot more than one plot
#in the same figure.
hold('on')

#make two arrays that contains the coordinates of a circle
#with center in (0,0) and radius 2.
t=linspace(0,2*pi,100)
X=<...>
Y=<...>

plot(X,Y)

"""
EXERICSE 2D.2.6
***
Calculate the final angular coordinate of the spaceship.

```

```

"""

#Calculate and print the final angular position.
#(Should be approximately 197.6 degrees).
finalphi=<...>
print "Final angular coordinate %f" %finalphi

"""

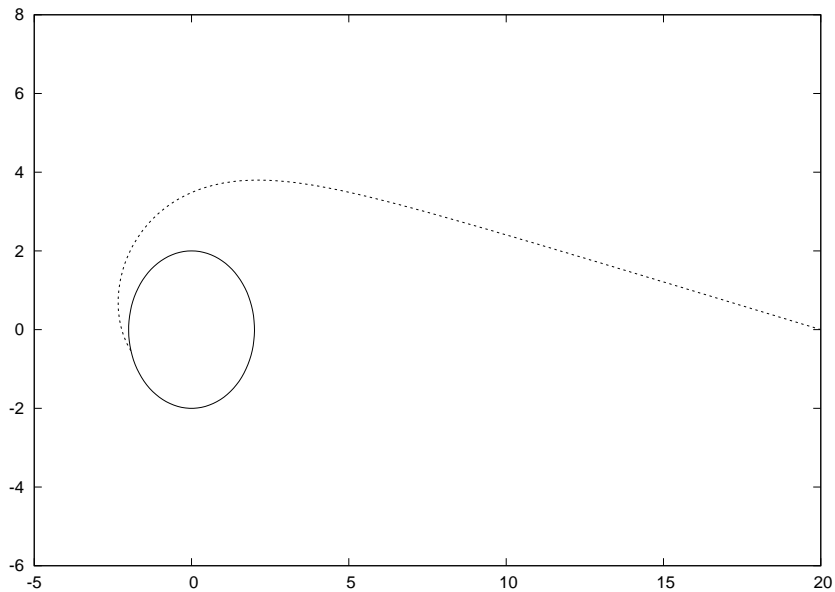
***

Save the plot as a ps file for printing.
The hardcopy() function saves the recent plot in
ps,eps or png format. The fileformat is automatically determined
from the extension of the filename.
"""

hardcopy('fig17_2.ps') #The plot is now saved as fig_17_2.ps

```

**Exercise 2D.2.7:** The plot. Units:  $M$ .



### Exercise 2D.3

In this exercise, we switch from normal stationary coordinates to a special coordinate system following an ingoing observer. The reason for this is that the light-cones from the shell observer will collapse at the event horizon.

This is not true for the observer falling into the black hole - the light-cone will rather be rotated, as depicted in the exercise text.

1. Shell observers using local coordinates see Lorentz geometry and we can use the normal Lorentz transformations, defining the  $r$  direction to be the  $x$  direction.
2. Inserting the relation between shell-coordinates and Schwarzschild-coordinates:

$$dr_{shell} = \frac{dr}{\sqrt{1 - 2M/r}}$$

and

$$dt_{shell} = dt\sqrt{1 - 2M/r}$$

into the expression from the Lorentz transformation in the previous question we obtain

$$dt' = -\frac{v\gamma dr}{\sqrt{1 - 2M/r}} + \gamma\sqrt{1 - 2M/r}dt \quad (0.1)$$

where  $v$  and  $\gamma$  are based on shell-velocities.

3. In the previous lectures, equation (8) states the velocity measured by a shell-observer of a freely falling particle (into a black hole). This was given as

$$v = -(1 - 2M/r)\sqrt{2M/r} \approx \sqrt{2M/r}$$

where the approximation is viable for a shell-observer sufficiently far away. Inserting this velocity into equation (0.1) yields

$$dt' = -\frac{1}{\sqrt{1 - GM/r}} \frac{\sqrt{2M/r}dr}{\sqrt{1 - 2M/r}} + \frac{1}{\sqrt{1 - GM/r}} \sqrt{1 - 2M/r}dt$$

such that

$$dt = dt' - \frac{\sqrt{2M/r}dr}{1 - 2M/r} \quad (0.2)$$

4. We continue by fluffing the substitute coordinates into the original Sewartzschild line element:

$$ds^2 = d\tau^2 = (1 - 2M/r)dt^2 - \frac{dr^2}{1 - 2M/r} + r^2d\Omega^2$$

where  $d\Omega^2$  is the spherical area element. Inserting equation (0.2) yields

$$ds^2 = d\tau^2 = (1 - 2M/r)\left(dt' - \frac{\sqrt{2M/r}dr}{1 - 2M/r}\right)^2 - \frac{dr^2}{1 - 2M/r} + r^2\Omega^2$$



expanding the terms gives

$$ds^2 = d\tau^2 = (1-2M/r) \left( dt'^2 - 2 \frac{\sqrt{2M/r}}{1-2M/r} dt' dr + \frac{2M/r dr^2}{(1-2M/r)^2} \right) - \frac{dr^2}{1-2M/r} + r^2 d\Omega^2$$

happily, the denominators cancel such that

$$ds^2 = d\tau^2 = (1-2M/r) dt'^2 - \sqrt{2M/r} dt' dr + 2 \frac{2M/r dr^2}{(1-2M/r)} - \frac{dr^2}{1-2M/r} + r^2 d\Omega^2$$

we now use the fact that

$$\frac{x}{1-x} - \frac{1}{1-x} = \frac{x-1}{1-x} = -1$$

on the third and fourth term. In the end,

$$ds^2 = d\tau^2 = (1-2M/r) dt'^2 - 2\sqrt{2M/r} dt' dr - dr^2 + r^2 d\Omega^2$$

So. What has happened? We have switched to a coordinate system in which the line element (metric) is **no longer orthogonal**. A line element (metric) is orthogonal if all  $dx_i dx_j = 0$  for all  $i \neq j$ . In our case, we have “restored normality” for the  $r$ -coordinate - the  $dr^2$  does no longer depend on  $r$ . The price we paid was coupling the new auxiliary time-coordinate  $t'$  to the  $r$  coordinate - the  $dt' dr$ -coupling. The reason - there is **no longer a singularity** at the Schwarzschild-radius  $r = 2M/r$ ! This is because the new coordinate system follows the ingoing observer - and therefore the event horizon is removed. Remember that the event horizon exists for a stationary observer *outside* the black hole, the freely falling observer will not register anything weird at the event horizon. That is, except a rather horrible imminent (but quick) death at some point after passing the horizon..

5. Light travel in world-lines corresponding to  $d\tau = dt' = 0$ . This means that photons don't experience time, which is true of all massless particles: the entire universe happened to exist in 0 time. Therefore, all spacetime distances are 0 for photons. The Schwarzschild-metric is thus

$$0 = (1-2M/r) dt'^2 - 2\sqrt{2M/r} dt' dr - dr^2$$

assuming radial velocity only ( $d\phi = 0$ ). Solving for  $dr$  yields a normal second-order equation. Using  $b = -2\sqrt{2M/r} dt'$ ,  $a = -1$  and  $c = (1-2M/r) dt'^2$  yields

$$\frac{dr}{dt'} = -\frac{1}{2} \left( b \pm \sqrt{b^2 + 4c} \right) = -\frac{1}{2} \left( 2\sqrt{2M/r} \pm \sqrt{4(2M/r) + 4(1-2M/r)} \right)$$

Giving

$$\frac{dr}{dt'} = \left( -\sqrt{2M/r} \pm \sqrt{(2M/r) + (1-2M/r)} \right) = \left( -\sqrt{2M/r} \pm 1 \right)$$

and done! Note that far off ( $r \gg 2M$ ) then  $dr/dt' = \pm 1$ , which yields “normal”  $45^\circ$  light-cones. At  $r = 2M$ , then  $dr/dt' = \{0, -2\}$ . The first solution giving a light speed of 0 (the outgoing world-line of light goes straight up ( $90^\circ$ )). The second solution shows a light beam going with 2 times the speed of light towards the center. For  $r < 2M$ , both solutions are always negative meaning that even the light beam emitted outwards away from the center is moving towards the center. We see that the speed of light in this case can be zero, but as discussed in the text for another case, this is not a speed which can be measured locally by any observer: It is the change in Schwarzschild coordinate as a function of time measured on the wristwatch of the infalling observer.

6. As discussed in the previous point, at  $r = 2M$ , the light speed for the outwards light beam is zero so the worldline goes straight upwards. Inside the horizon, the outwards light beam always has negative light speed, i.e. the beam goes inwards towards the center and the worldlines therefore go to the left. The inwards light beam will have speeds approaching infinity as  $r \rightarrow 0$  so the inclination of the worldline will be larger and larger and eventually horizontal very close to the center.
7. We see that even something which is sent out with the speed of light in the direction away from the center will move towards the center (immediately, no phase of deceleration and acceleration), one can view it like this: The spacetime geometry is now such that all directions point to the center, no matter in which direction you point, you point towards the center.