The following people have participated in creating these solutions: Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger
NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know

Partial solutions to problems: part 2E

Exercise 2E.1

The final results for this exercise is already given in the text.

Exercise 2E.2

The final results forthis exercise is already given in the text.

Exercise 2E.3

We use equation (3) in the lecture notes:

$$\frac{dr}{dt} = \pm(1-2M/r)\sqrt{1-(1-2M/r)\frac{b^2}{r^2}}$$

and square:

$$\left(\frac{dr}{dt}\right)^2 = (1-2M/r)^2\left(1-(1-2M/r)\frac{b^2}{r^2}\right)$$

Then switching to shell-coordinates

$$\frac{dr_{shell}}{dt_{shell}} = \frac{(1-2M/r)^{-1/2}dr}{(1-2M/r)^{1/2}dt} = \frac{dr}{(1-2M/r)dt}$$

such that

$$\left(\frac{dr_{shell}}{dt_{shell}}\right)^2 = 1-(1-2M/r)\frac{b^2}{r^2}$$

Divide by b^2 to obtain the desired equation.

We see that this equation is on the form of equation (4) in part 2D of the lecture notes. We identify $A = B = 1/b^2$, $x = dr_{\text{shell}}/dt_{\text{shell}}$ and $V^2(x) = (1-2M/r)/r^2$.

Exercise 2E.4

1. We differentiate the potential

$$V(r) = \sqrt{\frac{1-2M/r}{r^2}}$$

and find the extremal points:

$$\frac{d}{dr}V(r) = \frac{1}{2\sqrt{\frac{1-2M/r}{r^2}}} \cdot \left(-\frac{2}{r^3} + \frac{6M}{r^4}\right) = 0$$

such that $\frac{d}{dr}V(r) = 0$ if

$$\frac{2}{r^3} = \frac{6M}{r^4}$$

for $r = 3M$.

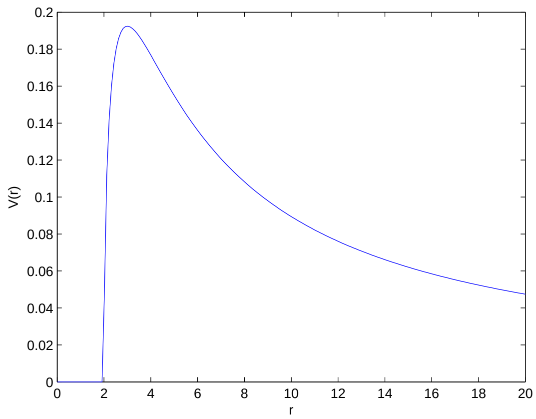

Figure 1: Potential V(r) for $M = 1$ with a maximum at $r = 3$.

As seen from figure 1, this extremal point is a maximum and not a minimum. As energy states stabilize towards minima, any perturbation from the maximum of the potential will rapidly decay towards lower energy states. Think of trying to balance a ball on top of the potential maximum - any perturbation to the ball will make it fall down either way. However, in a minimum, the ball would just roll back and forth in the potential. As this potential describes the orbits of light around a heavy object, we conclude that there are no stable orbits for light.

2. See the text to find the explanation for why $r = 3M$ is called the light sphere.

3. The critical point is when $1/b^2 \propto V(r)$ is larger or smaller than the peak in figure 1. At the maximum, the value of $V(r_{crit})$ is

$$V(r_{crit}) = V(3M) = \frac{1}{3M}\sqrt{1 - \frac{2M}{3M}} = \frac{1}{3\sqrt{3}M} = \frac{1}{b_{crit}}$$

Exercise 2E.5

The solution to this exercise is already given in the exercise.

Exercise 2E.6

1. The situation is depicted in figure 2 with an enlargement of the triangle ABC in figure 3.

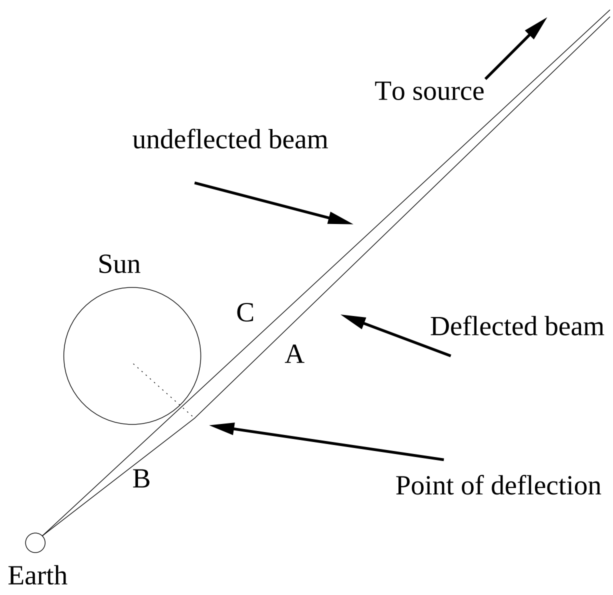

Figure 2: Deflection of light from a distant star by the Sun. 'Undeflected beam' refers to how a light beam from the star would have moved if the Sun had not been there to deflect it.

The angular shift on the sky is given by α, the deflection of light is $\Delta\phi$. First we observe that as the distance to the star goes to infinity $\gamma \to 90^\circ$. The star is much more distant than the Sun so it is a good approximation to set $\gamma \approx 90^\circ$. Then we see from the figure that $\beta = 90^\circ - \Delta\phi$. Using the small triangle on the left hand side in figure 3 we have that

$$\alpha + (90^\circ - \Delta\phi) + 90^\circ = 180^\circ$$

giving $\Delta\alpha = \Delta\phi$.

2. Inserting numbers for the mass and radius of the Sun (assuming that the light passes very close to the solar surface) $\Delta\alpha = \Delta\phi = 4M/R \approx 1.7'$

3. Similarly for the moon we get $\alpha = 6.3 \times 10^{-8}$ arc seconds.

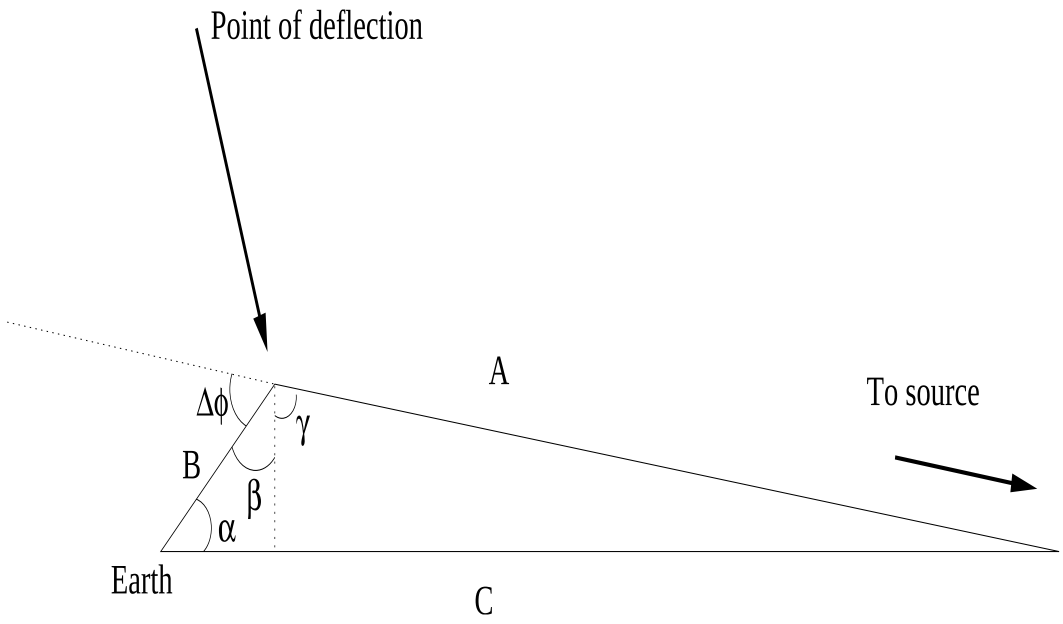

Figure 3: The triangle ABC in figure 2 enlarged.

Problem 2E.7.6

Problems 2E.7.1-2E.7.5 should be possible to solve using the equations and figures which are given. Here we only give the solution to problem 2E.7.6. We use that $d_S = 10^{10} ly, d_L = 10^9 ly$. The lensing formula is given as

$$\theta_E = \sqrt{\frac{4M(d_S - d_L)}{d_L d_S}}$$

solving for M:

$$M = \frac{\theta_E^2 d_L d_S}{4(d_S - d_L)} \times \frac{c^2}{G} = 1.35 \times 10^{15} M_\odot.$$