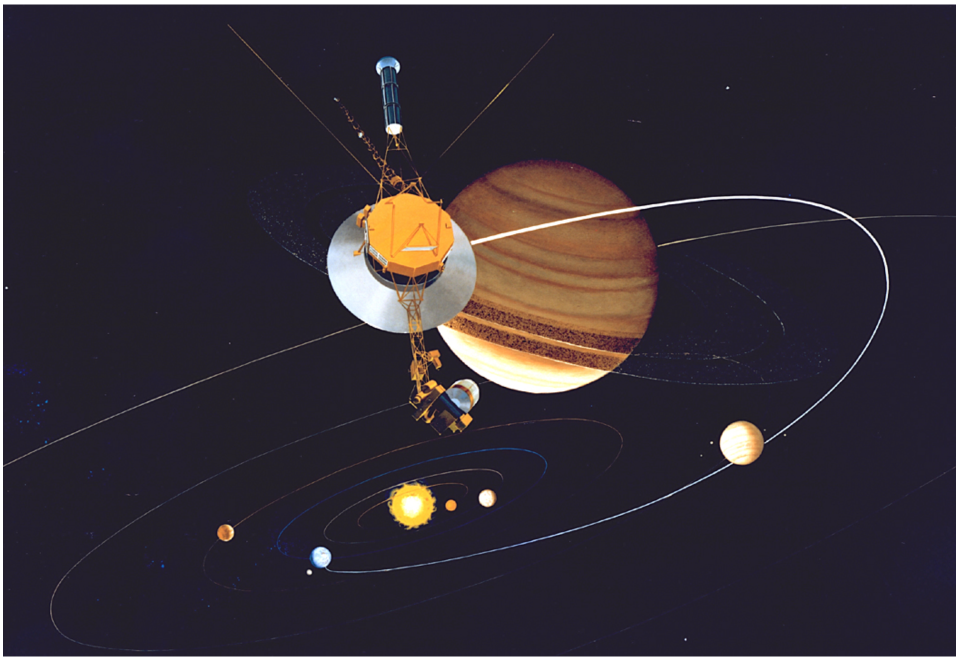
AST1100 Lecture Notes

Part 1B
Celestial Mechanics: calculating the orbits

Questions to ponder before the lecture

1. Can you write down Newton's second law and Newton's law of gravitation on their vectorial form?
2. If you want to find an analytical expression for the orbit of a planet around the Sun, how would you start? Which equation(s) would you set up?
3. How would you start to solve these equations?
4. Assume you observe two stars in the vicinity of each other. You find their positions and their velocities. How can you find out if these stars are orbiting each other or not?
5. A planet's orbit around the Sun is elliptical. Due to the pull of the planet, also the Sun moves in an orbit around their common center of mass. How does the orbit of the Sun look like?
6. About half of the known stars are binary stars, two stars orbiting each other. How would the orbit of a planet in such a system look like?



AST1100 Lecture Notes

Part 1B
Celestial Mechanics: calculating the orbits

Having launched our satellite into space, we now need to learn to calculate orbits: orbits of the planets where we want to land and the orbit of the spaceprobe in order to land it safely on the planet. We will first discuss how to do this analytically which is only possible in the simplest possible situations. Then we will do it numerically which allows for calculating orbits also in a more complicated settings. Before embarking on the details of orbital calculations, we will first remind you about/introduce you to some math

1 An important differential equation

We will now look at a differential equation which we will meet in this part as well as on other occations during this course. We will first look at a numerical solution and then at an analytical solution. The equation is

$$\frac{d^2 f(x)}{dx^2} = s(x) \tag{1}$$

where $s(x)$ is a known function and you need to find a solution for $f(x)$. Depending on the function $s(x)$, this equation may have an analytical solution or may have to be solved numerically.

1.1 General numerical solution

We will first look at the more general numerical solution. In order to find the numerical solution, we will first look at a slightly easier equation,

$$\frac{df(x)}{dx} = g(x)$$

where $g(x)$ is a known function. Starting with an initial value of x and $f(x)$, we can solve this equation iteratively by increasing x and thereby $f(x)$ step by step,

$$df(x) = g(x)dx$$

where the change $df(x)$ to $f(x)$ is calculated for a tiny increase dx in x. This can be continued until $f(x)$ is known for the desired range of x values. This is known as Euler's method. In a computer code one starts with initial values x_0 and f_0 and then step by step obtain the following values $f_1 = f(x_1)$, $f_2 = f(x_2)$ etc., using small increments Δx. One thus obtains

$$f_{n+1} = f(x_{n+1}) = f_n + g(x_n)\Delta x$$

Based on this, we can now solve equation 1 in two steps, rewriting the equation,

$$\frac{df'(x)}{dx} = s(x)$$

we see that we can first solve for $f'(x)$

$$f'_{n+1} = f'_n + s(x_n)\Delta x$$

and then use this solution to find $f(x)$. Knowing that

$$f'(x) = \frac{df(x)}{dx}$$

we can therefore write the solution for $f(x)$ knowing $f'(x)$ as

$$f_{n+1} = f_n + f'(x_{n+1})\Delta x$$

Note in the last step that we use x_{n+1} instead of x_n, this is known as the Euler-Cromer method.

1.2 Analytic solution for a special case

We will also encounter equation 1 in the form where $s(x) = -f(x) + C$,

$$\frac{d^2 f(x)}{dx^2} = -f(x) + C \quad (2)$$

where C is a known constant. This is a form of the *harmonic oscillator equation*. The equation of motion for a pendulum or for masses connected to springs can often be written on a similar form, giving an oscillating motion. Looking at the equation, can you see an easy analytic solution? Clearly we are looking for a function $f(x)$ which is such that the second derivative is propotional to the original function. $\cos(x)$ and $\sin(x)$ both fullfill this criterion, which makes sense given that they would represent an oscillating motion. We can write the solution as

$$f(x) = C + A\cos(x - \omega) \quad (3)$$

where A and ω are constants depending on the initial conditions. Now insert this solution into equation 2 to check that this is really a valid solution.

Now we have the math ready to start calculating orbits...

2 Kepler's Laws

Kepler used Tycho Brahe's detailed observations of the planets to deduce three laws concerning their motion:

1. The orbit of a planet is an ellipse with the Sun in one of the foci.
2. A line connecting the Sun and the planet sweeps out equal areas in equal time intervals.
3. The orbital period around the Sun and the semimajor axis (see figure 4 on page 9 for the definition) of the ellipse are related through:

$$P^2 = a^3, \quad (4)$$

where P is the period in years and a is the semimajor axis in AU (astronomical units, 1 AU = the distance between the Earth and the Sun).

Whereas the first law describes the shape of the orbit, the second law is basically a statement about the orbital velocity: When the planet is closer to the Sun it needs to have a higher velocity than when far away in order to sweep out the same area in equal intervals. The third law is a mathematical relation between the size of the orbit and the orbital period. As an example we see that when the semimajor axis doubles, the orbital period increases by a factor $2\sqrt{2}$ (do you agree?).

The first information that we can extract from Kepler's laws is a relation between the velocity of a planet and the orbit's distance from the Sun. When the orbit's distance from the Sun increases, does the orbital velocity increase or decrease? If we consider a nearly circular orbit, the distance traveled by the planet in one orbit is $2\pi a$, proportional to the semimajor axis. The mean velocity can thus be expressed as $v_m = 2\pi a/P$ which using Kepler's third law simply gives $v_m \propto a/(a^{3/2}) \propto 1/\sqrt{a}$ (check that you understood this!). Thus, the mean orbital velocity of a planet decreases the further away it is from the Sun.

When Newton discovered his law of gravitation,

$$\vec{F} = \frac{Gm_1m_2}{r^2}\vec{e}_r,$$

he was able to deduce Kepler's laws from basic principles. Here $\vec{F}$ is the gravitational force between two bodies of mass m_1 and m_2 at a distance r and G is the gravitational constant. The unit vector in the direction of the force is denoted by $\vec{e}_r$.

3 General solution to the two-body problem

Kepler's laws is a solution to the *two-body problem*: Given two bodies with mass m_1 and m_2 at positions $\vec{r}_1$ and $\vec{r}_2$ moving with speeds $\vec{v}_1$ and $\vec{v}_2$ (see figure 1). The only force acting on these two masses is their mutual gravitational attraction. How can we describe their future motion as a function of time? The rest of this lecture will be devoted to this problem.

Fact sheet: Our solar system consists of 8 planets: The four planets closest to the Sun, Mercury, Venus, Earth and Mars, are called terrestrial planets due to their similarity to the Earth: they have a solid surface and a thin atmosphere. The four outer planets, Jupiter, Saturn, Uranus and Neptun are called gas planets, or Jovian planets. The consist mainly of gas but may have an inner solid core. The asteroid belt between the Terrestrial and Jovian planets contains a large number of 'stones', bodies of various shapes and sizes consisting mainly of rock and metal with mean radii ranging from a few meters to hundreds of kilometers. The largest asteroid is Ceres. Beyond the orbit of Neptun is the Kuiper belt, a belt similar to the asteroid belt containing trans-Neptunian objects consisting mainly of frozen volatiles. The largest Kuiper belt object is Pluto.

Object	a (AU)	e	orb.vel.(km/s)	diam. (10^3km)	mass (10^{24}kg)
Mercury	0.39	0.2	47	4.9	0.3
Venus	0.72	0.007	35	12	5
Earth	1.0	0.02	30	13	6
Mars	1.5	0.09	24	6.8	0.6
Ceres	2.8	0.08	18	1.0	0.001
Jupiter	5.2	0.05	13	140	2000
Saturn	9.5	0.05	10	120	600
Uranus	19	0.05	6.8	51	90
Neptun	30	0.009	5.4	50	100
Pluto	40	0.25	4.7	2.4	0.01

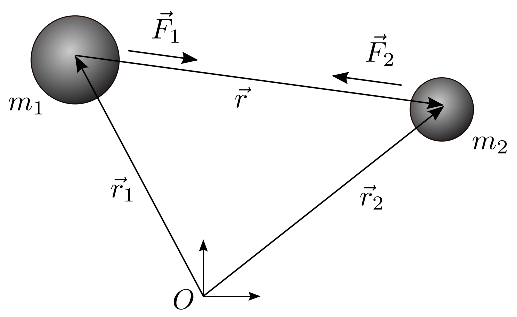


Figure 1: The two-body problem.

In order to solve the problem we will now describe the motion from the rest frame of mass 1: We will sit on m_1 and describe the observed motion of m_2, i.e. the motion of m_2 with respect to m_1. (As an example this could be the Sun-Earth system, from the Earth you view the motion of the Sun). The only force acting on m_2 (denoted $\vec{F}_2$) is the gravitational pull from m_1. Using Newton's second law for m_2, $\vec{F}_2 = m_2\vec{a}_2 = m_2\ddot{\vec{r}}_2$ with $\vec{F}_2$ taken from the law of gravitation, we get

$$\vec{F}_2 = -G\frac{m_1 m_2}{|\vec{r}|^3}\vec{r} = m_2\ddot{\vec{r}}_2, \quad (5)$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$ the vector pointing from m_1 to m_2 (or from the Earth to the Sun in our example). Overdots describe derivatives with respect to time,

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}$$
$$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2}$$

Sitting on m_1, we need to find the vector $\vec{r}(t)$ as a function of time (in our example this would be the position vector of the Sun as seen from the Earth). This function would completely describe the motion of m_2 and be a solution to the two-body problem (do you see this?).

Using Newton's third law, $\vec{F}_1 = -\vec{F}_2$, combined again with Newton's second law, we have a similar equation for the force acting on m_1

$$\vec{F}_1 = -\vec{F}_2 = G\frac{m_1 m_2}{|\vec{r}|^3}\vec{r} = m_1\ddot{\vec{r}}_1. \quad (6)$$

Subtracting equation (6) from (5), we can eliminate $\vec{r}_1$ and $\vec{r}_2$ and obtain an equation only in $\vec{r}$ which is the variable we want to solve for,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = -G\frac{m_1 + m_2}{|\vec{r}|^3}\vec{r} \equiv -m\frac{\vec{r}}{r^3}, \quad (7)$$

where $r = |\vec{r}|$ and $m = G(m_1 + m_2)$. This is the equation of motion of the two-body problem,

$$\ddot{\vec{r}} + m\frac{\vec{r}}{r^3} = 0. \quad (8)$$

We are looking for a solution of this equation with respect to $\vec{r}(t)$, this would be the solution to the two-body problem predicting the movement of m_2 with respect to m_1.

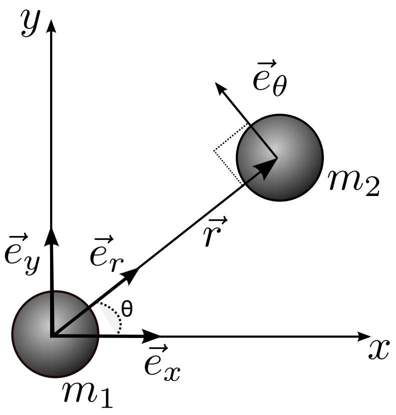


Figure 2: Geometry of the two-body problem.

To get further, we need to look at the geometry of the problem. We introduce a coordinate system with m_1 at the origin and with $\vec{e}_r$ and $\vec{e}_\theta$ as unit vectors. The unit vector $\vec{e}_r$ points in the direction of m_2 such that $\vec{r} = r\vec{e}_r$ and $\vec{e}_\theta$ is perpendicular to $\vec{e}_r$ (see figure 2). At a given moment, the unit vector $\vec{e}_r$ (which is time dependent) makes an angle θ with a given fixed (in time) coordinate system defined by unit vectors $\vec{e}_x$ and $\vec{e}_y$. From figure 2 we see that (do you really see this? Draw some figures to convince yourself!)

$$\vec{e}_r = \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \quad (9)$$
$$\vec{e}_\theta = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y \quad (10)$$

The next step is to substitute $\vec{r} = r\vec{e}_r$ into the equation of motion (equation 8). In this process we will need the time derivatives of the unit vectors. We obtain these by simply taking the time derivative of equations 9 and 10,

$$\begin{aligned}\dot{\vec{e}}_r &= -\dot{\theta}\sin\theta\vec{e}_x + \dot{\theta}\cos\theta\vec{e}_y \\ &= \dot{\theta}\vec{e}_\theta \\ \dot{\vec{e}}_\theta &= -\dot{\theta}\cos\theta\vec{e}_x - \dot{\theta}\sin\theta\vec{e}_y \\ &= -\dot{\theta}\vec{e}_r\end{aligned}$$

(check that you can do this yourself!). Using this, we can now take the derivative of the equation $\vec{r} = r\vec{e}_r$ twice,

$$\begin{aligned}\dot{\vec{r}} &= \dot{r}\vec{e}_r + r\dot{\vec{e}}_r \\ &= \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \\ \ddot{\vec{r}} &= \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta + r\dot{\theta}\dot{\vec{e}}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\vec{e}_\theta.\end{aligned}$$

(check again that you can do this!). Substituting $\vec{r} = r\vec{e}_r$ into the equation of motion (equation 8), we thus obtain

$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\vec{e}_\theta = -\frac{m}{r^2}\vec{e}_r.$$

Equating left and right hand sides, we have

$$\ddot{r} - r\dot{\theta}^2 = -\frac{m}{r^2} \quad (11)$$
$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad (12)$$

The vector equation (equation 8) has thus been reduced to these two scalar equations. Go back and check that you understood the transition.

The last of these equations indicates a constant of motion, something which does not change with time (why?). What constant of motion enters in this situation? Certainly the angular momentum of the system should be a constant of motion so let's check the expression for the angular momentum vector $\vec{h}$ (note that h is defined as angular momentum per mass, $(\vec{r}\times\vec{p})/m_2$ (remember that m_1 is at rest in our current coordinate frame)):

$$|\vec{h}| = |\vec{r}\times\dot{\vec{r}}| = |(r\vec{e}_r)\times(\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta)| = r^2\dot{\theta}.$$

So equation (12) just tells us that the magnitude of the angular momentum $h = r^2\dot{\theta}$ is conserved, just as expected.

To solve the equation of motion, we are left with solving equation (11). In order to find a solution we will

1. solve for r as a function of angle θ instead of time t. This will give us the distance of the planet as a function of angle and thus the orbit.
2. Make the substitution $u(\theta) = 1/r(\theta)$ and solve for $u(\theta)$ instead of $r(\theta)$. This will transform the equation into a form which can be easily solved.

In order to substitute u in equation (11), we need its derivatives. We start by finding the derivatives of u with respect to θ,

$$\begin{aligned}\frac{du(\theta)}{d\theta} &= \dot{u}\frac{dt}{d\theta} = -\frac{\dot{r}}{r^2}\frac{1}{\dot{\theta}} = -\frac{\dot{r}}{h} \\ \frac{d^2u(\theta)}{d\theta^2} &= -\frac{1}{h}\frac{d}{d\theta}\dot{r} = -\frac{1}{h}\ddot{r}\frac{1}{\dot{\theta}}.\end{aligned}$$

In the last equation, we substitute $\ddot{r}$ from the equation of motion (11),

$$\frac{d^2u(\theta)}{d\theta^2} = \frac{1}{h\dot{\theta}}(\frac{m}{r^2} - r\dot{\theta}^2) = \frac{m}{h^2} - \frac{1}{r} = \frac{m}{h^2} - u,$$

where the relation $h = r^2\dot{\theta}$ was used twice. We thus need to solve the following equation

$$\frac{d^2u(\theta)}{d\theta^2} + u = \frac{m}{h^2}$$

Now, pause for a moment and check that you understood the last transitions.

The equation we have obtained for $u(\theta)$ is just the equation for a harmonic oscillator (equation 2) for which we already found a solution (equation 3),

$$u(\theta) = \frac{m}{h^2} + A\cos(\theta - \omega),$$

where A and ω are constants depending on the initial conditions of the problem. Substituting back we now find the following expression for r:

The general solution to the two-body problem

$$r = \frac{p}{1 + e\cos f} \qquad (13)$$

where $p = h^2/m$, $e = (Ah^2/m)$ and $f = \theta - \omega$.

We recognize this expression as the general expression for a conic section.

4 Conic sections

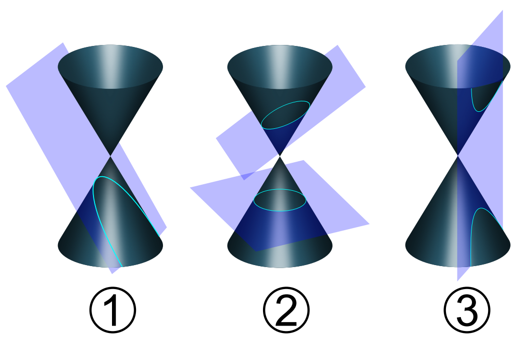


Figure 3: Conic sections: **Circle:** e=0,p=a, **Ellipse:** $0 \leq e < 1$, $p = a(1 - e^2)$, **Parabola:** $e = 1$, $p = 2a$, **Hyperbola:** $e > 1$ and $p = a(e^2 - 1)$

Conic sections are curves defined by the intersection of a cone with a plane as shown in figure 3. Depending on the inclination of the plane, conic sections can be divided into three categories with different values of p and e in the general solution to the two-body problem (equation 13),

1. the ellipse, $0 \leq e < 1$ and $p = a(1 - e^2)$ (of which the circle, $e = 0$, is a subgroup),
2. the parabola, $e = 1$ and $p = 2a$,
3. the hyperbola, $e > 1$ and $p = a(e^2 - 1)$.

In all these cases, a is defined as a positive constant $a \geq 0$. Of these curves, only the ellipse represents a bound orbit, in all other cases the planet just passes the star and leaves. We will discuss the details of an elliptical orbit later. First, we will check which conditions decides which trajectory an object will follow, an ellipse, parabola or hyperbola. Our question is thus: If we observe a planet or other object close to a star, is it in orbit around the star or just passing by? For two masses to be gravitationally bound, we expect that their total energy, kinetic plus potential, would be less than zero, $E < 0$. Clearly the total energy of the system is an important initial condition deciding the shape of the trajectory.

We will now investigate how the trajectory $r(\theta)$ depends on the total energy. In the exercises you will show that the total energy of the system can be written:

Total energy of a two-body system

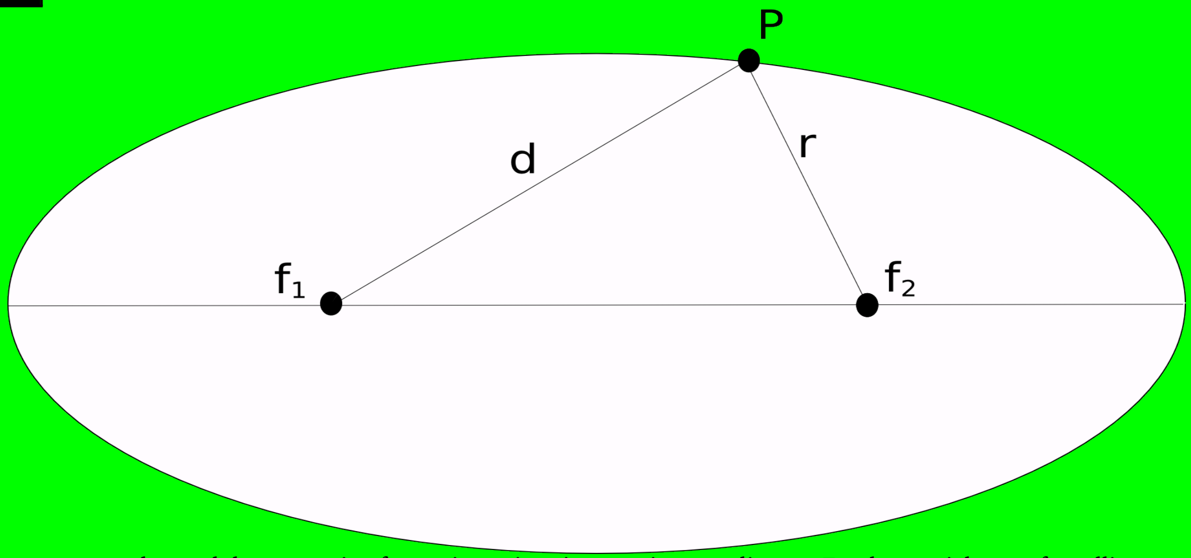
$$E = \frac{1}{2}\hat{\mu}v^2 - \frac{\hat{\mu}m}{r}, \qquad (18)$$

where $v = |\dot{\vec{r}}|$, the velocity of m_2 observed from m_1 (or vice versa) and $\hat{\mu} = m_1m_2/(m_1 + m_2)$.

We will now try to rewrite the expression for the energy E in a way which will help us to decide the relation between the energy of the system and the shape of the orbit. We will start by rewriting the velocity in terms of its radial and tangential components using the fact that $\vec{v} = \dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r$ (where did we derive this result?)

$$v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + (r\dot{\theta})^2, \qquad (19)$$

Help sheet:



In previous courses you learned the expression for conic sections in cartesian coordinates. For the special case of an ellipse we will now look at the relation between the expression in polar coordinates used in this course and the cartesian coordinates. The starting point for both expressions is the definition of the ellipse: Given two points (foci) f_1 and f_2 and a point P, where the distances to P from f_1 and f_2 are d and r respectively, an ellipse is the collection of all points P such that

$$r + d = 2a, \tag{14}$$

where a is a constant. In cartesian coordinates with origin in the middle point between the two foci, the relation between r and d and the (x,y) coordinates of the point P are given by (check that you can derive these two expressions combining the information in figure 4 with the figure above):

$$r^2 = (x - ae)^2 + y^2 \tag{15}$$

$$d^2 = (x + ae)^2 + y^2 \tag{16}$$

Inserting equations 15 and 16 in equation 14 eliminating r and d, you obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{17}$$

which is the cartesian expression for an ellipse known from other courses. If instead we switch to polar coordinates, using r and the angle f, we see from figure 4 (check!) that $x = ae + r\cos f$ and $y = r\sin f$. Inserting these two relations in equation 16, then inserting this equation in 14 eliminating d, you obtain

$$r = \frac{a(1 - e^2)}{1 + e\cos f}$$

which is the expression we use in this course.

decomposed into velocity along $\vec{e}_r$ and $\vec{e}_\theta$ (check that you got this!). We need the time derivative of r. Taking the derivative of equation (13),

$$\dot{r} = \frac{pe\sin f}{(1+e\cos f)^2}\dot{\theta},$$

we get from equation (19) for the velocity

$$v^2 = \dot{\theta}^2 \frac{p^2 e^2 \sin^2 f}{(1+e\cos f)^4} + r^2\dot{\theta}^2.$$

Next step is in both terms to substitute $\dot{\theta} = h/r^2$ (where did this come from?) and then using equation (13) for r giving

$$v^2 = \frac{h^2 e^2 \sin^2 f}{p^2} + \frac{h^2(1+e\cos f)^2}{p^2}.$$

Collecting terms and remembering that $\cos^2 f + \sin^2 f = 1$ we obtain

$$v^2 = \frac{h^2}{p^2}(1+e^2+2e\cos f).$$

We will now get back to the expression for E. Substituting this expression for v as well as r from equation (13) into the energy expression (equation 18), we obtain

$$E = \frac{1}{2}\hat{\mu}\frac{h^2}{p^2}(1+e^2+2e\cos f) - \hat{\mu}m\frac{1+e\cos f}{p} \quad (20)$$

Total energy is conserved and should therefore be equal at any point in the orbit, i.e. for any angle f. We may therefore choose an angle f which is such that this expression for the energy will be easy to evaluate. We will consider the energy at the point for which $\cos f = 0$,

$$E = \frac{1}{2}\hat{\mu}\frac{h^2}{p^2}(1+e^2) - \frac{\hat{\mu}m}{p}$$

We learned above (below equation 13) that $p = h^2/m$ and thus that $h = \sqrt{mp}$. Using this to eliminate h from the expression for the total energy we get

$$E = \frac{\hat{\mu}m}{2p}(e^2-1).$$

If the total energy $E = 0$ then we immediately get $e = 1$. Looking back at the properties of conic sections we see that this gives a parabolic trajectory. Thus, masses which have just too much kinetic energy to be bound will follow a parabolic trajectory. If the total energy is different from zero, we may rewrite this as

$$p = \frac{\hat{\mu}m}{2E}(e^2-1).$$

We now see that a negative energy E (i.e. a bound system) gives an expression for p following the expression for an ellipse in the above list of properties for conic sections (by defining $a = \hat{\mu}m/(2|E|)$. Similarly a positive energy gives the expression for a hyperbola. We have shown that the total energy of a system determines whether the trajectory will be an ellipse (bound systems $E < 0$), hyperbola (unbound system $E > 0$) or parabola ($E = 0$). We have just shown Kepler's first law of motion, stating that a bound planet follows an elliptical orbit. In the exercises you will also show Kepler's second and third law using Newton's law of gravitation.

5 The elliptical orbit

We have seen that the elliptical orbit may be written in terms of the distance r as

$$r = \frac{a(1-e^2)}{1+e\cos f}.$$

In figure (4) we show the meaning of the different variables involved in this equation:

- a is the *semimajor axis*
- b is the *semiminor axis*
- e is the *eccentricity* defined as $e = \sqrt{1-(b/a)^2}$
- m_1 is located in the *principal focus*
- the point on the ellipse closest to the principal focus is called *perihelion*
- the point on the ellipse farthest from the principal focus is called *aphelion*
- the angle f is called the *true anomaly*

The eccentricity is defined using the ratio b/a. When the semimajor and semiminor axis are equal, $e = 0$ and the orbit is a circle. When the semimajor axis is much larger than the semiminor axis, $e \to 1$.

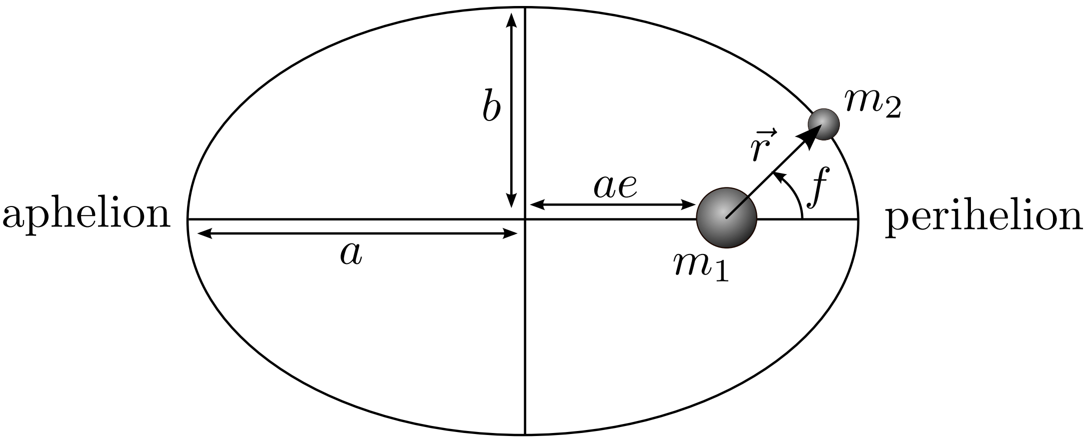
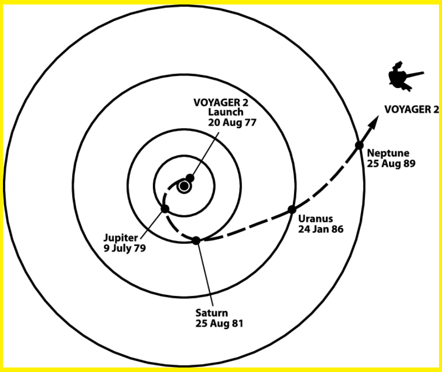


Figure 4: The ellipse.

Fact sheet: A diagram of the trajectory that enabled NASA's Voyager 2 spacecraft to tour all the four gas giants and achieve a large enough velocity to escape our solar system. Celestial mechanics obviously played an integral part in the extremely careful planning that was needed in order to carry out the probe's ambitious tour of the outer solar system. The planetary flybys not only allowed for close-up observations of the planets and their moons, but also accelerated the probe so that it could reach the next object. In 2012 Voyager 2 was at a distance of roughly 100 AU from the Sun, traveling outward at around 3.3 AU per year. It is expected to keep transmitting weak radio messages until at least 2025.



6 Center of mass system

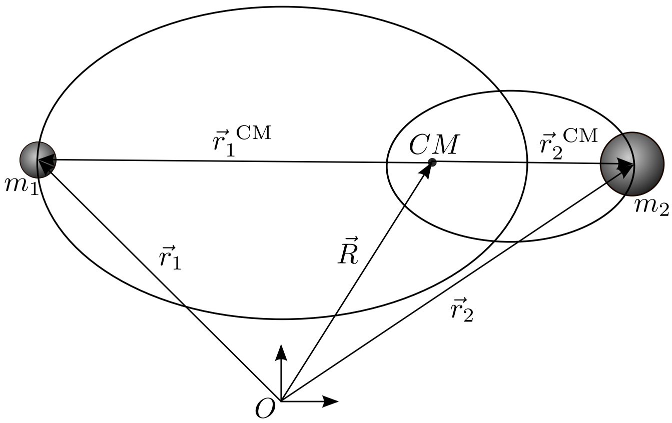


Figure 5: The center of mass system: The center of mass (CM) is indicated by a small point. The two masses m_1 and m_2 orbit the center of mass in elliptical orbits with the center of mass in one focus of both ellipses. The center of mass vectors $\vec{r}_1^{\,\rm CM}$ and $\vec{r}_2^{\,\rm CM}$ start at the center of mass and point to the masses.

In the previous section we showed that seen from the rest frame of one of the masses in a two-body system, the other mass follows an elliptical / parabolic / hyperbolic trajectory. How does this look from a frame of reference which is not at rest with respect to one of the masses? We know that both masses m_1 and m_2 are moving due to the gravitational attraction from the other. If we observe a distant star-planet system, how does the planet *and* the star move with respect to each other? We have only shown that sitting on either the planet or the star, the other body will follow an elliptical orbit.

An elegant way to describe the full motion of the two-body system (or in fact an N-body system) is to introduce *center of mass coordinates*. The center of mass position $\vec{R}$ is located at a point on the line between the two masses m_1 and m_2. If the two masses are equal, the center of mass position is located exactly halfway between the two masses. If one mass is larger than the other, the center of mass is located closer to the more massive body. The center of mass is a weighted mean of the position of the two masses:

$$\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2, \tag{21}$$

where $M = m_1 + m_2$. We can similarly define the center of mass for an N-body system as

$$\vec{R} = \sum_{i=1}^{N} \frac{m_i}{M}\vec{r}_i, \tag{22}$$

where $M = \sum_i m_i$ and the sum is over all N masses in the system. Newton's second law for one object in the system is

$$\vec{f}_i = m_i \ddot{\vec{r}}_i$$

where $\vec{f}_i$ is the total force on object i. Summing over all bodies in the system, we obtain Newton's second law for the full N-body system

$$\vec{F} = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i, \tag{23}$$

where $\vec{F}$ is the total force on all masses in the system. We may divide the total force on all masses into one contribution from internal forces between masses and one contribution from external forces,

$$\vec{F} = \sum_i \sum_{j \neq i} \vec{f}_{ij} + \vec{F}_{\text{ext}},$$

where $\vec{f}_{ij}$ is the gravitational force on mass i from mass j. Newton's third law implies that the sum over all internal forces vanish ($\vec{f}_{ij} = -\vec{f}_{ji}$). The right side of equation (23) can be written in terms of the center of mass coordinate using equation (22) as

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i = M\ddot{\vec{R}},$$

giving

$$M\ddot{\vec{R}} = \vec{F}_{\text{ext}}.$$

(Check that you followed this deduction!). If there are no external forces on the system of masses ($\vec{F}_{\text{ext}} = 0$), this equation tells us that the center of mass position does not accelerate, i.e. if the center of mass position is at rest it will remain at rest, if the center of mass position moves with a given velocity it will keep moving with this velocity. We may thus divide the motion of a system of masses into the motion of the center of mass and the motion of the individual masses with respect to the center of mass.

We now return to the two-body system assuming that no external forces act on the system. The center of mass moves with constant velocity and we decide to deduce the motion of the masses with respect to the center of mass system, i.e. the rest frame of the center of mass. We will thus be sitting at the center of mass which we define as the origin of our coordinate system, looking at the motion of the two masses. When we know the motion of the two masses with respect to the center of mass, we know the full motion of the system since we already know the motion of the center of mass position.

Since we take the origin at the center of mass location, we have $\vec{R} = 0$. Using equation (21) we get

$$0 = \frac{m_1}{M}\vec{r}_1^{\text{CM}} + \frac{m_2}{M}\vec{r}_2^{\text{CM}},$$

where CM denotes position in the center of mass frame (see figure 5). Combining this equation with the fact that $\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}_2^{\text{CM}} - \vec{r}_1^{\text{CM}}$ we obtain

$$\vec{r}_1^{\text{CM}} = -\frac{\hat{\mu}}{m_1}\vec{r}, \tag{24}$$

$$\vec{r}_2^{\text{CM}} = \frac{\hat{\mu}}{m_2}\vec{r}, \tag{25}$$

The **reduced mass** $\hat{\mu}$ is defined as

$$\hat{\mu} = \frac{m_1 m_2}{m_1 + m_2}.$$

The relative motion of the masses with respect to the center of mass can be expressed in terms of $\vec{r}_1^{\text{CM}}$ and $\vec{r}_2^{\text{CM}}$ as a function of time, or as we have seen before, as a function of angle f. We already know the motion of one mass with respect to the other,

$$|\vec{r}| = \frac{p}{1 + e\cos f}.$$

Inserting this into equations (24) and (25) we obtain

$$|\vec{r}_1^{\,\mathrm{CM}}| = \frac{\hat{\mu}}{m_1}|\vec{r}\,| = \frac{\hat{\mu}p}{m_1(1+e\cos f)}$$
$$|\vec{r}_2^{\,\mathrm{CM}}| = \frac{\hat{\mu}}{m_2}|\vec{r}\,| = \frac{\hat{\mu}p}{m_2(1+e\cos f)}$$

For a bound system we thus have

$$|\vec{r}_1^{\,\mathrm{CM}}| = \frac{\frac{\hat{\mu}}{m_1}a(1-e^2)}{1+e\cos f} \equiv \frac{a_1(1-e^2)}{1+e\cos f}$$
$$|\vec{r}_2^{\,\mathrm{CM}}| = \frac{\frac{\hat{\mu}}{m_2}a(1-e^2)}{1+e\cos f} \equiv \frac{a_2(1-e^2)}{1+e\cos f}$$

We see from these equations that for a gravitationally bound system, *both* masses move in elliptical orbits with the center of mass in one of the foci (how do you see this?). The semimajor axis of these two masses are given by

$$a_1 = \frac{\hat{\mu}a}{m_1},$$
$$a_2 = \frac{\hat{\mu}a}{m_2},$$
$$a = a_1 + a_2$$

(check that you understand how these equations come about) where a_1 and a_2 are the semimajor axis of m_1 and m_2 respectively and a is the semimajor axis of the elliptical orbit of one of the masses seen from the rest frame of the other. Note that the larger the mass of a given body with respect to the other, the smaller the ellipse. This is consistent with our intuition: The more massive body is less affected by the same force than is the less massive body. The Sun moves in an ellipse around the center of mass which is much smaller than the elliptical orbit of the Earth. Figure (5) shows the situation: the planet and the star orbit the common center of mass situated in one common focus of both ellipses.

7 Exercises

Exercise 1B.1 You need to read all sections from 1 to 5 to be able to solve this exercise. The scope of this problem is to deduce Kepler's second law. Kepler's second law can be written mathematically as

$$\frac{dA}{dt} = \text{constant},$$

i.e. that the area A swept out by the vector $\vec{r}$ per time interval is constant. We will now show this step by step:

1. Show that the infinitesimal area dA swept out by the radius vector $\vec{r}$ for an infinitesimal movement dr and $d\theta$ is $dA = \frac{1}{2}r^2 d\theta$. (hint: draw a figure: can you approximate the area in the figure as a triangle?)
2. Divide this expression by dt and you obtain an expression for dA/dt in terms of the radius r and the tangential velocity v_θ.
3. By looking back at the above derivations, you will see that the tangential velocity can be expressed as $v_\theta = h/r$.
4. Show Kepler's second law.

Exercise 1B.2 The scope of this problem is to deduce Kepler's third law. Again we will solve this problem step by step:

1. In the previous problem we found an expression for dA/dt in terms of a constant. Integrate this equation over a full period P and show that
$$P = \frac{2\pi ab}{h}$$
(Hint: the area of an ellipse is given by πab).
2. Use expressions for h and b found in the text to show that
$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)}a^3 \qquad (26)$$
3. This expression obtained from Newtonian dynamics differs in an important way from the original expression obtain empirically by Kepler (equation 4). How? Why didn't Kepler discover it?

Exercise 1B.3

1. How can you measure the mass of a planet in the solar system by observing the motion of one of its satellites? Assume that we know only the semimajor axis and orbital period for the elliptical orbit of the satellite around the planet. **Hint 1:** Kepler's third law (the exact version that you deduce in exercise 1B.2). **Hint 2:** You are allowed to make reasonable approximations.
2. Look up (using Internet or other sources) the semimajor axis and orbital period of Jupiter's moon Ganymede.
   - (a) Use these numbers to estimate the mass of Jupiter.
   - (b) Then look up the mass of Jupiter. How well did your estimate fit? Is this an accurate method for computing planetary masses?
   - (c) Which effects could cause discrepancies from the real value and your estimated value?

Exercise 1B.4 You need to have read also section 6 in order to solve this problem.

1. Show that the total energy of the two-body system in the center of mass frame can be written as
$$E = \frac{1}{2}\hat{\mu}v^2 - \frac{GM\hat{\mu}}{r},$$
where $v = |d\vec{r}/dt|$ is the relative velocity between the two objects, $r = |\vec{r}|$ is their relative distance, $\hat{\mu}$ is the reduced mass and $M \equiv m_1 + m_2$ is the total mass. **Hint:** make the calculation in the center of mass frame and use equation (24) and (25).
2. Show that the total angular momentum of the system in the center of mass frame can be written
$$\vec{P} = \hat{\mu}\vec{r} \times \vec{v},$$
3. Looking at the two expressions you have found for energy and angular momentum of the system seen from the center of mass frame: Can you find an equivalent two-body problem with two masses m_1' and m_2' where

the equations for energy and momentum will be of the same form as the two equations which you have just derived? What are m_1' and m_2'? If you didn't understand the question, here is a rephrasing: If you were given these two equations without knowing anything else, which physical system would you say that they describe?

Exercise 1B.5

1. At which points in the elliptical orbit (for which angels f) is the velocity of a planet at maximum or minimum?

2. Using only the mass of the Sun, the semi-major axis and eccentricity of Earth's orbit (which you look up in Internet or elsewhere), can you find an estimate of Earth's velocity at aphelion and perihelion?

3. Look up the real maximum and minimum velocities of the Earth's velocity. How well do they compare to your estimate? What could cause discrepancies between your estimated values and the real values?

4. Use Python (or Matlab or any other programming language) to plot the variation in Earth's velocity during one year.

Hint 1: Use one or some of the expressions for velocity found in section (4) as well as expressions for p and h found in later sections (including the above problems). **Hint 2:** You are allowed to make reasonable approximations.

Exercise 1B.6

1. Find our maximum and minimum distance to the center of mass of the Earth-Sun system.

2. Find Sun's maximum and minimum distance to the center of mass of the Earth-Sun system.

3. How large are the latter distances compared to the radius of the Sun?

Exercise 1B.7

In this problem you can use the commands `dumpToXml` and `DualStarXml`from the SolarSystemViewer class combined with the SolarSystemviewer application to visualize and check your results as explained below.

1. In this exercise you will solve the equation of motion numerically to obtain the orbits of the planets in your solar system. Use Newton's second law,

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} = \vec{F},$$

to solve the 2-body problem numerically. Use the Euler-Cromer method for differential equations as explained in section 1. It is more obvious how you can do this if you decompose Newtons's second law in x and y components: Write Newton's second law in terms of the velocity vector.

$$m\left(\frac{dv_x}{dt}\vec{e}_x + \frac{dv_y}{dt}\vec{e}_y\right) = F_x\vec{e}_x + F_y\vec{e}_y$$

Then we have the following relation between the change in the components of the velocity vector and the components of the force vector;

$$\frac{dv_x}{dt} = \frac{F_x}{m}$$
$$\frac{dv_y}{dt} = \frac{F_y}{m}$$

These equations can be solved directly by the Euler-Cromer method (see again section 1) and the given initial conditions: the initial positions and velocities for the planets at $t = 0$ in your solar system can be obtained from the SolarSystemViewer class. For each timestep (use a for- or while-loop), calculate the velocity $v_{x/y}(t+dt)$ (Euler's method) and the position $x(t + dt)$, $y(t + dt)$ (standard kinematics) for each planet (or even easier: use vectors). You may use the following approach:

(a) Assume that the gravitational force between the planets is negligible. Assume further that the star is fixed at the origin and is not influenced by the gravitational pull of the planets. This assumption will be relaxed in a coming exercise

Fact sheet: 433 Eros was the target of the first long-term, close-up study of an asteroid. After a four year journey the NEAR-Shoemaker space probe was inserted into orbit around the 33 km long, potato-shaped asteroid in February 2000 and encircled it 230 times from various distances before touching down on its surface. The primary scientific objective was to return data on the composition, shape, internal mass distribution, and magnetic field of Eros. Asteroids are a class of rocky small solar system bodies that orbit the Sun, mostly in the asteroid belt between Mars and Jupiter. They are of great interest to astronomers as they are leftover material from when the solar system formed some 4.6 billion years ago.

in part 1C, but for now you must make this assumption in order for the viewer to work properly.

(b) Initiate planet positions and velocities with your given initial values.

(c) Define a time interval dt and a number of time steps suitable for your solar system. You will have to experiment a bit to find these numbers. Start with just a few time steps until the code works.

(d) For each time step, calculate the force of the star on each planet.

(e) Update velocity and then position using Euler-Cromer for all the planets and store these in an array.

(f) Continue until at least one of the outer planets has made one full orbit.

(g) Send the array to the AST1100 SolarSystemView class using the `dumpToXml` method (as explained in the documentation to the SolarSystemViewer class) to create an xml to view with SolarSystemViewer. Do the orbits look correct?

2. This part of the exercise is **optional**, but it will count favourably on the grading of the exercise if you include it. We will use our code to study the 3-body problem. There is no analytical solution to the 3-body problem, so in this case we are forced to use numerical calculations. The fact that most problems in astrophysics consider systems with a huge number of objects strongly underlines the fact that numerical solutions are of great importance. In this problem you can use the `DualStarXml` method of the SolarSystemView class to visualize the orbit of the planet around the two stars.

About half of all the stars are binary stars, two stars orbiting a common center of mass. Binary star systems may also have planets orbiting the two stars. Here we will look at one of many possible shapes of orbits of such planets. We will consider a planet with the mass identical to the mass of Mars. One of the stars has a mass identical to the mass of the Sun (2×10^{30} kg), the other has a mass 4 times that of the Sun.

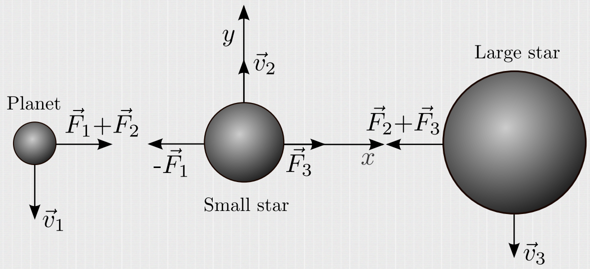


Figure 6: The binary star system with the planet at time t = 0.

The initial positions are $[x_1 = -1.5$ AU, $y_1 = 0$, $z_1 = 0]$ (for the planet), $[x_2 = 0$, $y_2 = 0$, $z_2 = 0]$ (for the small star) and $[x_3 = 3$ AU, $y_3 = 0$, $z_3 = 0]$ (for the large star) (Figure 3). The initial velocity vectors are $\vec{v_1} = -1\ \frac{\text{km}}{\text{s}}\vec{j}$ (for the planet), $\vec{v_2} = 30\ \frac{\text{km}}{\text{s}}\vec{j}$ (for the small star) and $\vec{v_3} = -7.5\ \frac{\text{km}}{\text{s}}\vec{j}$ (for the large star).

Plot the orbit of the planet and the two stars in the same figure. Use timestep $dt = 400$ seconds and make 10^6 calculations. It should now be clear why it is impossible to find an analytical solution to the 3-body problem. Note that the solution is an approximation. If you try to change the size and number of time steps you will get slighly different orbits, small time steps cause numerical problems and large time steps is too inaccurate. The given time step is a good trade-off between the two problems but does not give a very accurate solution. Accurate methods to solve this problem is outside the scope of this course. Play around and try some other starting positions and/or velocities. Insert your positions to the `DualStarXml` method of the SolarSystemView class as explained in the documentation for the SolarSystemView class in order to see how your planet moves between the two stars. Looking at the movement of the planet using the viewer gives you a very good understanding of the physics at work here. **NOTE: The best way to see it is to click on the planet such that the planet is in focus and zoom in and out depending on whether the stars are far away or not.**

Hints: There is really not much more code you need to add to the previous code to solve this problem. Declare arrays and constants for the three objects. In your for-/while-loop, calculate the total force components for each object. Since we have a 3-body problem we get two contributions to the total force for each object. In other words, you will have to call the function of gravitation three times for each time-evaluation. For each time step, first calculate the force components between the planet and the small star, then the force components between the planet and the large star, and finally the force components between the small and the large star. Then you sum up the contributions that belong to each object.

Look at the trajectory and try to imagine how the sky will look like at different epochs. If we assume that the planet has chemical conditions for life equal to those on earth, do you think it is probable that life will evolve on this planet? Use your tracetory to give arguments.

Exercise 1B.8 In this exercise you should choose the planet where you want to land your spaceprobe. You can use the `landingSat` method from the SolarSystemViewer class to visualize your landing and check if it is correct as explained below. Assume that you have already reached orbit around your chosen planet. The satellite is orbiting 40000km above the surface when the lander unit is sent towards the planet. Here is the plan,

- You can choose yourself the exact position around your planet from where the lander is launched, but the distance to the surface has to be 40000km (note that this is the distance to the **surface** of the planet, not to center of the planet).
- Assume the weight of the lander to be 100kg.
- Assume the position of the planet to be fixed at the origin.
- You know the surface density ρ_0 of your atmosphere which you can obtain from the SolarSystemViewer class. Assume the density profile of your atmosphere is given by

  $$\rho(h) = \rho_0 e^{-h/h_{\text{scale}}}$$

  where h is the height above the surface and h_{scale} is the scale height of the atmosphere given by $h_{\text{scale}} = 75200/g$ m, where g is the gravitational acceleration at the surface of your planet (in SI units). The drag force on your lander (including its parachute) is given as

  $$F_D = \frac{1}{2}\rho A v^2$$

  where ρ is the density of the atmosphere, A is the area of the parachute and v is the current velocity of the lander. Note that this force works in the direction opposite of the velocity vector. One of your tasks here will be to find the size of the parachute.
- The heat shield of the lander unit can withstand frictional forces (drag forces) from the atmosphere up to 25000N. If at any point during the landing, the drag force exceeds this value, the lander will be ripped apart.
- In order to have a soft landing, the component of your velocity vector pointing radially

inwards towards the planet needs to be less than 3m/s.

- You need to choose the initial angle and speed with which the lander is launched from the orbiting satellite (you might need some trial and error here) as well as the size of the parachute such that you manage to get a soft landing without destroying the lander from the frictional forces of the atmosphere. Explain your strategy to lower the frictional forces on the lander unit.

- You need use Euler-Cromer again with the extra force included. Try to find a suitable time step dt such that the code is quick but still sufficiently accurate.

- In order to check the accuracy of the time step: when you have found a solution, you should reduce dt to 1/10 of its original value and check if you still get a similar answer with this more exact calculation. If you do, you have found a time step which is sufficiently small. You may need a time step as small as $dt = 0.1$ seconds (or maybe even smaller for some planets) to get reliable results.

To solve this exercise you therefore need to:

1. Find the initial launch position and velocity vector of the lander and the necessary surface area of the parachute (as small as you can manage, but for some planets it may need to be unrealistically big) in order to get a soft landing on the planet and at the same time avoiding too large frictional forces in the atmosphere. Describe how you went about to find these numbers (probably some trial and error, describe what you tried which did not work before you found your solution).

2. Note that on some planets it may be almost impossible to find a trajectory where the drag forces are less than 25000N. If this is the case for your planet, describe and show the attempts you made and present the trajectory with the lowest drag force.

3. Simulate the trajectory of the lander down to the surface of the planet.

4. Pass your positions coordinates to the `landingSat` method from the SolarSystemViewer class as explained in the SolarSystemViewer documentation to visualize your landing and check if the trajectory looks reasonable.

Hints - You can use most of the code from the previous exercise. First, we write Newton's second law in terms of the cartesian components;

$$\frac{dv_x}{dt} = \frac{F_x + f_x}{m}$$

$$\frac{dv_y}{dt} = \frac{F_y + f_y}{m}$$

(or use the vector form directly if you prefer, this makes the code much shorter). The best approach is to make one more function that calculates the force of friction with the lander's velocity components as arguments. In this problem you should use a while-loop. For each evaluation, first call the gravitational function (as before) and then the friction function. Remember to send the space craft's velocity components from the previous timestep. In the friction function, first calculate the total force F, and then (if you do not use vectors) the components F_x and F_y (use simple trigonometry) with correct positive/negative-sign by checking the sign of the velocity components. Then return the force components to the loop. For each evaluation (in the while-loop) check whether the spaceship has landed or not.