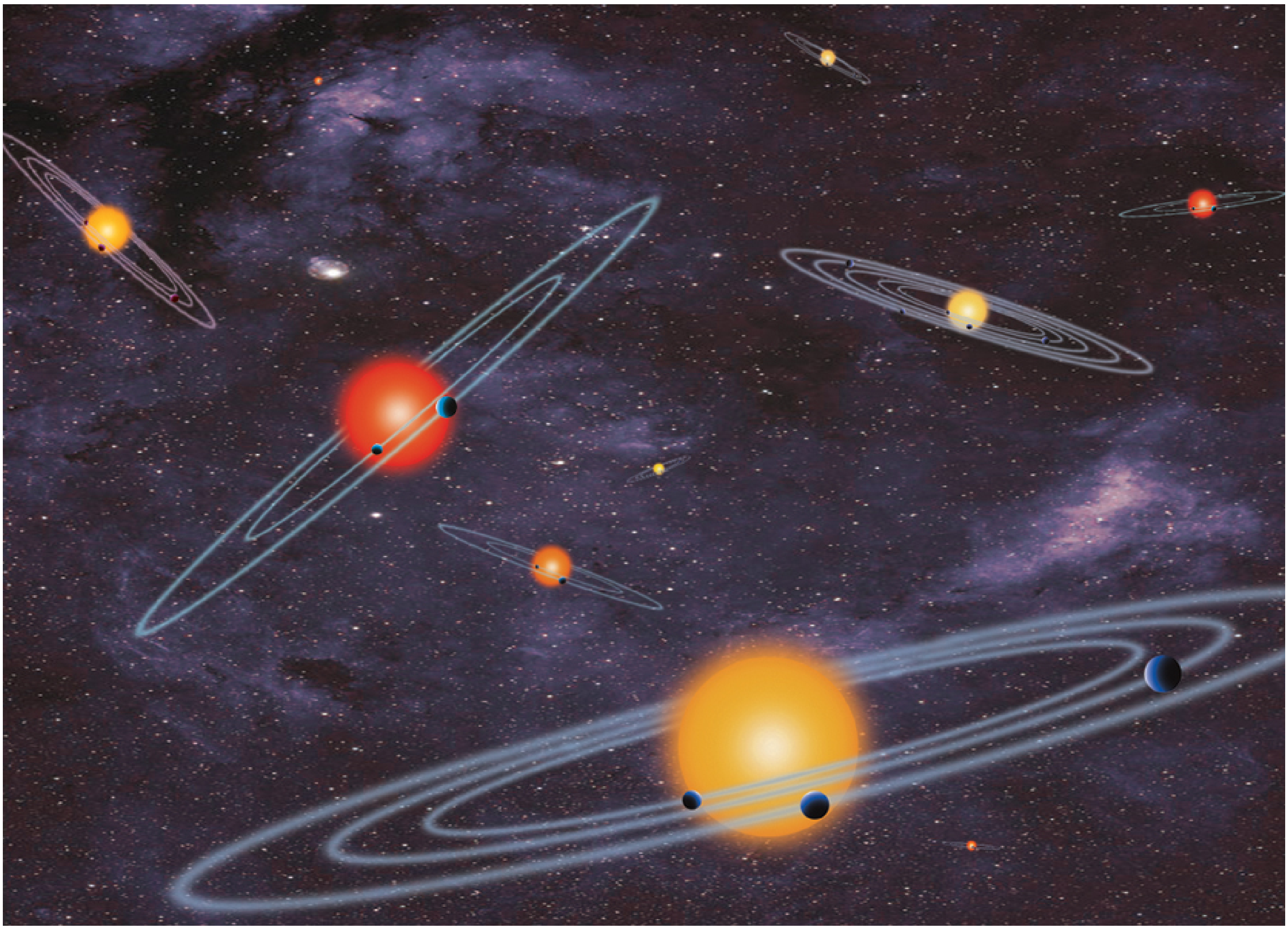
AST1100 Lecture Notes

Part 1C
Extrasolar planets

Questions to ponder before the lecture

1. Why is it only during recent years that we have started to detect planets orbiting stars outside our solar system? Why has it been so difficult?
2. Did you ever see a (real) picture of a planet orbiting another star? If not, can you imagine why?
3. Most of these exoplanets have never been seen directly, but how do we still know so much about them?



AST1100 Lecture Notes

Part 1C
Extrasolar planets

We will now discuss how distant solar systems can be discovered by simple observational methods. Our goal is twofold, first of all we need this in order to be able to detect solar systems to visit with our rocket and second, we would be interested in understanding how extraterrestrials far away can discover that the star in your solar system has planets orbiting it.

1 Detecting extrasolar planets

Most models of star formation tell us that the formation of planets is a common process. We expect most stars to have planets orbiting them. Why then, has only a very few planets (about 18 by fall 2014) around other stars been seen directly? There are two main reasons for this:

1. The planet's orbit is often close to the star. If the star is far away from us, the angular distance between the star and the planet is so small that the telescope cannot separate the two objects.

2. The light from the star is much brighter than the starlight reflected from the planet. It is very difficult to detect a faint signal close to a very bright source.

How large is the angular distance on the sky between Earth and Sun seen from our closest star, Proxima Centauri 4.22 light years away? Look at the geometry in figure 1.

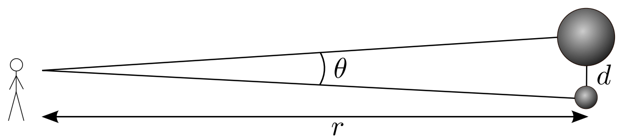


Figure 1: The angular extension of a distant planet's orbit around its star.

The distance r is 4.22 light years, the distance Sun-Earth d is 150×10^6 km. Using the small angle formula from geometry (and this is indeed a very small angle),

$$d = r\theta$$

we find $\theta = 0.00021°$ (check!). In astrophysics we usually specify small angles in terms of *arcminutes* and *arcseconds*, denoted $'$ and $''$. There are 60 arcminutes in one degree and 60 arcseconds in one arcminute. Thus the angular distance between Sun and Earth as seen from Proxima Centauri is $0.77''$. From the ground, the best resolution a normal telescope can reach is $0.4''$ under very good atmospheric conditions (actually using so-called adaptive optics better resolutions may be attained). This means that two objects with a smaller angular distance on the sky cannot be separated by the telescope. So the green men on a planet orbiting our nearest star would just be able to see the Earth with the best telescopes under very good atmospheric conditions (provided the atmosphere on this planet is similar to the Earth's)! The Hubble Space Telescope which is not limited by the atmosphere can reach a resolution of $0.1''$ (in reality, using statistical analysis, one can resolve objects which are closer than the resolution limit). For the people on a planet orbiting a star located 100 light years away from Earth, the angular distance between the Earth and the Sun would be 0.03″. From this planet, our green friends would barely be able to see the Earth using the Hubble Space Telescope! A huge advance in optics and telescope technology is needed in the future in order to resolve planets which are orbiting close to their mother star.

Still, about 1800 planets orbiting other stars have been detected (by fall 2014, but the number is now rapidly increasing after the launch of the Kepler satellite in 2009 (`http://kepler.nasa.gov/`)). The reason for this can be found in the previous lecture: In a star-planet system, the planet *and* the star are orbiting their common center of mass. Thus, the star is moving in an elliptical orbit. If the velocity of the star can be measured, then a regular variation of the star's velocity as it orbits the center of mass should be detected. This was, untill 2009, the way most of the extrasolar planets were discovered. This is now changing with Kepler which discovers extrasolar planets by eclipses which we will come back to later.

One way to measure the velocity of a star is by the Doppler effect, that electromagnetic waves (light) from the star change their wavelength depending on whether the star is moving towards us or away from us. When the star is approaching, we observe light with shorter wavelength, the light is *blueshifted*. On the contrary, when the star is receding, the light is *redshifted*. By measuring the displacement of spectral lines in the stellar spectra (more details about this in a later lecture), we can measure velocities of stars by the impressive precision of 1 m/s, the walking speed of a human being. In this way, even small variations in the star's velocity can be measured. Recall the formula:

Change in wavelength due to the Doppler effect

$$\frac{\lambda - \lambda_0}{\lambda_0} = \frac{v_r}{c},$$

where λ is the observed wavelength and λ_0 is the wavelength seen from the rest frame of the object emitting the wave.

There is one drawback of this method: only radial velocity can be measured. Tangential velocity, movements perpendicular to the line of sight, does not produce any Doppler effect. The orbital plane of a planet (which is the same as the orbital plane of the star) will have a random orientation. We will therefore only be able to measure one component of the star's velocity, the radial velocity.

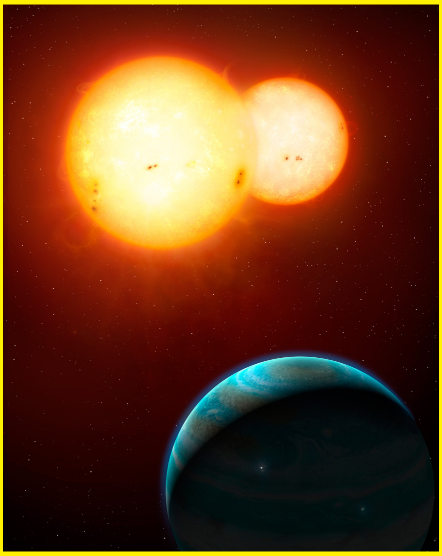
In figure 2 we have plotted the situation. The angle i is called the *inclination* of the orbit. It is simply the angle between the line of sight and a line perpendicular to the orbital plane (see figure 2). When the inclination $i = 90°$, the plane of the orbit is aligned with the line of sight and the velocity measured from the Doppler effect is the full velocity. For an inclination $i = 0°$, there is no radial component of the velocity and no Doppler effect is seen. A regular variation in a star's radial velocity could be the sign of a planet orbiting it.

We will in the following assume circular orbits (i.e. the eccentricity $e = 0$). This will make calculations easier, the distance from the center of mass a is always the same and more importantly, the velocity v is the same for all points in the orbit. In figure 3 we show how the radial velocity changes during the orbit of the star around the center of mass. If the inclination is $i = 90°$, then the radial velocity v_r equals the real velocity v in the points B and D in the figure. For other inclinations, the radial velocity v_r in points B and D is given by

$$|v_r| = v \sin i. \tag{1}$$

This is found by simple geometry, it is the component of the velocity vector taken along the line of sight (do you see this?). Note: The velocity v discussed here is the orbital velocity of the star, i.e. the velocity of the star with respect to the center of mass. Normally the star/planet system, i.e. the center of mass, has a (approximately) constant velocity with respect to the observer. This velocity v_{pec} is called the peculiar velocity and must be subtracted in order to obtain the velocity with respect to the center of mass. Recall from the previous lecture that the velocity of the star can be decomposed into the velocity of the center of mass (peculiar velocity) and the velocity of the star with respect to the center of mass (which is

Fact sheet: Artistic rendition of Kepler-35b, a Saturn-sized planet orbiting a pair of Sun-like stars. The first confirmed detection of a planet orbiting a main-sequence star other than the Sun was made in 1995. Since then hundreds of exoplanets have been discovered; see http://exoplanet.eu for a complete and up-to-date list. Astronomers employ several methods for finding exoplanets, e.g. the radial velocity or Doppler method, the transit method, gravitational microlensing, astrometry, pulsar timing, and even direct imaging. Recent surveys have shown that planets around stars in the Milky Way are the rule rather than the exception! Exoplanet research is one the hottest fields in astronomy today.(M. Garlick)



the one we need).

Another way to see this is by writing the position of the star with respect to the observer $\vec{r}$ (defining the origin to be the position of the observer),

$$\vec{r} = \vec{R} + \vec{r}_*$$

where $\vec{R}$ is the position of the center of mass (using the notation of part 1B) and $\vec{r}_*$ is the position of the star with respect to the center of mass. Taking the time derivative of this equation we have

$$\vec{v} = \vec{V} + \vec{v}_* \qquad (2)$$

which means that the observed velocity $\vec{v}$ is the sum of the velocity of the center of mass (peculiar velocity) $\vec{V}$ (i.e. of the whole system) with respect to us and the velocity $\vec{v}_*$ of the star with respect to the center of mass. Note that the vector $\vec{r}$ is pointing **from the observer to the object** which has important consequences for the interpretation of the sign of the observed velocity: positive velocity means that $|\vec{r}|$ (the distance to the observer) is increasing and negative velocity means the opposite.

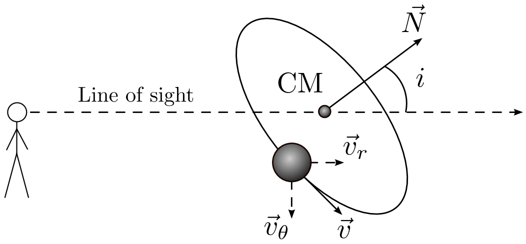


Figure 2: Inclination: The angle between the line of sight and the normal $\vec{N}$ to the orbital plane is called the inclination i. The maximum radial velocity of the star equals $v \sin i$.

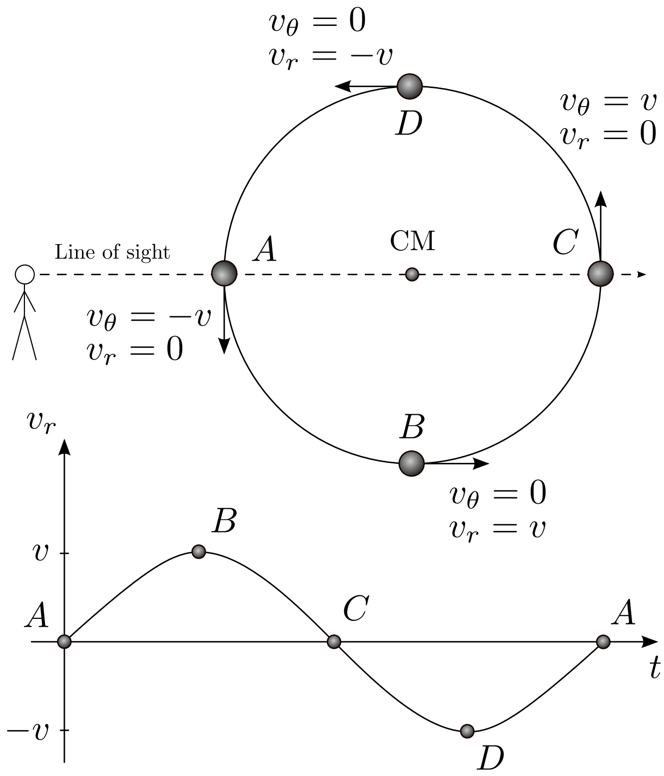


Figure 3: The velocity curve of a star orbiting the common center of mass with a planet. The points where the component of the velocity vector along the line of sight is zero (A and C) as well as the points where the radial component equals the full velocity (B and D) are indicated. In the figure, we have assumed an inclination of 90°.

2 Determining the mass of extrasolar planets

We know that Kepler's third law (Newton's version as you deduced it in the exercises of the previous lectures) connects the orbital period P, the semimajor axis a (radius in the case of a circular orbit) and mass m of the planet/star (do you remember how?). From observations of the radial velocity of a star we can determine the orbital period of the star/planet system. Is there a way to combine this with Kepler's laws in order to obtain the mass of the planet? The goal of this section is to solve this problem. We will deduce a way to determine the mass of an extrasolar planet with as little information as possible.

In the following we will use m_*, a_*, v_* for mass, radius of the orbit and velocity of the star in its orbit around the center of mass. Similarly we will use m_p, a_p and v_p for the corresponding quantities regarding the planet. The constant velocities may be expressed as,

$$v_* = \frac{2\pi a_*}{P} \qquad v_\mathrm{p} = \frac{2\pi a_\mathrm{p}}{P}. \tag{3}$$

Note again that this is velocity with respect to center of mass, any peculiar velocity has been subtracted. In part 1B, section 6, we found expressions for the position of the two bodies m_1 and m_2 taken in the center of mass frame, $\vec{r}_1^{\,\mathrm{CM}}$ and $\vec{r}_2^{\,\mathrm{CM}}$. Before reading on, look back at these lecture notes now and make sure you remember how these expressions were obtained!

Did you check those lecture notes? Ok, then we can continue. Take these masses to be the star and the planet. Using these expressions, we obtain (check!)

$$\frac{|\vec{r}_*^{\,\mathrm{CM}}|}{|\vec{r}_\mathrm{p}^{\,\mathrm{CM}}|} = \frac{m_\mathrm{p}}{m_*} = \frac{a_*}{a_\mathrm{p}},$$

where the expressions for the semimajor axes a_1 and a_2 from lecture 1–2 were used. Using equation (3), we also have that

$$\frac{a_*}{a_\mathrm{p}} = \frac{v_*}{v_\mathrm{p}} = \frac{v_{*r}/\sin i}{v_{\mathrm{p}r}/\sin i} = \frac{v_{*r}}{v_{\mathrm{p}r}},$$

where equation (1) was used. Note: Here, the radial velocities v_{*r} and $v_{\mathrm{p}r}$ refer to the velocity at the point B in figure 3, the point for which the radial velocity is maximal. We may use these two equation to eliminate the unknown velocity of the planet

$$v_{\mathrm{p}r} = v_{*r}\frac{m_*}{m_\mathrm{p}}. \tag{4}$$

We will now return to Kepler's third law,

$$m_* + m_\mathrm{p} = \frac{4\pi^2 a^3}{P^2 G},$$

where we have used the exact expression for Kepler's third law, derived in problem 2 in part 1B. From section 6 in those notes, we also had that

$$a = a_* + a_\mathrm{p},$$

the semimajor axis a (of the orbit of the planet seen from the star or vice versa) equals the sum of the semimajor axes of the orbits of the planet and star about the center of mass. We can now express these in terms of velocities (equation 3)

$$a = \frac{P}{2\pi}(v_* + v_\mathrm{p}).$$

Inserting this into Kepler's third law, we have

$$m_* + m_\mathrm{p} = \frac{P}{2\pi G}(v_* + v_\mathrm{p})^3.$$

Normally we are only able to measure radial velocities, not the absolute velocity. We thus use equation (1) as well as equation (4) to obtain

$$\begin{aligned} m_* + m_\mathrm{p} &= \frac{P}{2\pi G}\frac{(v_{*r} + v_{\mathrm{p}r})^3}{\sin^3 i} \\ &= \frac{P v_{*r}^3}{2\pi G \sin^3 i}\left(1 + \frac{m_*}{m_\mathrm{p}}\right)^3. \end{aligned}$$

Assuming that the star is much more massive than the planet (which is normally the case, for instance $m_\mathrm{Jupiter}/m_\mathrm{Sun} \sim 10^{-3}$) we get

$$m_* = \frac{P v_{*r}^3}{2\pi G \sin^3 i}\frac{m_*^3}{m_\mathrm{p}^3},$$

which solved for the mass of the planet (which is the quantity we are looking for) gives

$$m_\mathrm{p} \sin i = \frac{m_*^{2/3} v_{*r} P^{1/3}}{(2\pi G)^{1/3}}. \qquad (5)$$

Normally, the mass of the star is known from spectroscopic measurements. The radial velocity of the star and the orbital period can both be inferred from measurements of the Doppler effect. Thus, the expression $m_\mathrm{p} \sin i$ can be calculated. Unfortunately, we normally do not know the inclination angle i. Therefore, this approach for measuring the planet's mass can only put a lower limit on the mass. By setting $i = 90°$ we find $m_\mathrm{p}^\mathrm{min}$. If the inclination angle is smaller, then the mass is always greater than this lower limit by a factor of $1/\sin i$. In the next section however, we will discuss a case in which we can actually know the inclination angle.

Before proceeding, we will first ask ourselves a question: Look at the velocity curve in figure 3. It looks like a sine or a cosine. It actually *is* a sine/cosine. We can write the curve as

$$v(t) = v_{*r} \cos \frac{2\pi}{P} t.$$

if we define the time $t = 0$ at point B (this is just to make it easier to see, if we define it at the point A it would be a sine instead). Comparing the expression with the figure, can you see that it is correct? Check for $t = -P/4$ (point A), $t = 0$ (point B), $t = P/4$ (point C) and $t = P/2$ (point D). But why is this curve a cosine?

Did you find the reason? This is pure geometry: Looking again at the upper part of figure 3, we see that the line of sight velocity component v_r is just $v \cos \theta$ where θ is the angle between the velocity vector and the line of sight. But since we only look at circular orbits, the angular velocity $\dot{\theta}$ is constant (why?). Where therefore have

$$\frac{d\theta}{dt} = \text{constant}$$

Giving

$$\theta = X_1 + X_2 t,$$

where X_1 and X_2 are constants. We see from the figure that X_2 must be $2\pi/P$ and the constant X_1 just shifts the curve. If we write the full expression as

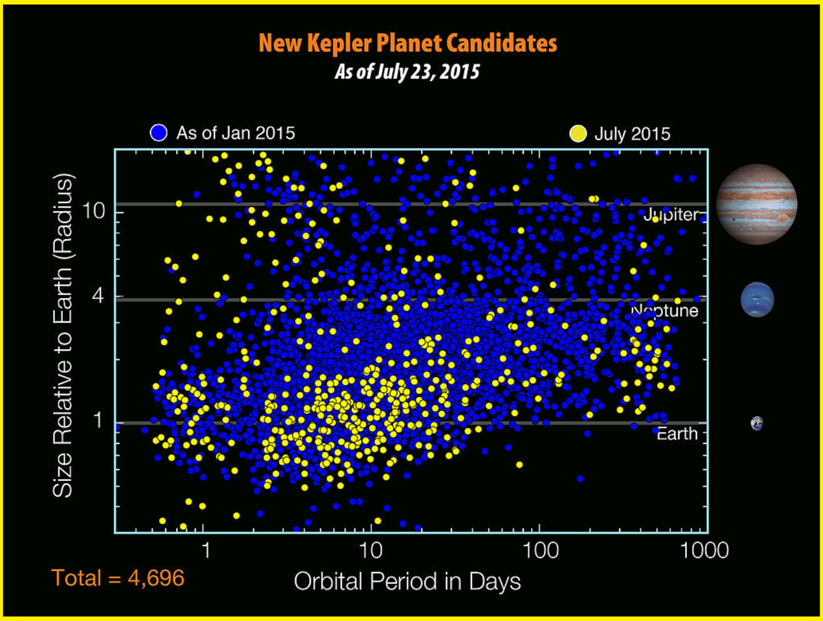
$$v(t) = v_{*r} \cos \frac{2\pi}{P}(t - t_0), \qquad (6)$$

(thus setting $X_2 = 2\pi t_0/P$) we see that t_0 can be interpreted as the time when the cosine is at maximum (when $v(t) = v_{*r}$). Now make sure you understand the relation between this equation and figure 3 and why it has this form. If the orbit had been an ellipse, would this expression still hold? Why/why not?

3 Measuring the radius and the density of extrasolar planets

If the inclination is close to $i \sim 90°$, the planet passes in front of the stellar disc and an eclipse occurs: The disc of the planet obscures a part of the the light from the star. When looking at the light curve of the star, a dip will occur with regular intervals corresponding to the orbital period. In figure 4 we show a typical light curve. When the disc of the planet enters the disc of the star, the light curve starts falling. When the entire disc of the planet is inside the disc of the star, the light received from the star is now constant but lower than before the eclipse. When the disc of the planet starts to leave the disc of the star,

Fact sheet: Kepler is a space observatory launched by NASA to discover Earth-like planets orbiting other stars. The spacecraft, named after the German Renaissance astronomer Johannes Kepler, was launched on March 7, 2009. Designed to survey a portion of our region of the Milky Way to discover dozens of Earth-size extrasolar planets in or near the habitable zone and estimate how many of the billions of stars in our galaxy have such planets, Kepler's sole instrument is a photometer that continually monitors the brightness of over 145,000 main sequence stars in a fixed field of view. This data is transmitted to Earth, then analyzed to detect periodic dimming caused by extrasolar planets that cross in front of their host star. The Kepler satellite has significantly increased the number of detected exoplanets. (from Wikipedia)



the light curve starts rising again. When such a light curve is observed for a star where a planet has been detected with the radial velocity method described above, we know that the inclination of the orbit is close to $i = 90°$ and the mass estimate above is now a reliable estimate of the planet's mass rather than a lower limit.

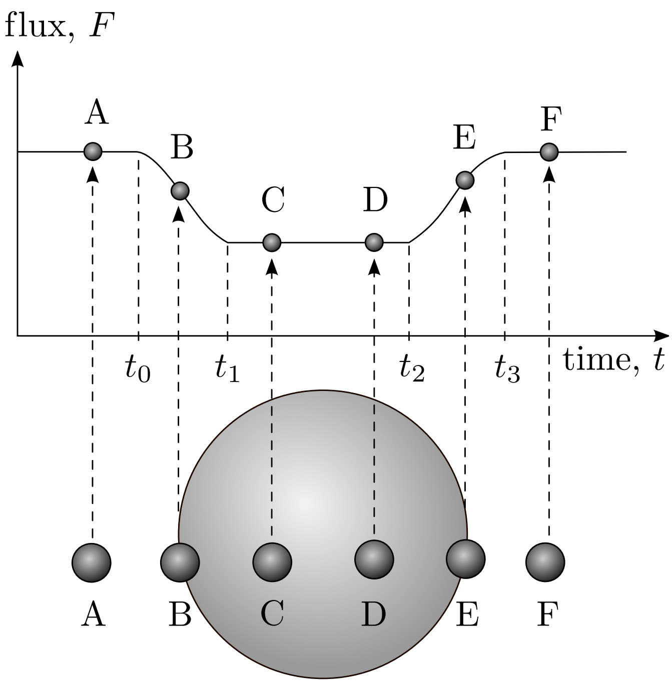


Figure 4: The lower part of the figure shows a planet eclipsing a star. The upper part shows a plot of the flux variation with time at the different points during the eclipse. The moments at which the eclipse starts $t = t_0$ and ends $t = t_3$ as well as the moments when the full disc of the planet enters $t = t_1$ and leaves $t = t_2$ the star are indicated.

In these cases, where the effect of the eclipse can be seen, the radius of the planet may also be measured. If we know the time of first contact (time t_0 in figure 4), the time when the disc of the planet has fully entered the disc of the star (time t_1) as well as the velocity of the planet with respect to the star, we can measure the radius of the planet. If the radius of the planet is R_p, then it took the disc of the planet with diameter $2R_\mathrm{p}$ a time $t_1 - t_0$ to fully enter the disc of the star. The planet moves with a velocity $v_* + v_\mathrm{p}$ with respect to the star (the velocity v_p is only the velocity with respect to the center of mass). Using simply that distance equals velocity times interval, we have

$$2R_\mathrm{p} = (v_* + v_\mathrm{p})(t_1 - t_0)$$

As we have seen, we can obtain t_1 and t_0 from the light curve. We can also obtain the velocity of the planet (the velocity of the star is measured directly by the Doppler effect) by using equation (4),

$$v_\mathrm{p} = v_* \frac{m_*}{m_\mathrm{p}}. \tag{7}$$

Here the mass of the planet m_p has been calculated since we know that the inclination is $i \sim 90°$. Thus, the radius R_p of the planet is easily obtained. Combining the measured mass and radius of the planet we get an estimate of the mean density

$$\rho_\mathrm{p} = \frac{m_\mathrm{p}}{4/3\pi R_\mathrm{p}^3}.$$

We can use this to determine whether the detected planet is *terrestrial planet* with a solid surface like the inner planets in the solar system, or a *gas planet* consisting mainly of gas and liquids like the outer planets in our solar system. The terrestrial planets in our solar system have densities of order $4 - 5$ times the density of water whereas the gas planets have densities of order $0.7 - 1.7$ times the density of water. If the detected planet is a terrestrial planet, it could also have life.

Finally, note that also the radius R_* of the observed star can be obtain by the same method using the time it takes for the planet to cross the disc of the star,

$$2R_* = (v_* + v_\mathrm{p})(t_2 - t_0). \tag{8}$$

We have discussed two ways of discovering extrasolar planets,

- by measuring radial velocity
- by measuring the light curve

In the following problems you will also encounter a third way,

- by measuring tangential velocity

For very close stars, the tangential movement of the star due to its motion in the orbit about the center of mass may be seen directly on the sky. The velocity we measure in this manner is the projection of the total velocity onto the plane perpendicular to the line of sight. There are two more methods which will briefly be discussed in later lectures,

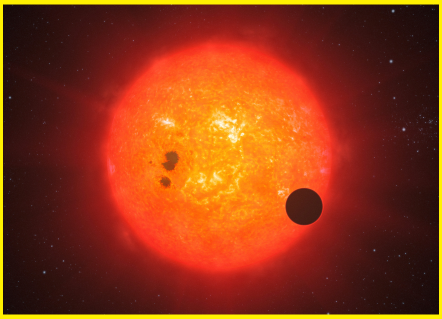
- by gravitational lensing

- by pulsar timing

4 Example exercise: The atmosphere of extrasolar planets

In figure 5 we show observations of the radial velocity of a star over a large period of time. We assume that these data is a collection of data from several telescopes around the world. Real data contain several additional complicated systematic effects which are not included in this figure. For instance, changes in the velocity of the Earth need to be corrected for in velocity measurements. Here we assume that these corrections have already been made. Even if this plot does not show you all the complications of real life, it does give an impression of how data from observations may look like and how to use them to say something about extrasolar planets. You see that this is not a smooth curve, several systematic effects as for instance atmospheric instabilities give rise to what we call 'noise'.

Fact sheet: This artist's impression shows how the super-Earth surrounding the star GJ1214 may look. The planet was discovered by the transit method: the brightness of its host star decreased by a tiny amount as the (unseen) planet crossed in front of it. Spectroscopic follow-up observations, i.e. radial velocity measurements, were needed in order to confirm the planetary nature of the object and to obtain its mass. The planet is the second super-Earth (defined as a planet between one and ten times the mass of the Earth) for which astronomers have determined the mass and radius, giving vital clues about its structure. It is also the first super-Earth around which an atmosphere has been found. The planet is too hot to support life as we know it.(ESO/L. Calcada)



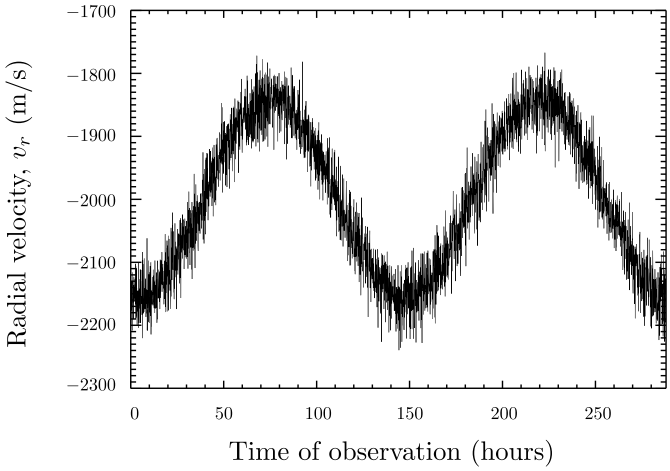


Figure 5: Velocity measurements of a star.

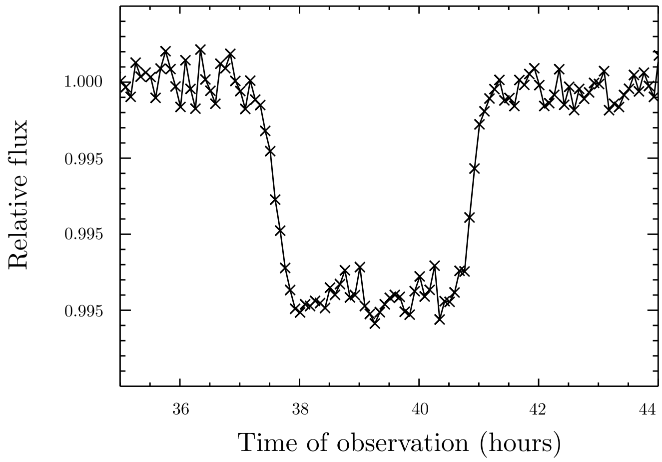


Figure 7: The light curve of a star at 1450 nm. There are 5 minutes between each cross.

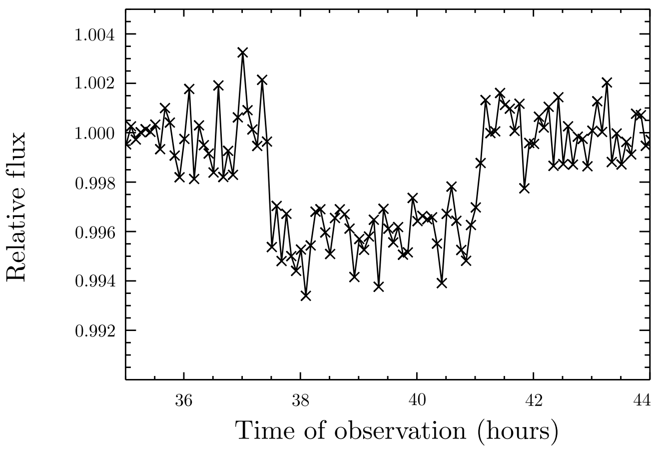


Figure 6: The light curve of a star at 600 nm. There are 5 minutes between each cross.

The answers to the following questions are given below. Try to find them yourself before looking.

1. Does this star move towards us or away from us? Use the figure to give an estimate of the peculiar velocity.
2. Use the curve to find the maximum radial velocity v_{r*} of the star (with respect to the center of mass) and the orbital period of the planet.
3. Spectroscopic measurements have shown the mass of the star to be 1.1 solar masses. Give an estimate of the lower bound for the mass of the planet. The result should be given in Jupiter masses.
4. In figure 6 we show a part of the light curve (taken at the wavelength 600 nm) of the star for the same period of time. Explain how this curve helps you to obtain the real mass of the planet, not only the lower bound, and give an estimate of this mass.
5. Use the light curve to find the radius of the

planet. Note: there are 5 minutes between each cross in the plot.

6. In figure 7 we show a part of the light curve taken at the same time as the previous light curve but at a wavelength 1450 nm which is an absorption line of water vapor. Use the figure to determine if this planet may have an atmosphere containing water vapor and estimate the thickness of the atmosphere.

Here are the answers. Please try to do the exercise before looking.

1. Towards us with roughly 2000m/s. From equation 2 we see that the observed velocity is a sum of the peculiar velocity and the velocity with respect to the center of mass. We know from figure 3 that v_* (its component along the line of sight) will fluctuate between positive and negative values, thus the component of $\vec{v}$ along the line of sight will fluctuate above and below the peculiar velocity. We therefore need to look at the velocity in the middle, just between the positive and negative fluctuations of v_* to find the peculiar velocity. For the sign of the velocity, see the discussion just above equation 2.

2. $v_{r*} = 150$m/s. The top of the curve (maximum velocity along the line of sight) is found at $v_{r*} = 2150$m/s, but in order to find the velocity with respect to the center of mass, the peculiar velocity has to be subtracted (see equation 2). As discussed above, the noise makes the curve 'thick' due to the rapid fluctuations. As the noise fluctuates both upwards and downwards, **the real signal therefore has to be in the middle of the thick line.**. The orbital periode can be found by looking at the time it takes for the star to return to the same place in the orbit, which corresponds to it having the same velocity. We can for instance choose the distance between two peaks, one is roughly at $t = 75$h and the other at $t = 225$h giving $P = 150$h.

3. Using equation 5 inserting the period and maximum radial velocity, we found $m_{\mathrm{min}} = 1.5 m_{\mathrm{jup}}$. We do not know the inclination of the orbit, but choosing $i = 90°$ gives the largest possible value of the sine function and therefore the smallest possible mass on the planet.

4. The fact that the light curve diminishes at the same time as the planet is at the point in the orbit when it is closest to the observer (see if you can see this from the velocity curve) suggests that it is passing in front of the star. The inclination therefore has to be $i = 90°$. This is exactly the number we used in the previous question, so the number we obtained there is now confirmed to be the actual mass, not only the minimum mass.

5. We see that at time $t = 37.5$h, the curve starts dropping. It seems to take roughly 2 crosses before the full disk of the planet is in front of the star (the ecplise is at maximum). Looking at the other end, when the eclipse is finishing, it still seems to be about 2 crosses. We can therefore conclude that it takes about 10 minutes from the disk of the planet enters the disk of the star, until the full disk of the planet is inside the disk of the star. In order to use equation 8, we see that we also need the velocity of the planet. Using equation 7 we get $v_{\mathrm{p}} \approx 120$km/s. Using equation 8, we therefore conclude that the radius of the planet is roughly 36000km.

6. Repeating the last calculation using the information from this spectral line, we get a radius of about 126000km. A larger radius measured with the absorption line for water vapour means that the extension of the area of the planet absorbing the water vapour line is larger than for other spectral lines. This can only occur if the planet has an outer shell which only absorbs certain frequencies. This would happen if the planet has a large atmosphere with thickness 90000km containing water vapour.

5 Statistical model fitting: how to extract information from noisy data

In the previous section you saw an example of noisy data: the velocity measurement, like any measurement, have uncertainties caused by various effects, from disturbances in the atmosphere to design of the telescope and noise in the detectors. We used 'by-eye' measurements to account for the noise in the curves. In this section we will see how we can use statistics to properly take into account the noise in the data when estimating parameters like the mass of an extrasolar planet.

We can write the velocity measurements as

$$v(t) = v_{\text{real}}(t) + \delta v(t),$$

where $v_{\text{real}}(t)$ is the actual velocity of the star and δv is the random noise. The random noise, like any random quantity, has a probability distribution. Thanks to the central limit theorem (part 1A), the probability distribution of the noise is often Gaussian. We will furthermore assume that the random noise is independent from one time step to the other (this is not always the case in real life). This means that if there is a huge positive fluctuation at one time step, the fluctuation in the next time step could be anything from a huge negative fluctuation to another huge positive fluctuation. We will also assume, as is often the case, that the noise fluctuates about zero, i.e. that the mean value $\langle \delta v \rangle = 0$. Then we only need the standard deviation $\sigma(t)$ to be able to model the uncertainty caused by the noise. Often the standard deviation σ will be independent of the time step t if the measurement set up and conditions are unchanged during a specific time period. To start with, we will still consider the possibility of a time dependence of σ.

Assume we have a set of N velocity measurements v_i at a set of different times t_i where $i = [1, N]$. Assume that the noise for measurement i at time t_i has a standard deviation σ_i. Again, we assume the noise to be independent between observations and a zero mean value for the noise. Then we can define v_i as a random variable,

$$v_i = v_{\text{real}}(t_i) + \delta v$$

Since $v_{\text{real}}(t_i)$ here is a constant (it is not random, it is the real velocity of the planet), the mean value of v_i is

$$\langle v_i \rangle = v_{\text{real}}(t_i) + \underbrace{\langle \delta v \rangle}_{=0} = v_{\text{real}}(t_i)$$

We can therefore write the Gaussian probability for observation i to have the value v_i as

$$P(v_i) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{(v_i - \langle v_i \rangle)^2}{2\sigma_i^2}} = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{(v_i - v_{\text{real}}(t_i))^2}{2\sigma_i^2}}$$

exactly as in part 1A. We also remember from part 1A (the law of multiplication of probabilities) that the probability for v_0 to have a certain value, v_1 to have another value, etc. can be written as

$$P(v_0, v_1, v_2, ..., v_N) = P(v_0)P(v_1)P(v_2)...P(v_N)$$
$$= \frac{1}{\Pi_{i=1}^N \sigma_i (2\pi)^{N/2}} e^{-\frac{1}{2}\sum_{i=1}^N \frac{(v_i - v_{\text{real}}(t_i))^2}{\sigma_i^2}}$$

Check that you manage to arrive at this equation by yourself. This equation gives the probability to observe the full velocity curve (like the one in figure 5) including the noise fluctuations. Note while v_i here are the known observed velocities, the real velocities $v_{\text{real}}(t_i)$ are unknown and the ones we would like to find. The standard deviation σ_i of the observations can often be measured or modelled.

Our task is therefore to find the real star velocities $v_{\text{real}}(t_i)$ which gives the best fit to the noisy observed values v_i. Or with other words, we want to find the values for $v_{\text{real}}(t_i)$ which are **most probably** the true ones, given our observed noisy values v_i. This is now reduced to a purely mathematical problem: we need to find a set of values $v_{\text{real}}(t_i)$ which **maximize the total probability** function $P(v_0, v_1, v_2, ..., v_N)$. (Read through the last sentences a few times to make sure you got it.) In statistics this approach is called the method of *maximum likelihood.*

We can simplify our maximum likelihood problem significantly using equation 6. We know that the velocity curve should follow this cosine form. We call this equation our *model.* Our model has 3 free unknown parameters, v_{*r}, P and t_0 which we need to find. Our problem has therefore now simplified to the following: We need to find the combination of 3 values, v_{*r}, P and t_0 which maximize our probability (make the observed curve the most probable one)

$$P(v_0, v_1, v_2, ..., v_N) = P(v_0)P(v_1)P(v_2)...P(v_N)$$
$$= \frac{1}{\Pi_{i=1}^N \sigma_i (2\pi)^{N/2}} e^{-\frac{1}{2}\sum_{i=1}^N \frac{(v_i - v_{*\mathrm{r}} \cos\frac{2\pi}{P}(t_i - t_0))^2}{\sigma_i^2}} \quad (9)$$

Can we simplify the problem further? Look at the expression and think about it as a purely mathematical problem. Can it be written in a simpler way?

Did you think about it? Ok, here we go. Do you agree that maximizing equation 9 with respect to the parameters v_{*r}, P and t_0 equals minimizing the function

$$\chi^2(v_{*r}, P, t_0) = \sum_{i=1}^N \frac{(v_i - v_{*\mathrm{r}} \cos\frac{2\pi}{P}(t_i - t_0))^2}{\sigma_i^2}$$

with respect to the same 3 parameters? If not, look at it again and again until you get it.

We now have a simpler problem at hand: we need to find a set of values v_{*r}, P and t_0 which minimize χ^2. We now make it even simpler. As mentioned above, the standard deviation as a function of time can often be approximated as constant, such that $\sigma_i = \sigma$, the same at every time step. With this assumption, can you see an even easier way to formulate the mathematical problem? And if so, can you recognize this even simpler expression as a method which you have been working with in previous courses?

Did you find a simpler expression? Do not read on before you have. Since σ is now a constant, it can be taken out of the sum and is not important in the minimization process. We can therefore simply minimize

$$\Delta(v_{*r}, P, t_0) = \sum_{i=1}^N (v_i - v_{*\mathrm{r}} \cos\frac{2\pi}{P}(t_i - t_0))^2$$

with respect to v_{*r}, P and t_0. We need to find a set of values v_{*r}, P and t_0 which minimize Δ. Do you recoginize this method? Yes, that is correct, it is the method of *least squares* which we have deduced using Gaussian statistics. Look at the last expression and compare it with figure 5. The v_i is the noisy curve while the real velocity $v_{\mathrm{real}}(t_i)$ is a smooth cosine curve passing through the middle of the noisy one. The least square function is a sum of the squares of the difference between the observed noisy points and the smooth model curve. The least squares method therefore tells us to find the set of paramters v_{*r}, P and t_0 which minimize the difference between the (noisy) observed and smooth model velocity curve. It is a method to find the model (or the parameters of the model) which fits the noisy data points as close as possible, given our knowledge about the noise.

The slightly more advanced form of the least squares method is the χ^2 minimization approach in the equation above. When the noise properties vary with time, this is a more exact approach as it takes into account these varying noise properties: when σ_i is big, meaning that the noise has large fluctuations (remember the shape of the Gauss function), the data point is given a low weight (it is divded by σ_i^2) whereas when the σ_i is low, meaning that the noise has small fluctuaions (and is therefore close to 0), the data point is given a high weight as it has more information.

In this course we will mainly use the least squares approach (except an exercise in part 1D), but now you see where it comes from and how it can be made more exact.

6 Exercises

Exercise 1C.1 In order to be able to solve this exercise, you need to read section 1 as well as part 1B first.

1. The precision in measurements of radial velocities by the Doppler effect is currently 1 m/s. Can a Jupiter like planet orbiting a star similar to the Sun at a distance from the mother star equal to the Sun-Jupiter distance be detected? (Use `www` or other sources to find the mass of Jupiter, the Sun and the distance between the two which are the only data you are allowed to use).

2. What about an Earth like planet in orbit at a distance 1 AU from the same star?

3. Using the radial velocity method, is it easier to detect planets orbiting closer or further away from the star?

4. In what distance range (from the mother star) does an Earth like planet need to be in order to be detected with the radial velocity method? (Again use a star similar to the Sun). Compare with the distance Sun-Mercury, the planet in our solar system which is closest to the Sun.

Exercise 1C.2 In order to be able to solve this exercise, you need to read sections 1 and 2. For stars which are sufficiently close to us, their motion in the orbit about a common center of mass with a planet may be detected by observing the motion of the star directly on the sky. A star will typically move with a constant velocity in some given direction with respect to the Sun. If the star has a planet it will also be wobbling up and down (see figure 8). We will now study the necessary conditions which might enable the observation of this effect.

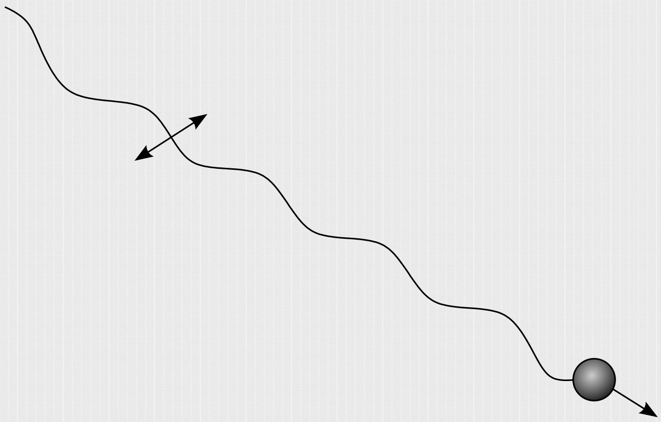


Figure 8: The transversal wobbling of a nearby star due to its orbital motion about the common center of mass with a planet. The angular extension of the orbit is indicated by two small arrows.

1. The Hubble Space Telescope (HST) has a resolution of about $0.1''$. Assuming now that this angle is really the limit to how close two objects can be in order for HST to distinguish the objects. How close to us does a star similar to the Sun with a Jupiter like planet (at the distance from the mother start equal to the Sun-Jupiter distance) need to be in order for the HST to observe the tangential wobbling of the star?

2. What about an Earth like planet at the distance of one AU from the same star?

3. The closest star to the Sun is Proxima Centauri at a distance of 4.22 l.y.. How massive does a planet orbiting Proxima Centauri at the distance of 1 AU need to be in order for the tangential wobbling of the star to be observed?

4. What about a planet at the distance from Proxima Centauri equal to the Sun-Jupiter distance?

5. If we can measure the tangential velocity (perpendicular to the line of sight) component of a star, we can get an estimate of the mass of the planet not only a lower limit. Show that the exact mass of the planet can be expressed as (for any inclination i)

$$m_\mathrm{p} = \left(\frac{m_*^2 P}{2\pi G}\right)^{1/3} v_{t*}$$

(tangential velocity v_{t*} here is measured when the radial velocity is zero).

Exercise 1C.3 In order to be able to solve this exercise, you need to read sections 1 to 4. In figure 9 we show observations of the radial velocity of a star over a large period of time. We assume that these data is a collection of data from several telescopes around the world. Real data contain several additional complicated systematic effects which are not included in this figure. For instance, changes in the velocity of the Earth need to be corrected for in velocity measurements. Here we assume that these corrections have already been made. Even if this plot does not show you all the complications of real life, it does give an impression of how data from observations may look like and how to use them to say something about extrasolar planets. You see that this is not a smooth curve, several systematic effects as for instance atmospheric instabilities give rise to what we call 'noise'.

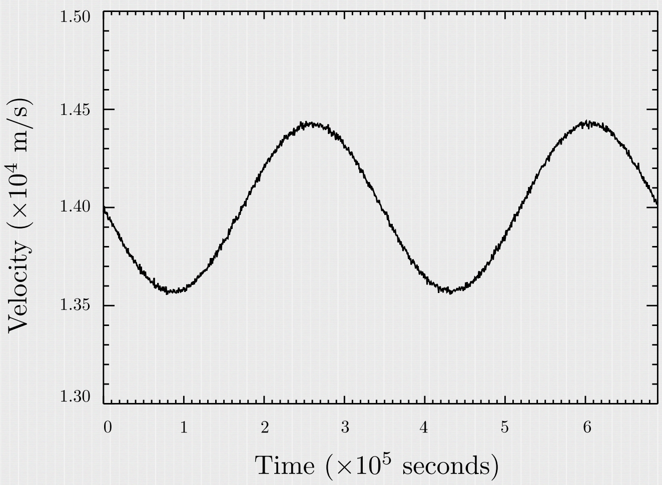


Figure 9: Velocity measurements of a star.

1. This plot shows a curve with a wave like shape, can you explain the shape of the curve?
2. Use this plot to give an estimate for the the 'peculiar velocity' of the star. 'Peculiar velocity' is a term used to describe the average motion of the star with respect to us, not taking into account oscillations from planets.
3. Use the curve to find the maximum radial velocity v_{r*} of the star (with respect to the center of mass) and the orbital period of the planet.
4. Spectroscopic measurements have shown the mass of the star to be 1.3 solar masses. Give an estimate of the lower bound for the mass of the planet. The result should be given in Jupiter masses.
5. In figure 10 we show the light curve of the star for the same period of time. Explain how this curve helps you to obtain the real mass of the planet, not only the lower bound, and give an estimate of this mass.
6. In figure 11 we have zoomed in on a part of the light curve. Use the figure to give a rough estimate of the density of the planet.
7. Is this a gas planet or a terrestrial planet?

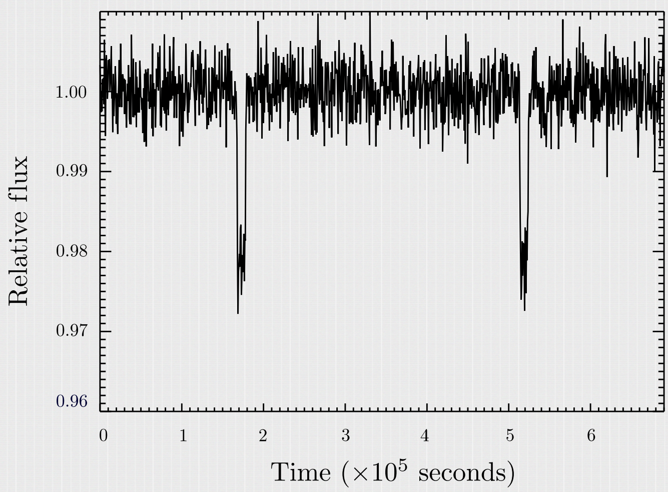


Figure 10: The light curve of a star.

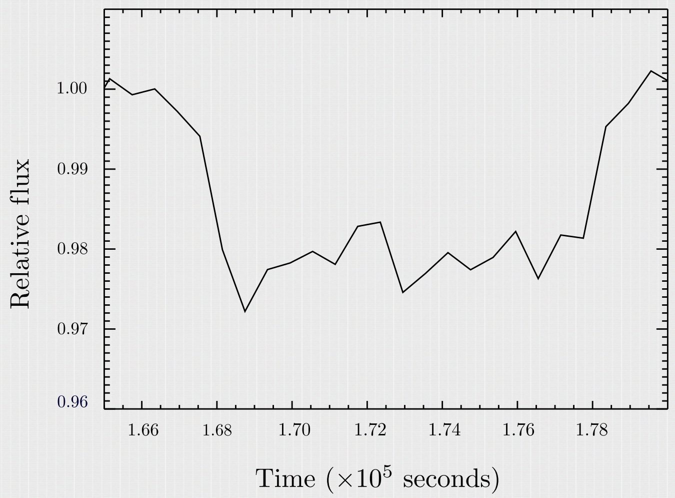


Figure 11: The light curve of a star.

Exercise 1C.4 In order to solve this exercise, you need to read **all** of part 1C. At the following link you will find directories containing some files with velocity and light curves:

`http://folk.uio.no/frodekh/AST1100/part1C/`

In order to find your directory, you should look at the two last digits of your seed, i.e. if your seed is 42325 then you should enter the directory called `seed25`. Inside your directory you will find 5 files, each with velocity and light curves of a star. The name of the file indicates the mass of the star, i.e. `star2_1.34.txt` is the second star for that seed, the mass of the star is 1.34 solar masses.

Real data contain several additional complicated systematic effects which are not included in these files. For instance, changes in the velocity of the Earth need to be corrected for in velocity measurements. Here we assume that these corrections have already been made. Even if these data do not show you all the complications of real life, they will still give an impression of how data from observations may look like and how to use them to say something about extrasolar planets. Each file contains three rows, the first row is the time of observation, counted in days from the first observation which we define to be $t = 0$. We assume that these data is a collection of data from several telescopes around the world, studying these stars intensively for a given period of time. The second row gives the observed wavelength λ of a spectral line (The $H\alpha$ line) at $\lambda_0 = 656.3$ nm in nm $= 10^{-9}$ m. You need to use the Doppler formula to obtain radial velocities yourself. You will see that this is not a smooth curve, several systematic effects, i.e. atmospheric instabilities give rise to what we call 'noise'. As you will see, this noise makes exact observations difficult. The third row shows the measured flux of light relative to the maximum flux for the given star. Again, also these data contain noise.

Use Python, Matlab or other software/programming languages to solve the following problems:

1. Estimate the peculiar velocity for each of the 5 stars, taking the mean of the velocity over all observations. Plot the velocity curves (subtract the mean velocity from the velocity for each observation) and light curves for the 5 stars. Which of the stars appear to have a planet orbiting? Which of these planets are eclipsing their mother star? Explain how you reach your conclusions.

2. Can you, by looking at the velocity curves (velocity as a function of time), find the lower limit for the mass of the planet for the stars where you detected a planet. Find the numbers for the periods and max radial velocities by eye. Explain/draw how you extract these numbers.

3. If you, by looking at the light curve, discovered that some of the planets are actually eclipsing the star, you may also estimate the radius and density of these planets. Again, you will need to estimate the time of eclipse by eye. For some light curves you do not have sufficient information to estimate the radius of the planet. If this is the case with your stars, explain how you can see that you cannot estimate planet radius from your data.

4. You have made estimates of mass and radius using 'by-eye' measurements. This is not the way that astrophysicists are working. Often, advanced signal processing methods are employed in order to get the best possible estimates. Also, scientific measurements always have uncertainties. The detailed methods for analyzing these data are outside the scope of this course, but you will encounter this in

more advanced courses in astrophysics. Here we will use the least squares approach to obtain estimates which are more exact than the 'by-eye' observations above. We employ the model from equation 6,

$$v_r^{\text{model}}(t) = v_r \cos(\frac{2\pi}{P}(t - t_0)), \qquad (10)$$

where $v_r^{\text{model}}(t)$ is the theoretical model of the radial velocity as a function of time. The unknown parameters in this model are v_r, P and t_0. Only the two first parameters, v_r and P, are necessary in order to estimate the mass of the planet, but we need to estimate all three in order to have consistent estimates of the first two. We will now try to find a combination of these three parameters, such that equation (10) gives a good description of the data. To do this, you need to write a computer code which calculates the least squares function of the data and your model for a large number of values for the three parameters t_0, P and v_r. You need to define a function (an array in you computer) $\Delta(t_0, P, v_r)$ given as

$$\Delta(t_0, P, v_r) = \sum_t (v_r^{\text{data}}(t) - v_r^{\text{model}}(t, t_0, P, v_r))^2$$

where you sum over all the observed data points. This function gives you the difference between the data and your model for different values of t_0, P and v_r. What you want to find is the function which best fits your data, that is, the model which gives the minimum difference between the data and your model. You simply want to find for which parameters t_0, P and v_r that the function $\Delta(t_0, P, v_r)$ is minimal. How do you find the parameters t_0, P and v_r which minimize Δ? In this case it is quite easy, try to follow these steps:

(a) Choose one of your stars which clearly has a planet orbiting.

(b) Look at your data: You know that for $t = t_0$, the velocity is maximal. Look for the first peak in the curve and define a range in time around this curve for which you think that the exact peak must be. Define a minimum possible t_0 and a maximum possible t_0 (being sure that exact peak is somewhere between these two values). Then define a set of, say 20 (you choose what is more convenient in each case) values of t_0 which are equally spaced between the minimum and maximum value.

(c) Do the same for v_r, try to find a minimum and a maximum v_r which are such that you see by eye that the real exact v_r is between these two values. Then divide this range into about 20 equally spaced values (maybe less depending on the case).

(d) Do the same thing for the period. Look at the time difference between two peaks, and find a set of possible periods.

(e) Now, calculate the function Δ for all these values of t_0, P and v_r which you have found to be possible values. Find which of these about 20^3 combination of values which gave the smallest Δ, thus the smallest difference between data and model. These values are now your best estimates of P and v_r.

(f) Calculate the mass of the planet again with these values for P and v_r and compare with your previous 'by-eye' estimates. How well did you do in estimating 'by-eye'?

(g) Now repeat the procedure to estimate the exact mass for one other stars with planets and compare again with your 'by-eye' estimates. (**only for those who want a challenge:** can you write your code in an automatic way, such that you do not need to find any initial values for your code bye eye?

Exercise 1C.5 In order to solve this exercise, you need to read **all** of part 1C. This exercise can only be done if you have already done exercise 1B.1 in part 1B. In that exercise you calculated the orbits of the planets in your solar system with respect to the star fixed at the origin. In this exercise your first task is to redo the calculation of the planetary orbits taking into account also the forces of the planets on the star, **but:** choose the 3-4 most massive planets in your system and ignore the other planets. You can still ignore the forces between the planets. Assume that the position and velocity vector of the star at time $t = 0$ is $(0, 0)$. You will then obtain the orbits of the planets and the star around the center of mass. Note that, as the initial position of the star is at the origin, the center of mass will be slightly displaced from the origin. Also note that since you assume a zero initial velocity of the star, you have transformed yourself into a frame where you have a non-zero movement with respect to the center of mass and you will therefore experince that your solar system has a peculiar velocity with respect to the origin and is therefore slowly drifting. When you have obtained the new orbits of (some of) the planets and the star, your task is to:

1. calculate the radial velocity curve of your star as seen from an extraterrestrial far away. You can choose peculiar velocity, line of sight and inclination angle yourself (but the inclination cannot be 0 such that there is no motion). Plot the velocity curve for a time which covers at least one full orbit of one of the outer planets.
2. simulate real data by adding noise to your velocity curve. Assume that the noise is Gaussian with zero mean and independent from observation to observation. The standard deviation of the noise is 1/5 of your maximum value for v_{*r}.
3. **optional:** Now assume an inclination of 90° and plot the light curve of your solar system. Assume the flux to have the value 1 when no planet eclipses. Remember that the flux is proportional to the area which emits light.
4. **optional:** add Gaussian noise also to your light curve. Use a standard deviation of 0.2. Would any of your planets be visible through eclipses?