AST1100 Lecture Notes

Part 1D
Electromagnetic radiation

Questions to ponder before the lecture

1. Unlike physicists, astrophysicists cannot make direct experiments in the laboratory. We rely completely on information transmitted by signals from the universe. What kind of signals? The title of this part clearly gives a hint, but are there other kinds of signals?
2. In the picture below, what kind of telescopes do you see?
3. How do you think the pictures they take look like?
4. If you were to calculate the amount of energy per square meter which the Earth receives from the Sun, how would you start? Or, equivalently, if your satellite needs solar panels to operate at a large distance from the Sun, how large surface area does it need in order to have sufficient power?
5. If you were to determine the temperature of a star, which kind of observation/measurement would you do?

AST1100 Lecture Notes

Part 1D
Electromagnetic radiation

In this part we will look at the sources of information we have about the distant universe. In particular we will study electromagnetic radiation at all wavelengths, which is by far our most important source of information. For our satellite, we need this to calculate the amount of energy which its solar panels will recieve when far away from the star. We also need how to know how to analyse the spectrum of electromagnetic radiation and in particular how to interpret spectral lines to gain information about the atmosphere of our planet.

1 The electromagnetic spectrum

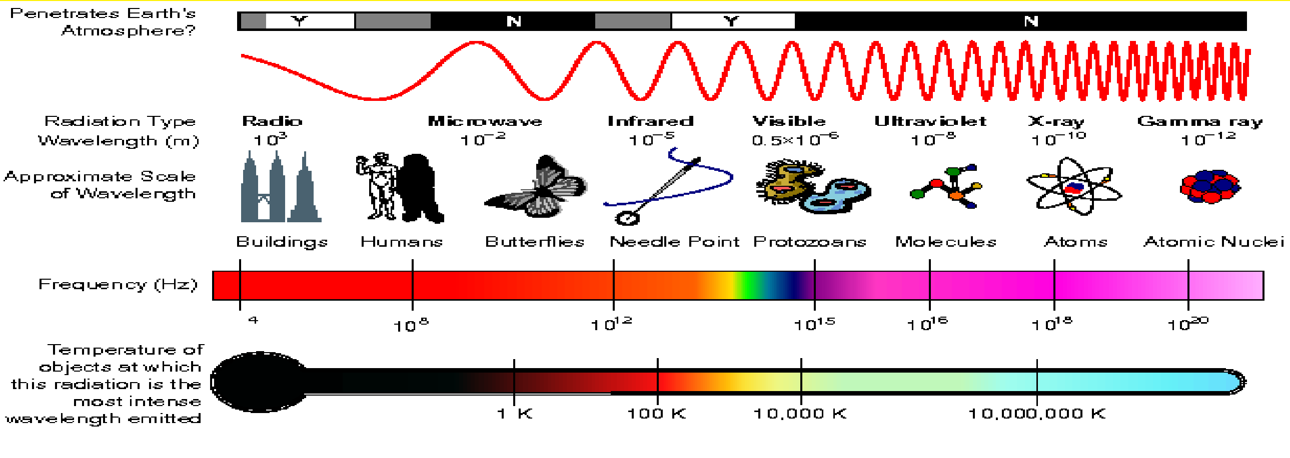
To obtain information about the distant universe we have the following
sources available:

1. **electromagnetic waves** at many different wavelengths.
2. **cosmic rays:** high energy elementary particles arriving from supernovae or black holes in our galaxy as well as from distant galaxies. The galactic magnetic field changes the direction of these particles making it impossible to determine the incoming direction and therefore the exact sources of the rays.
3. **neutrinos:** these extremely light elementary particles interact very rarely with other particles and can therefore arrive from huge distances without being scattered on the way. This property also makes neutrinos very difficult to detect and therefore a source of information with limited usefulness until better detection methods are discovered.
4. **gravitational waves:** spacetime distortions traveling through space as a wave. These are predicted by Einstein's general theory of relativity. Gravitational waves have still not been directly detected, but experiments are on their way.

Of these sources, electromagnetic waves is by far the most important. Practical problems limit the amount of information we can obtain from other sources with current technology. Since electromagnetic radiation is almost the only source which we use to get information about the distant universe, it is of high importance in astrophysics to know the processes which produce this kind of radiation. Here we will discuss some of the most important processes along with some discussion on how the radiation from these different processes is used to obtain information about the universe. Some important types of radiation are

- **thermal radiation:** the thermal motion of atoms produces electromagnetic radiation at all wavelengths. For a *black body* (see later), the radiation emitted at a given frequency is distributed according to Planck's law of radiation.
- **synchrotron radiation:** radiation produced by energetic charged particles accelerated in a magnetic field. This process emits electromagnetic radiation at different wavelengths depending on the energies involved in the process. Our own galaxy emits synchrotron radiation as radio waves due to the

Fact sheet: A diagram of the electromagnetic spectrum, showing various properties across the range of frequencies and wavelengths. The spectrum is a continuum, but is often divided into the following main regions of decreasing wavelength and increasing energy: radio, microwave, infrared, visible, ultraviolet, X-ray, and gamma-ray. Note that the Earth's atmosphere is transparent only to visible light, a part of the radio spectrum and a few narrow wavelength intervals in the infrared, thus limiting the types of celestial objects and astrophysical processes that can be studied using ground-based telescopes. (Figure:Wikipedia)



acceleration of cosmic ray electrons in the magnetic field of the galaxy.

- **Bremsstrahlung:** radiation produced by the 'braking' of a charged particle, usually an electron, by another charged particle, typically a proton or atomic nucleus. Due to electromagnetic forces from ions, electrons are deflected, and hence accelerated, producing electromagnetic radiation at all wavelengths. The space between galaxies in the clusters of galaxies is called the *intergalactic medium (IGM)*. It contains a very hot plasma of electrons and ions emitting brehmsstralung mainly as X-rays. These X-rays constitute an important source of information about distant clusters of galaxies.

- **21 cm radiation:** Neutral hydrogen emits radiation with wavelength 21 cm due to a so-called spin-flip: The quantum spin of the electron and proton may change direction such that the spin vectors go from having their orientation in the same direction to having their orientation in opposite directions. In this process, the total energy of the atom decreases and the energy difference between the two states is emitted as 21 cm radiation. This is a so-called forbidden transition, meaning that it occurs very rarely. For a single atom one would on average need to wait about 10 millions years for the process to occur. However, in huge clouds of gas the number of hydrogen atoms is so large that the intensity of 21 cm radiation can be quiet large even for such a rare process.

2 Solid angles

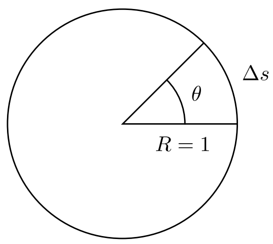


Figure 1: The angle measured in radians is defined as the length taken along the rim of the unit circle.

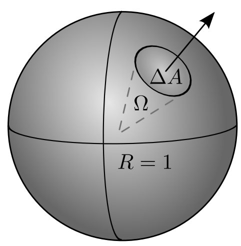


Figure 2: The solid angle measured in steradians is defined as the area taken on the surface of the unit sphere.

Before embarking on the properties of radiation, we will first introduce a new concept which will be widely used: *the solid angle*. The solid angle is a generalization of the concept of an *angle* from one to two dimensions. Looking at figure 1, we see that an angle measured in radians is simply a distance Δs taken along the rim of the unity circle

$$\theta = \Delta s.$$

To convince you about this, remember that the circumference of the unity circle, the full distance taken around the circle, is 2π. Now, the solid angle is measured in units of *steradians*, for short *sr*, and is a part of the *area* of the surface of the unit sphere as seen in figure 2. Thus,

$$\Omega = \Delta A.$$

The solid angle corresponding to the full unit sphere is then 4π sr which is the full area of the surface of the unit sphere. If we imagine a source of radiation in the center of the unit sphere, the solid angle can be used to describe the amount of radiation going in a certain direction as the energy transported per steradian. This is widely used in the study of radiative processes in stars.

3 Black body radiation

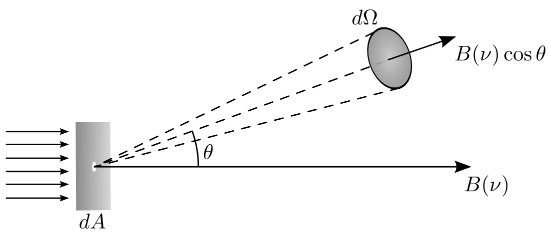


Figure 3: Intensity is the energy of radiation passing through area dA into a solid angle $d\Omega$ per time, per wavelength.

Thermal radiation is emitted from an object of temperature T because of the thermal motion of atoms at this temperature. Black body radiation is thermal radiation from a black body. A black body is defined as a body which absorbs all radiation it receives, no radiation is reflected or can pass through. Many objects in astrophysics are close to being a black body, a star is a typical example. For a black body, an expression for the intensity of the thermal radiation as a function of wavelength/frequency can be obtained analytically. A black body emits thermal radiation at all frequencies, but which frequency has the largest intensity depends on the temperature of the black body. To calculate the distribution of radiation per frequency quantum physics is needed. We will therefore not make the calculation here (you will come to this in physics courses later), but rather state the result:

Planck's law of radiation

$$B(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}.$$

where ν is the frequency, T is the temperature of the black body, h is Planck's constant and k is the Boltzmann constant.

The quantity $B(\nu)$ is *intensity* defined such that

$$\Delta E = B(\nu)\cos\theta\Delta\nu\Delta A\Delta\Omega\Delta t \qquad (1)$$

is the small energy passing through a small area ΔA into a small solid angle $\Delta\Omega$ (see figure 3) per small time interval Δt in the small frequency range $[\nu, \nu+\Delta\nu]$. Intensity is measured in units of $\mathrm{W/m^2/sr/Hz}$. Here the factor $\cos\theta$ comes from the fact that energy per solid angle per area is lower by a factor $\cos\theta$ for an observer making an angle θ with the normal to the area emitting radiation. Example: Imagine you have a light bulb which emits black body radiation at a certain temperature. You set up a wall between you and the light bulb and let light pass only through a small hole in the wall of area $\Delta A = 0.1$ mm^2. Just around the hole you construct a unit sphere and put a detector at an angle $\theta = 30°$ with a line orthogonal to the wall. The detector occupies about 1/1000 of the unit sphere and thus absorbes light from $\Delta\Omega = 4\pi/1000$ sr. Finally, the detector contains a material which only absorbes and measures radiation in the wavelenght range 600–600.1 nm , such that $\Delta\nu = 0.1$ nm. The energy that the detector measures from the light during a period of $10^{-3}s$ is then:

$$\Delta E = B(600\text{ nm}) \times \cos(30°) \times 0.1\text{ mm}^2$$
$$\times (4\pi/1000)\text{ sr} \times 0.1\text{ nm} \times 10^{-3}\text{ s}$$

In reality, the definition is made when we let all Δ be infinitesimally small, such that the definition reads

$$dE = B(\nu)\cos\theta\, d\nu\, dA\, d\Omega\, dt \tag{2}$$

When we use differentials instead of finite differences Δ, we can use integrals to obtain the energy over large intervals in area, frequency, solid angle or time.

Note that in order to write Planck's law in terms of wavelength λ instead of frequency ν one can *not* simply replace $\nu = c/\lambda$. $B(\nu)$ is defined in terms of differentials, so we need to take these into account. When changing from frequency to wavelength, the energy must be the same, we are only changing variables, not the physics. Using that the energy ΔE is the same, we get from equation 2 that $B(\nu)d\nu = -B(\lambda)d\lambda$ (the minus sign comes from the fact that λ and ν increase in opposite directions, $\lambda + |\delta\lambda| \rightarrow \nu - |\delta\nu|$). We can write

$$B(\nu)d\nu = -B(\nu)\frac{d\nu}{d\lambda}d\lambda \equiv B(\lambda)d\lambda,$$

We therefore obtain

$$B(\lambda) = -B(\nu)\frac{d\nu}{d\lambda} = -B(\nu)\left(-\frac{c}{\lambda^2}\right) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(kT\lambda)} - 1}.$$

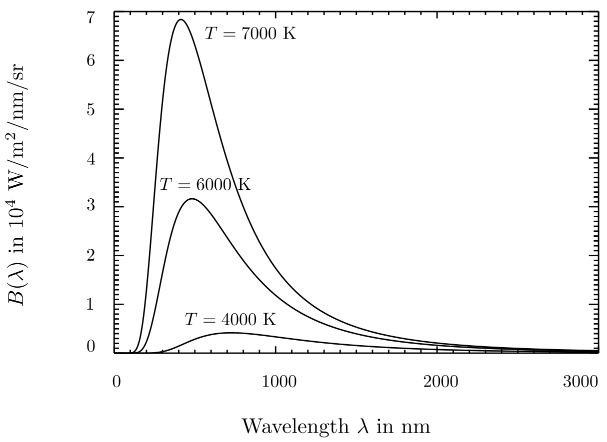


Figure 4: Planck's law for different black body temperatures.

Figure (4) shows the intensity as a function of wavelength for black bodies with different temperature T. We see that the wavelength of maximum intensity is different for different temperatures. We can use the position of this peak to determine the temperature of a black body. We can find an analytical expression for the position of the peak by setting the derivative of Planck's law equal to zero,

$$\frac{dB(\lambda)}{d\lambda} = 0$$

In the exercises you will show that the result gives:

Wien's displacement law

$$T\lambda_{\text{max}} = 2.9 \times 10^{-3}\ \text{Km}.$$

Another way to obtain the temperature of a black body is by taking the area under the Planck curve, i.e. by integrating Planck's law over all wavelengths. This area is also different for different temperatures T. Integrating this over all solid angles $d\Omega$ and frequencies $d\nu$, we obtain an expression for the *flux*, energy per time per area,

$$F = \frac{dE}{dA\, dt}.$$

The integral can be written as (here we are just integrating equation (2) over $d\nu$ and $d\Omega$)

$$F = \int_0^\infty d\nu \int d\Omega B(\nu)\cos\theta.$$

Using that $d\Omega = d\phi \sin\theta d\theta = -d\phi(d\cos\theta)$ and substituting $u = h\nu/kT$, we get

$$\begin{aligned} F &= \int_0^{2\pi} d\phi \int_0^1 d\cos\theta\cos\theta \int d\nu \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1} \\ &= \frac{2k^4T^4\pi}{h^3c^2}\int \frac{u^3 du}{e^u - 1} \\ &= \frac{2\pi k^4T^4}{h^3c^2}\underbrace{\zeta(4)}_{\pi^4/90}\underbrace{\Gamma(4)}_{3!} \\ &= \underbrace{\frac{2\pi^5k^4}{15h^3c^2}}_{\equiv\sigma} T^4. \end{aligned}$$

Here the solution of the u-integral can be found in tables of integrals expressed in terms of ζ, the Riemann zeta-function and Γ, the gamma-function, both of which can be found in tables of mathematical functions. The final result is thus:

Stefan-Boltzmann law
The flux emitted from a black body is proportional to the temperature to the fourth power.

$$F = \sigma T^4,$$

where σ is a constant.

We see that we have two ways of measuring the temperature of a star, by looking for the wavelength were the intensity is maximal, or by measuring the energy per area integrated over all wavelengths. If a star had been a black body, these two temperatures would have agreed. However, a star is not a perfect black body. A star has different temperatures at different depths in the star's atmosphere. At different wavelengths we receive radiation from different depths and the final radiation is a combination of Planck radiation at several temperatures. Since the intensity as a function of wavelength is not a perfect Planck curve at a fixed temperature T, the two ways of measuring the temperature will also disagree,

- From Wien's displacement law, we get the *color temperature*, $T = \text{constant}/\lambda_{\text{max}}$.
- From Stefan-Boltzmann's law we get the *effective temperature*, $T = (F/\sigma)^{1/4}$.

The first temperature is called the color temperature since it shows for which wavelength the radiation has it's maximal intensity and hence which color the star appears to have. The second temperature is based on the total energy emitted.

We have so far introduced two measures for the energy of electromagnetic radiation:

Intensity

$$I(\nu) = \frac{dE}{cos\theta\, d\nu\, dA\, d\Omega\, dt}$$

energy received per frequency, per area, per solid angle and per time.

Flux (or total flux)

$$F = \frac{dE}{dA\, dt}$$

total energy received per area and per time.

You will now soon meet the following expressions:

Flux per frequency

$$F(\nu) = \frac{dE}{dA\, dt\, d\nu}$$

total energy received per area, per time and per frequency.

Luminosity

$$L = \frac{dE}{dt}$$

total energy received per unit of time.

Luminosity per frequency

$$L(\nu) = \frac{dE}{dt\, d\nu}$$

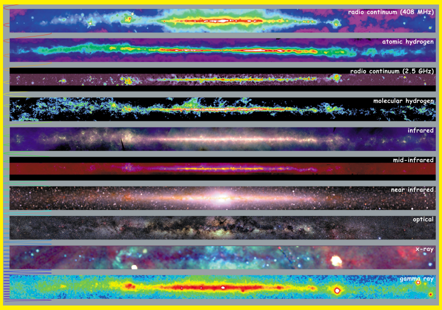
total energy received per frequency per time.

You will soon see more uses of all these expressions in practise, but it is already now a good idea to memorize the meaning of intensity, flux and luminosity.

4 Spectral lines

When looking at the spectra of stars you will discover that they have thin dark lines at some specific wavelengths. Something has obscured the radiation at these wavelengths. When the radiation leaves the stellar surface it passes through the stellar atmosphere which contains several atoms/ions absorbing the radiation at specific wavelengths corresponding to energy gaps in the atoms. According to Bohr's model of the atom, the electrons in the atom may only take certain energy levels $E_0, E_1, E_2, \ldots$. The electron cannot have an energy between these levels. This means that when a photon with energy $E = h\nu$ hits an atom, the electron can only absorb the energy of the photon if the energy $h\nu$ corresponds exactly to the difference between two energy levels $\Delta E = E_i - E_j$. Only in this case is the photon absorbed and the electron is excited to a higher energy level in the atom. Photons which do not have the correct energy will pass the atom without being absorbed. For this reason, only radiation at frequency ν with photon energy $E = h\nu$

Fact sheet: The Milky Way band observed in several wavelength regions (ultraviolet light is missing, though). The development of new detectors and, in particular, space telescopes has enabled us to study the universe at all wavelengths. We can now learn about celestial objects and physical processes that were completely unknown to astronomers only a few decades ago.(Figure: NASA)



corresponding to the difference in the energy level of the atoms in the stellar atmosphere will be absorbed. We will thus have dark lines in the spectra at the wavelengths corresponding to the energy gaps in the atoms in the stellar atmosphere (see figure 5). By studying the position of these dark lines, the *absorption lines*, in the spectra we get information about which elements are present in the stellar atmosphere.

The opposite effect also takes place. In the hotter parts of the stellar atmospheres, electrons are excited to higher energy levels due to collisions with other atoms. An electron can only stay in an excited energy level for a limited amount of time after which it spontaneously returns to the lowest energy level, emitting the energy difference as a photon. In these cases we will see bright lines, *emission lines*, in the stellar spectra at the wavelength corresponding to the energy difference, $h\nu = \Delta E$ (see figure 6).

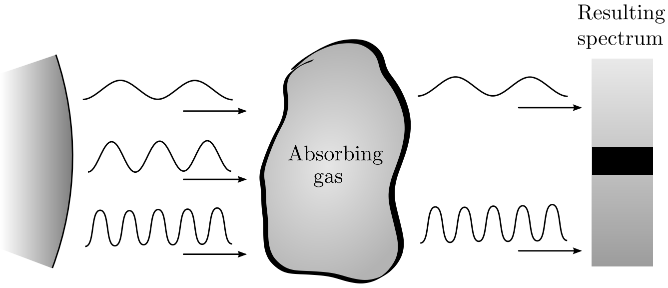


Figure 5: Formation of absorption lines.

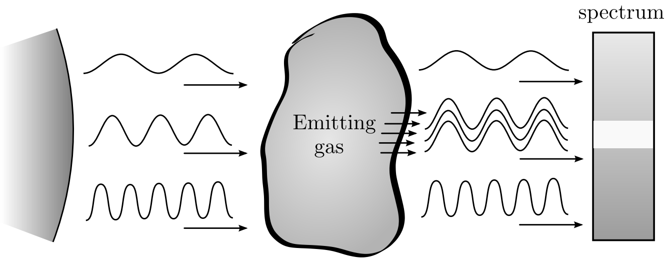


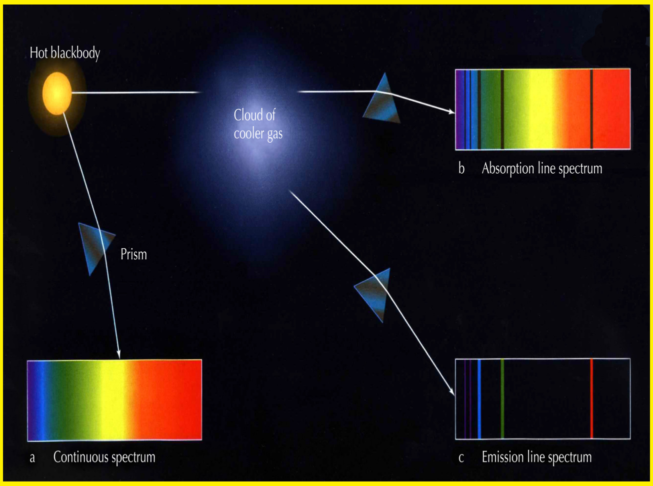
Figure 6: Formation of emission lines.

The exact energy levels in the atoms and thus the wavelengths of the absorption and emission lines can be calculated using quantum physics, or they can be measured in the laboratory. However, the actual wavelength where the spectral line is found in a stellar spectrum may differ from the predicted value. One reason for this could be the Doppler effect. If the star has a non-zero radial velocity with respect to the Earth, all wavelengths and hence also the position of the spectral lines will move according to

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v_r}{c},$$

where v_r is the radial component of the velocity. By taking the difference $\Delta\lambda$ between the observed wavelength (λ) and predicted wavelength (λ_0) of the spectral line, one can measure the velocity of a star or any other astrophysical object as we discussed in the lecture on extrasolar planets.

Fact sheet: By studying the spectra of objects in the universe, you can do "remote learning" that is, from millions and even billions of light-years away you can figure out the object's chemical composition and velocity. a) If you look directly at a blackbody through a prism or a modern spectrograph, you will see a continuous spectrum. b) Clouds of gas absorb certain wavelengths of light. A continuous spectrum that hits a cloud of cool gas will be partially absorbed. The transmitted spectrum is called an absorption line spectrum, and is continuous except for the wavelengths that were absorbed by the gas. c) Anything that absorbs also emits. A cloud of cool gas that absorbs certain wavelengths from a blackbody will emit exactly those wavelengths as the gas atoms de-excite. If we look at the cloud without the blackbody in our line of sight, we will see an emission line spectrum. (Figure: www.nthu.edu.tw)



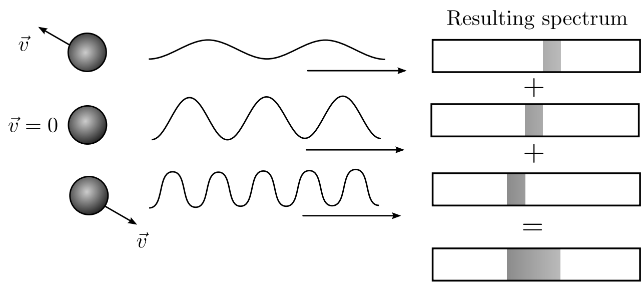


Figure 7: Broadening of spectral lines due to thermal motion.

Note that even if the star has zero-velocity with respect to Earth, we will still measure a Doppler effect: The atoms in a gas are always moving in random directions with different velocities. This thermal motion of the atoms will induce a Doppler effect and hence a shift of the spectral line. Since the atoms have a large number of different speeds and directions, they will also induce a large number of different Doppler shifts $\Delta\lambda$ with the result that a given spectral line is not seen as a narrow line exactly at $\lambda = \lambda_0$, but as a sum of several spectral lines with different Doppler shifts $\Delta\lambda$. The total effect of all these spectral lines is one single broad line centered at $\lambda = \lambda_0$ (see figure 7). The width of the spectral line will depend on the temperature of the gas, the higher the temperature, the higher the dispersion in velocities and thus in shifts $\Delta\lambda$ of wavelengths.

If we knew the velocities of the gas particles, we could calculate the width of the spectral line. Going back to part 1A, we already learned how to calculate the velocity of particles in a gas: the Maxwell-Boltzmann distribution. Looking at the figure in part 1A showing the Maxwell-Boltzmann distribution, we see that the peak of this distribution, i.e. the velocity that the largest number of atoms have, depends on the temperature of the gas,

$$\frac{dn(v)}{dv} = 0 \rightarrow \frac{d}{dv}(e^{-mv^2/(2kT)}v^2) = 0.$$

Taking the derivative and setting it to zero gives the following relation

$$v_{\max}^2 = \frac{2kT}{m},$$

i.e. the most probable velocity for an atom in the gas is given by $v_{\max}$ (Note: 'max' does *not* mean highest velocity, but highest *probability*). Most of the atoms will have a velocity close to this velocity).

This version of the Maxwell-Boltzmann distribution only tells you the absolute value v of the velocity. When measuring the Doppler effect, only the radial (along the line of sight) component v_r has any effect. The atoms in a gas have random directions and therefore atoms with absolute velocity v will have radial velocities scattered uniformly in the interval $v_r = [-v, v]$ (why this interval? do you see it?). Since the most probable absolute velocity is $v_{\max}$ the most probable radial velocity will be all velocities in the interval $v_r = [-v_{\max}, v_{\max}]$ (you see that for instance $v_r = 0$ is in this interval, do you understand why $v_r = 0$ is at as common as $v_r = v_{\max}$?). The atoms with absolute velocity $v_{\max}$ will thus give Doppler shifts uniformly distributed between

$\Delta\lambda/\lambda_0 = -v_{\max}/c$ and $\Delta\lambda/\lambda_0 = v_{\max}/c$. Few atoms have a much higher velocity than $v_{\max}$ and therefore the spectral line starts to weaken (less absorption/emission) after $|\Delta\lambda|/\lambda_0 = v_{\max}/c$. We will thus see a spectral line with the width given roughly by

$$2\Delta\lambda = \frac{2\lambda_0}{c} v_{\max} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT}{m}},$$

using the expression for $v_{\max}$ above. Do you see how this comes about? Try to imagine how the spectral line will look like, thinking how atoms at different velocities (above and below the most probable velocity) will contribute to v_r and thereby to the the spectral line. Try to make a rough plot of how $F(\lambda)$ for a spectral line should look like. Do not proceed until you have made a suggestion for a plot for $F(\lambda)$.

Of course, there are atoms at speeds other than $v_{\max}$ contributing to the spectral line as well. The resulting spectral line is thus not seen as a sudden drop/rise in the flux at $\lambda_0 - \Delta\lambda$ and a sudden rise/drop again at $\lambda_0 + \Delta\lambda$. Contributions from atoms at all different speeds make the spectral line appear like a Gaussian function with strongest absorption/emission at $\lambda = \lambda_0$. We say that the *line profile* is Gaussian. More accurate thermodynamic calculations show that we can approximate an absorption line with the Gaussian function

$$F(\lambda) = F_{\text{cont}}(\lambda) + (F_{\min} - F_{\text{cont}}(\lambda))e^{-(\lambda-\lambda_0)^2/(2\sigma^2)}, \qquad (3)$$

where $F_{\text{cont}}(\lambda)$ is the *continuum flux*, the flux $F(\lambda)$ which we would have if the absorption line had been absent. As the line is Gaussian, we can define the width using either σ or FWHM (see part 1A). The latter is given by

$$\text{FWHM} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT\ln 2}{m}}, \qquad (4)$$

We see that this exact line width differs from our approximate calculations above only by $\sqrt{\ln 2}$. With this expression we also have a tool for measuring the temperature of the elements in the stellar atmosphere.

5 Stellar magnitudes

The Greek astronomer Hipparchus (about 150 BC) made a catalogue of about 850 stars and divided them into 6 *magnitude* classes, depending on their brightness: the brightest stars were classified as magnitude 1 stars, and the stars which could barely be seen were classified as magnitude 6. Little did Hipparchus know about the fact that more than 2000 years later his system would still be used, and not only that, it would be used by all astronomers in the (now much bigger) world. Whereas Hipparchus classified the stars by eye, a more scientific method is used today. The eye reacts to differences in the logarithm of the brightness. For this reason, the magnitude classification is logarithmic in the flux that we receive (energy received per area per time $F = \frac{dE}{dt\,dA}$). For a difference in magnitude of 5 between two stars, the ratio of the fluxes of these stars is defined to be exactly 100.

The flux we receive from a star depends on the distance to the star. We define the *luminosity* L of a star to be the total energy emitted by the whole star per unit time (dE/dt). This energy is radiated equally in all directions. If we put a spherical shell around the star at distance r, the energy received per unit area on this shell would equal the total energy L divided by the surface area of the shell,

$$F = \frac{L}{4\pi r^2}.$$

Thus, the larger the distance r, the larger the surface area of the shell $4\pi r^2$ and the smaller the energy received per unit area (flux F). If we have two stars with observed fluxes F_1 and F_2 and magnitudes m_1 and m_2, we have learned that if $F_1 = F_2$ then $m_1 = m_2$ (agree?). We have also learned that if $F_1 = 100F_2$ then $m_2 - m_1 = 5$ (remember that in Hipparchus' system $m = 1$ stars were the brightest and $m = 6$ stars were the faintest).

The magnitude scale is logarithmic, thus we obtain the following general relation between magnitude and flux

$$\frac{F_1}{F_2} = 100^{(m_2-m_1)/5},$$

or

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right).$$

(Check that you can go from the previous equation to this one!) Given the difference in flux between two stars, we can now find the difference in magnitude.

We have so far discussed the *apparent magnitude* m of a star which depends on the distance r. If you change the distance to the star, the flux and hence the magnitude changes. We can also define *absolute magnitude M* which only depends on the total luminosity L of the star. The absolute magnitude M does not depend on the distance. It is defined as the star's apparent magnitude if we had moved the star to a distance of exactly 10 parsec (pc) (remember that 1pc=3.26ly). We can find the relation between apparent and absolute magnitude of a star,

$$\frac{F_r}{F_{r=10pc}} = \frac{L/(4\pi r^2)}{L/(4\pi(10\text{ pc})^2)} = \left(\frac{10\text{ pc}}{r}\right)^2 = 100^{(M-m)/5},$$

giving

$$m - M = 5 \log_{10}\left(\frac{r}{10\text{ pc}}\right).$$

(here we used a distance of $r = 10$ pc to calculate the flux for the absolute magnitude, this comes directly from the definition of absoluter magnitude: read it again if you did not understand this point). With this new more precise definition, stars can have magnitudes lower than 1. The brightest star in the sky, Sirius, has apparent magnitude -1.47 (note that the logarithmic dependence actually gives the brightest stars negative apparent magnitude). The planet Venus at maximum brightness has apparent magnitude -4.7 and the Sun has magnitude -26.7. The faintest object in the sky visible with the Hubble Space Telescope has apparent magnitude of about 30, about 100^5 times fainter than the faintest star visible with the naked eye. Originally the zero point of the magnitude scale was defined to be the star Vega. This has now been slightly changed with a more technical definition (outside the scope of this course).

Note: In order to define the magnitude we use the flux which we receive on Earth, the *received flux*. In some situations you will also need the *emitted flux*, the flux measured on the surface of the star emitting the radiation. It is important to keep these apart as they are calculated in a different manner (what is the difference?).

6 Exercises

Exercise 1D.1 You need to read sections 1 to 3 in order to be able to solve this exercise. At very large ($h\nu \gg kT$) and very small ($h\nu \ll kT$) frequencies, Planck's law can be written in a simpler form. The first limit is called the Wien limit and the second limit is called the Rayleigh-Jeans limit or simply the Rayleigh-Jeans law.

1. Show that Planck's law can be written as

$$B(\nu) = \frac{2h\nu^3}{c^2} e^{-h\nu/(kT)}$$

in the Wien limit.

2. Show that Planck's law can be written as

$$B(\nu) = \frac{2kT}{c^2}\nu^2$$

in the Rayleigh-Jeans limit. What kind of astronomer do you think uses Rayleigh-Jeans' law regularly?

Exercise 1D.2 You need to read sections 1 to 3 in order to be able to solve this exercise. Now we will deduce Wien's displacement law by finding the peak in $B(\lambda)$.

1. Use the expression in the text for $B(\lambda)$ and take the derivative with respect to λ. After taking the derivative, eliminate λ everywhere using

$$x = \frac{hc}{kT\lambda}.$$

2. To find the peak in $B(\lambda)$, we need to set the derivative equal to zero. Show that this gives us the following equation

$$\frac{xe^x}{e^x - 1} = 5.$$

3. We now want to solve this equation numerically. We see that all we need to do is to find a value for x such that the expression on the left hand side equals 5. The easiest way to do this is to try a lot of different values for x in the expression on the left hand side. When the expression on the left hand side has got a value very close to 5, we have found x.

   (a) The solution to x will be in the range $x = [1, 10]$. Define an array x in Python with 1000 elements going from 1.0 as the lowest value to 10.0 as the highest value. Make a plot of the expression on the left hand side as a function of the array x. Can you see by eye at which value for x the curve crosses 5? Then you have already solved the equation.

   (b) To make it slightly more exact, we try to find which x gives us the closest possible value to 5. We define the difference Δ between our expression and the value 5 which we want for this expression

$$\Delta = \left(\frac{xe^x}{e^x - 1} - 5\right)^2,$$

   where we have taken the square to get the absolute value. Define an array in Python which contains the value of Δ for all the values of x. Plot Δ as a function of x. By eye, for which value of x do you find the minimum?

   (c) Use Python to find the exact value of x (from the 1000 values defined above) which gives the minimum Δ.

   (d) Now use the definition of x to obtain the constant in Wien's displacement law. Do you get a value close to the value given in the text?

Exercise 1D.3 You need to read sections 1 to 3 in order to be able to solve this exercise. Here we will assume that a star is a perfect black body.

1. At which wavelength λ does the star in your solar system (the one you have been given for the numerical exercises) radiate most of its energy? (you will need to extract the temperature of the star from the Python class using `system.temperature`)

2. Plot $B(\lambda)$ for your star. What kind of electromagnetic radiation dominates?

3. What is the total energy emitted per time per surface area (flux) from the surface of the star?

4. Use this flux to find the luminosity L (total energy emitted per time) of your star? (Here you need the radius of the star which you can extract from the Python class).

5. What is the flux (energy per time per surface area) that your destination planet (the one you have chosen to visit) receives from its star? (Here you need the distance between the star and the planet, you can choose yourself at which point in the orbit you calculate this distance). (See figure 8 which shows the situation for the Sun-Earth system).

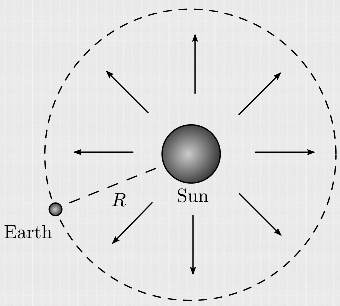


Figure 8: Radiation from the Sun. The flux is constant on a spherical surface with center at the Sun's center of mass.

6. Assume that your lander unit needs 40W of energy for its instruments and that it gets this energy from its solar panels. Assume that the efficiency of solar panels is 12%, i.e. that the electric energy that solar panels can produce is 12% of the energy that they receive. How many square meters of solar panel does your lander unit need in order to get 40W of energy when at the surface of the planet.

Exercise 1D.4 You need to read sections 1 to 3 in order to be able to solve this exercise. We will now study a simple climate model. You will need the results from question 1-6 in the previous problem.

1. We assume that the atmosphere of your destination planet is transparent for all wavelengths. How much energy per second arrives at the surface of your planet? The flux that you calculated in the previous exercises is the flux received by an area located at the planet's surface with orientation perpendicular to the distance-vector between the star and the planet. **Hint** - Since the planet is (close to) a sphere, the flux is not at all constant over the surface. However, we do not need to calculate the density for each square meter (fortunately). We can just look at the size of the effective absorption area (shadow area) which is shown in figure 9 (which shows the situation for the Sun-Earth system). Since the distance between the star and the planet is so large, we can assume that the rays arriving at the planet are traveling in the same direction (parallell). The radius r of the shadow area is then equal to planet's radius. The rest should be straight forward.

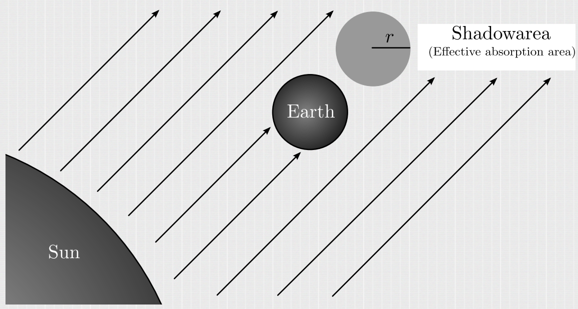


Figure 9: Shadowarea (effective absorption area).

2. You are now going to estimate your planet's temperatur by using a simple climate model that only takes into account the radiation from the star and the planet itself. We still assume that the atmosphere is transparent for all wavelengths. The model says that the planet is a blackbody with a constant temperature. This means that it absorbs all incoming radiation and emits the same amount (in energy/time) in *all* directions. Calculate the planet's temperature by using the simple climate model. You will need the result from the previous question and Stefan Boltzmann's law. Do you think this is a realistic measure of the surface temperature? Which important factors did we not take into account?

Exercise 1D.5 You need to read sections 1 to 3 in order to be able to solve this exercise. Here you will deduce a general expression for the flux per wavelength, $F(\lambda) = dE/(dA\,dt\,d\lambda)$, that we receive from a star with radius R at a distance r with surface temperature T. Assume that the star is a perfect black body. You can solve this problem in two steps,

1. Find the luminosity per wavelength $L(\lambda) = dE/(d\lambda\,dt)$, i.e. the energy per time per wavelength, emitted from the star. The intensity $B(\lambda)$ is defined as $dE/(dA\,d\Omega\,d\lambda\,dt\,cos\theta)$. You need to integrate over solid angle and area to obtain the expression for the luminosity. **Hint:** Look at the derivation of Stefan-Boltzmann's law in the text.

2. Find the flux $F(\lambda)$ using $L(\lambda)$.

3. Does the expression for the flux peak at the same wavelength as for Planck's law? Can we simply use the maximum wavelength from flux measurements to obtain $\lambda_{\max}$ to be used in Wien's displacement law?

Exercise 1D.6 You need to read sections 1 to 4 as well as part 1C in order to be able to solve this exercise. At the following link you will find directories containing some files with 10 different observations of the spectrum of a star taken at different times:

`http://folk.uio.no/frodekh/AST1100/part1D_1/`

In order to find your directory, you should look at the two last digits of your seed, i.e. if your seed is 42325 then you should enter the directory called `seed25`. Inside your directory you will find 10 files, each with the spectrum of the star taken at the day specified in the filename.

The filename indicates the time of observation given in days from the first observation taken at $t = 0$. The first column of the file is the wavelength of observation in nm, the second column is the flux relative to the continuum flux around the spectral absorption line $H\alpha$ at $\lambda_0 = 656.3$ nm. Due to the Doppler effect, the exact position of the spectral line is different from λ_0. You will also see that this difference changes in time. As we have seen before, real life observations are noisy. It is not so easy to see exactly at which wavelength the center of the spectral line is located.

1. Plot each of the spectra as a function of wavelength. Can you see the absorption line?

2. Make a bye-eye estimate of the position of the center of the spectral line for each observation. Use the Doppler formula to convert this into relative velocity of the star with respect to Earth for each of the 10 observations (neglect the fact that the velocity of the Earth changes with time).

3. Now we will make a more exact estimate of the spectral line position using a least squares fit. As discussed in the text, we can model the spectral line as a Gaussian function (see equation 3),

$$F^{\text{model}}(\lambda) = F_{\max} + (F_{\min} - F_{\max})e^{-(\lambda-\lambda_{\text{center}})^2/(2\sigma^2)}.$$

When $\lambda = \lambda_{\text{center}}$, the model gives $F^{\text{model}}(\lambda) = F_{\min}$. When λ is far from λ_{center} the model becomes $F^{\text{model}}(\lambda) = F_{\max}$ as expected (check!). Thus the flux in this wavelength range if there hadn't been any spectral line would equal $F_{\max}$. The flux at the wavelength for which the absorption is maximal is $F_{\min}$. The spectra are normalized to the continuum radiation meaning that $F_{\max} = 1$. We are left with three unknown parameters, $F_{\min}$, σ and λ_{center}. The first parameter gives the flux at the center of the spectral line, the second parameter is a measure of the width of the line and the third parameter gives the central wavelength of the spectral line. In order to estimate the speed of the star with the Doppler effect, all we need is λ_{center}. But in order to get the best estimate of this parameter, we need to find the best fitting model to the spectral line, so we need to estimate all parameters in order to find the one that interests us. Again we will estimate the parameters using the method of least squares. We wish to minimize

$$\Delta(F_{\min}, \sigma, \lambda_{\text{center}}) = \tag{5}$$

$$\sum_{\lambda} \left(F^{\text{obs}}(\lambda) - F^{\text{model}}(\lambda, F_{\min}, \sigma, \lambda_{\text{center}})\right)^2, \tag{6}$$

where $F^{\text{obs}}(\lambda)$ is the observed flux from the file and the sum is performed over all wavelengths available.

(a) Choose one of the 10 spectra. Plot the spectrum as a function of wavelength and identify the range of possible values for each of the three parameters we are estimating. Define three arrays `fmin`, `sigma` and `lambdacenter` in Python which contain the range of values for each of F_{min}, σ and λ_{center} where you think you will find the true values. Do not include more values of the parameters than necessary, but make sure that the true value of the parameter must be within the range of values that you select. Do not use more than 50 values for each parameter, preferably less (often 20 will be sufficient).

(b) Define a 3-dimensional array `delta` where you calculate Δ for all the combinations of parameters which you found reasonable.

(c) Find for which combination of the parameters F_{min}, σ and λ_{center} that Δ is minimal. These are your best estimates.

(d) **optional: It is completely sufficient to do this for just one of the spectra. However if you do the optional exercises it will count favourably on the grade.** Repeat this procedure for all 10 spectra and obtain 10 values for the Doppler velocity v_r. Can you manage to write the code in an automated way such that you do not need to look at each of the spectra to find the grid values to search over?

4. **optional:** Make an array of the 10 values you have obtained for the velocities and plot it as a function of time.

5. **optional:** Assume that the change of velocity with time indicates the presence of a planet around the star (is there something in your observations which indicates this?). The mass of the star was found to be 1 solar mass. Find the minimum mass of this planet (find v_r and the period 'by eye' looking at the velocity curve). **Hint:** Remember that you need to subtract the peculiar velocity (velocity of the center of mass of the system), found by taking the mean of the velocity.

Exercise 1D.7 You need to read sections 1 to 4 in order to be able to solve this exercise. At the following link

`http://folk.uio.no/frodekh/AST1100/part1D_2/`

you find the spectrum of the atmosphere of the planet your are going to visit. In order to find which file to use, you should look at the two last digits of your seed, i.e. if your seed is 42325 then you should use the file with `seed25` in the name.The file contains flux measurements in the rang 600nm to 3000nm normalized such that the background flux is 1. The noise is strongly varying between the measurements. A file called `sigma_noise.txt` with the standard deviation σ_n of the random noise fluctuations for each observed wavelength is found in the same directory. Note that all the files are compressed with `gzip` in order to make quicker downloads.

Your task is to look for spectral lines in order to determine which molecules you can expect to find in the atmosphere of your planet. You should look for the following gases:

- Oxygen O_2 has absorption lines at 630nm, 690nm and 760nm. The oxygen atom has 8 protons and 8 neutrons.
- Water vapour H_2O has absorption lines at 720nm, 820nm and 940nm. The hydrogen atom consists of only one proton.
- Carbon dioxide CO_2 has absorption lines at 1400nm and 1600nm. The carbon atom has 6 protons and 6 neutrons.
- Methane CH_4 has absorption lines at 1660nm, 2200nm. If you find methane on your planet, this could be a sign of life (or geological activity).
- Carbon monoxide CO has an absorption line at 2340nm.
- Nitrous oxide N_2O, also known as laughing gas, has an absorption line at 2870nm and would also be a possible sign of biological

activity. Nitrogen consists of 7 protons and 7 neutrons.

The observation is made from a satellite which can have a motion up to 10km/s with respect to the planet and would therefore induce a Doppler effect of the spectral lines. Assume a Gaussian line profile as in exercise 1D.6. Your task now is to:

1. Identify which of the spectral lines above may be present and therefore which gases you find in your atmosphere.
2. Find the velocity of the satellite during the measurement of the spectrum.
3. For the lines you find, estimate the width of the line and find the temperature of the given gas using equation 4.
4. Assuming a temperature profile $T(h) = 450\text{K}e^{-h/10000\text{m}}$ for a given height h above the surface for your atmosphere, sketch roughly in which layers of your atmoshere you expect which gases based on the temperatures obtained in the previous point.

As the standard deviation of the noise is varying, you cannot use least squares minimization, you need to minimize χ^2 as explained in part 1C. This is exactly the same, but you need to divide each square by the σ_n of the noise given in the file. The procedure would therefore be the following:

1. For each possible line, try different models for λ_{center}, F_{min} and σ exactly as in exercise 1D.6 (except now you use χ^2). But this time, due to strongly varying σ_n you may not be able to see the spectral line by eye. You need to explore a set of possible values values for λ_{center}, F_{min} and σ without being able to see the line visually. **Note the difference between σ as the width of your spectral line and σ_n as the standard deviation of the noise fluctuations**. You can find the range for σ by considering that the temperature of the gas is expected to be in the range 150K to 450K. The depths cannot be smaller than $F_{\text{min}} > 0.7$. You have already been given the possible velocities so this limits the possible positions of λ_{center}.
2. Optimally you should try 300 positions for λ_{center} and 30 for σ and F_{min}. If you do not program this in a very clever manner, this may take too long on your laptop. In this case, reduce the number of points until you get an acceptable speed. Remember that the more calculations (in this case sums) you can do outside of the loop the better. Also moving a sum such that it is inside only one or two for-loops instead of three, makes a huge difference in speed.
3. Having obtained Doppler velocity (from the λ_{center} values), line depth F_{min} and temperature (obtained from your σ values) for all possible lines, you should now look at your results to find which lines are actually present and which are not.

Even if there is no line present, your statistical procecure will still find some values for λ_{center}, σ and F_{min}. This is due to statistical fluctuations. If you just look at a part of the data, you will see lots of fluctuations which could easily be mistaken for spectral lines. Our algorithm is doing this mistake. It is therefore not an easy task to determine which lines are actual lines and which are just statistical flukes. A professional astronomer would use advanced statistics to, in each case, obtain the probability that a line is really detected or not. Now you will instead need to use some logical reasoning to determine which lines you think are really detected. You should consider:

- remember that the Doppler shift will be equal for the real lines, but due to uncertainties in your estimated numbers, the velocities you obtain may differ a bit from line to line even if the underlying velocity is the same.
- a real line would often be deeper (have a lower F_{min}) than a fluke, but this is not always the case.
- A real detected gas would be expected to have temperature in the range $T = [150K, 450K]$, but uncertainties in your estimated numbers may give you higher or lower values also for real lines.
- It can be that you find for instance only one of the O_2 lines. The fact that you detect one O_2 line does not automatically mean that

the other O_2 lines are present, this depends a bit on temperature and other atmospheric conditions.

- Plotting your best fit model line on top of the observed spectrum may, by focusing on the wavelengths within the detected line, indicate if this seems to be a very good fit to a real spectral line or just the algorithm trying to fit a smooth spectral line to noise. This is not very exact, but in a few cases when the noise is low you may be able to identify a very clear line by eye.

For some planets (seeds) the detection of spectral lines by the above approach may be rather easy with several lines showing clear signs of being real lines, for other planets it may be very hard and maybe even just one line may be possible to identify. When you think you have detected a line, classify it as 'highly probable detection' or 'possible detection' and explain well the reasons for your classification in each case.

Exercise 1D.8 You need to read **all** sections of part 1D in order to be able to solve this exercise. In the text you find the apparent magnitudes of Sirius, Vega and the Sun. Look up the distances to these objects (again, wikipedia is a useful source of information) and calculate the absolute magnitude. Which of these three stars is actually the brightest?

Exercise 1D.9 You need to read **all** sections of part 1D in order to be able to solve this exercise.

1. Use the flux calculated in Problem 3.6 to check that the apparent magnitude of the Sun used in the text is correct. In order to calibrate the magnitude you also need to know that the star Vega has been defined to have zero apparent magnitude (actually with newer definitions it has magnitude 0.03) and that the absolute magnitude of Vega is 0.58. You also need to know the luminosity of Vega: Look it up in Wikipedia. All other quantities that you may need (for instance the distance to Vega) should be calculated using these numbers.
2. The faintest objects observed by the Hubble Space Telescope (HST) have magnitude 30. Assume that this is the limit for HST. How far away can a star with the same luminosity as the Sun be for HST to see it? (here you will need the luminosity of the Sun calculated in problem 3.5)
3. Assume that the luminosity of a galaxy equals the luminosity of 2×10^{11} Suns. How far away can we see a galaxy using HST?