AST1100 Lecture Notes

Part 1E
Hydrostatic equilibrium: modelling the planetary atmosphere

Questions to ponder before the lecture

1. Why doesn't the Sun collapse to a black hole due to its own gravitation?
2. Why doesn't our atmosphere completely fall to the ground? Or why doesn't it evaporate away from the planet?
3. Why is the atmosphere getting gradually thinner, the higher we get?
4. If you were to model how density and temperature of our atmosphere vary with height which equations would you put up?

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Part 1E
Hydrostatic Equilibrium: modelling the planetary atmosphere

In figure 1 we show a mass element with mass dm in the atmosphere of a planet (or inside a star) at a distance r from the center. We know that gravity pulls this element towards the center. But the size of the atmosphere (or the radius of a stable star) does not change with time, so there must be a force working in the opposite direction keeping this mass element stable at distance r. This force is the pressure. On a planet with a stable atmoshere (or in a main sequence star) the pressure forces must exactly equal the force of gravity, otherwise the atmosphere would retract or expand (or the star would change its radius). This fact, called *hydrostatic equilibrium*, gives us an invaluable source of information about several astrophysical objects, from planetary atmospheres to stellar interiors. We can't observe the interior of a star directly, but the equation of hydrostatic equilibrium together with other thermodynamic relations combined with observations of the star's surface allow detailed computer modelling of the interior of stars. Here we will deduce this important equation.

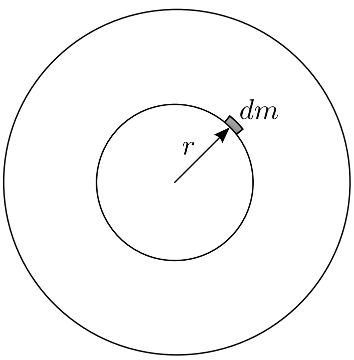

Figure 1: The mass element dm inside the atmosphere is not moving radially.

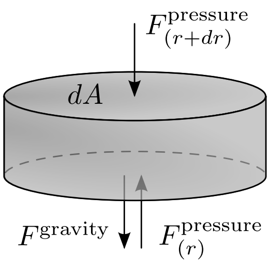

Figure 2: The mass element dm inside the atmosphere is not moving: The forces add to zero.

In figure 2 we have zoomed in on the mass element dm. Because of the symmetry of the problem (the fact that gravitation only works radially), we can assume spherical symmetry, i.e. that density, pressure and temperature are all a function only of the distance r from the center. We show the forces of pressure pushing the mass element from above and below, as well as the force of gravity. Assuming that the element is infinitesimally small, there are no gravitational forces pushing on the sides and the pressure forces on the sides will be equal since the distance r from the center is the same on both sides. The forces on the sides must therefore sum up to zero. We will now look at a possible radial movement of the mass element. Newton's second law on the mass element gives

$$dm\frac{d^2r}{dt^2} = -F^{\text{grav}} - F^{\text{pressure}}(r+dr) + F^{\text{pressure}}(r),$$

where all forces are defined to be positive. The minus sign on the two first forces show that they push towards the center in negative r direction. The area of the upper and lower sides of the element is dA. Pressure is defined as force per area, so

$$P = \frac{F^{\text{pressure}}}{dA},$$

giving

$$dm\frac{d^2r}{dt^2} = -G\frac{M(r)dm}{r^2} - P(r+dr)dA + P(r)dA,$$

(check that you understand where each term comes from here) where $M(r)$ is the total mass inside radius r:

$$M(r) = \int_0^r dr' 4\pi (r')^2 \rho(r') \qquad (1)$$

The infinitesimal difference in pressure between r and $r+dr$ is $dP = P(r+dr) - P(r)$. We have

$$\frac{dm}{dA}\frac{d^2r}{dt^2} = -\frac{dm}{dA}\frac{GM(r)}{r^2} - dP$$

We write the mass of the element as the density $\rho(r)$ at radius r times the volume $dA\,dr$ of the mass element $dm = \rho dA\,dr$. Dividing by dr on both sides gives

$$\rho\frac{d^2r}{dt^2} = -G\frac{\rho M(r)}{r^2} - \frac{dP}{dr}.$$

(Did you understand all parts of the deduction?) For a stable atmoshpere (or a main sequence star) the radial size is not changing so the mass element cannot have any acceleration in r direction giving $d^2r/dt^2 = 0$. This gives the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho(r)g(r),$$

where $g(r)$ is the local gravitational acceleration

$$g(r) = G\frac{M(r)}{r^2}.$$

The equation of hydrostatic equilibrium tells us how the pressure $P(r)$ must change as a function of height in an atmosphere or as a function of radius in order for a star to be stable. In the following we will study what kind of pressure we might experience in an atmosphere or inside a star and which effect it has.

You showed in part 1A the the equation of state for an ideal gas can be written as $P = nkT$ where n is number density, the number of gas particles per volume. We can express this in terms of mass density ρ, the mass per volume, as

Ideal gas law

$$P = \frac{\rho kT}{\mu m_H}.$$

Here μ is the *mean molecular weight* and m_H is the mass of a hydrogen atom. The mean molecular weight is the mean mass of a particle in the gas, measured in hydrogen atom masses. In other words,

Mean molecular weight

$$\mu = \frac{1}{N}\sum_{i=1}^{N} f_i \frac{m_i}{m_H},$$

where N is the number of different molecules present and the sum is taken over all types of molecules. The mass of molecule type i is m_i and the fraction of this kind of molecules in the atmosphere is f_i. In this way, the mean molecular weight of pure hydrogen is 1, whereas a gas of pure H_2 has $\mu = 2$ and a gas of pure CH_4 has $\mu = 16$.

In the stellar interior, there is a high density of photons traveling in all possible directions. The photons behave like the atoms or molecules in a gas. So we may consider the collection of photons as a *photon gas*. This photon gas also has a pressure in the same way as a normal gas has. Thermodynamics tells us that the pressure of a photon gas is given by

$$P_{\mathrm{r}} = \frac{1}{3}aT^4,$$

where $a = 7.56 \times 10^{-16}$ J/m^{3}K^4 is the *radiation constant*.

1 Exercises

Exercise 1E.1

You need to read all of the text in order to be able to solve this exercise. In this exercise we will model the pressure, density and temperature variations in the atmosphere of your destination planet. Before doing this exercise you should do exercise 1D.4 in part 1D in order to get an estimate of the mean surface temperature of your destination planet.

You will need two equations to solve this problem: (1) the equation of hydrostatic equilibrium: assuming the the atmosphere is stable (not contracting or expanding), this is a safe assumption, (2) the equation of state for an ideal gas.

In addition you will need the density of your atmosphere at the surface which you can extract from the SolarSystem class. You can also use the (reasonable) approximation that the gravitational acceleration is the same throughout the whole atmosphere (use the one at the surface).

1. In order to combine the two equations, we need the mean molecular weight μ of the atmosphere. If you did exercise 1D.7 in part 1D, you already know which gases are present in your atmosphere (include all those which you suspected are present). In this case, assume that there are equal amounts of all of the gases that you found and **use this to find the mean molecular weight in the atmosphere of your destination planet**. If you did not do that exercise, you could (1) go back and do it now, or (2) assume a composition of the atmosphere equal to the one on earth: 78% nitrogen (each atom, 7 protons, 7 neutrons) N_2, 21% oxygen (each atom, 8 protons, 8 neutrons) O_2 and about 1% argon Ar (18 protons, 22 neutrons). In this case, find the mean molecular weight of the Earth's atmosphere.

2. We will first make a very rough approximation: we assume that your atmosphere is isothermal (the temperature is the same everywhere). Use the temperature that you obtained in exercise 1D.4. Combine the two equations and solve these analytically to show that the pressure as a function of heigh h above the surface in your atmosphere can be written as
$$P(h) = P(0)e^{-\frac{h}{h_0}}$$
where h_0 is the the *scale height*, the height difference for which the corresponding pressure difference is a factor e. What is the scale height for the atmosphere at your destination planet?

3. Find a similar expression for the density $\rho(h)$ as a function of height h above the surface?

4. Now assume that the temperature varies with height h above the surface as
$$T(h) = T(0)e^{-\frac{h}{4h_0}}$$
where the scale height h_0 is the one you obtained for pressure above and T_0 is the temperature at the surface which you obtained in exercise 1D.4. Use this expression to find again the pressure $P(h)$ and density $\rho(h)$, but this time you need to solve the equations numerically. What method should you use? Plot the results $P(h)$ and $\rho(h)$ up to heights where the pressure has fallen to 1% of the pressure at the surface. **Hint:** You have been given the atmospheric density at the surface. You can use that to find the atmospheric pressure at the surface.

5. We now assume that the atmosphere is adiabatic (that the gas can change temperature without loosing or gaining heat from the environment, this is a good approximation away from the surface), meaning that we can use the adiabatic law (you will show this in the course on thermodynamics):
$$p^{1-\gamma}T^{\gamma} = \text{constant}$$
Assume the adiabatic index γ to have a value of 1.4 in your final result, but use the general value γ in your calculations. Combine the adiabatic law with the equation for hydrostatic equilibrium and ideal gas law to obtain more realistic height profiles for temperature, pressure and density analytically.

6. Plot the three different profiles you found for pressure in one plot. Do the the same for temperature and density. How well do they match?