# AST1100 Lecture Notes

## Part 2A
## The special theory of relativity: Basic principles

### Questions to ponder before the lecture

1. You have already used the Lorentz transformations. Do you know where they come from? Which basic principles/formulas would you use if you wanted to deduce the Lorentz transformation?

2. You may have heard about the twin paradox: one of the twins is launched into space, travels with a speed close to the speed of light and returns to the Earth. After returning, which of the twins are older?

3. You have already learned how time goes slower when travelling close to the speed of light. So in principle, it is not a paradox that the two twins have different ages after the space trip. What is then the paradox of the twin paradox?

# AST1100 Lecture Notes

## Part 2A
## The special theory of relativity: Basic principles

## 1  Simultaneity

We all know that 'velocity' is a relative term. When you specify velocity you need to specify velocity *with respect to something*. If you sit in your car which is not moving (with respect to the ground) you say that your velocity is zero with respect to the ground. But with respect to the Sun you are moving at a speed of 30 km/s. From the point of view of an observer passing you in his car with a velocity of 100 km/h with respect to the ground, your speed is −100 km/h (see figure 1). Even though you are not moving with respect to the ground, you are moving backwards at a speed of 100 km/h with respect to the passing car.

In the following we will use the expression *'frame of reference'* to denote a system of observers having a common velocity. All observers in the same frame of reference have zero velocity with respect to each other. An observer always has velocity zero with respect to his own frame of reference.

An observer on the ground measures the velocity of the passing car to be 100 km/h with respect to his frame of reference. On the other hand, the driver of the car measures the velocity of the ground to be moving at −100 km/h with respect to his frame of reference. We will also use the term *'rest frame'* to denote the frame of reference in which a given object has zero velocity. In our example we might say: In the rest frame of the passing car, the ground is moving backwards with 100 km/h.
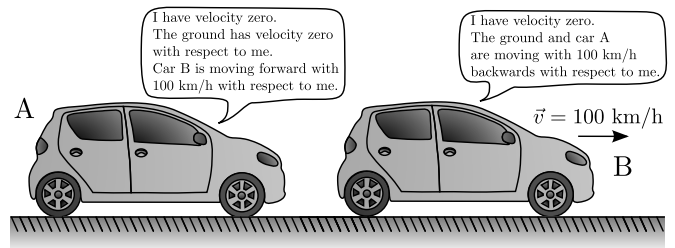


Figure 1: Velocities are relative.

You are observing a truck coming towards you with a speed of $v_{\text{truck}}^{\text{ground}} = -50$ km/h with respect to the ground (see figure 2, velocities are defined to be positive to the right in the figure). From your frame of reference, which is the same frame of reference as the ground, the speed of the truck is $|v_{\text{truck}}^{\text{ground}}| = 50$ km/h towards you. Now you start driving your car in the direction of the truck with a speed of $v_{\text{car}}^{\text{ground}} = +50$ km/h with respect to the ground (see again figure 2). From your frame of reference you observe the ground to be moving backwards with a velocity of $v_{\text{ground}}^{\text{car}} = -50$ km/h. Again, from your frame of reference you now observe the velocity of the approaching truck to be $v_{\text{truck}}^{\text{car}} = v_{\text{truck}}^{\text{ground}} - v_{\text{car}}^{\text{ground}} = (-50 \text{ km/h}) - (50 \text{ km/h}) = -100$ km/h (whereas from the frame of reference of an observer on the ground, the truck still has $v_{\text{truck}}^{\text{ground}} = -50$ km/h). Now you make a turn so that you drive in the opposite direction: Now your velocity is −50 km/h with respect to the ground, but now you are driving in the same direction as the truck. You are now moving in the same direction as the truck with exactly the same speed with respect to the ground. From your frame of reference (which is now the same frame of reference as the truck) the truck is not moving.

So far, so good. This was just stating some ob-

vious facts from everyday life in a difficult way. Now, replace the truck with a beam of light (a photon) and the car with the Earth. The situation is depicted in figure 3. You observe the speed of light from a distant star at two instants: One at the 1st of January, another at the 1st of July. In January you are moving away from the photons approaching you from the distant star. In July you are moving towards the photons arriving from the star. If the speed of light with respect to the distant star is $c$, then in January you expect to measure the speed of the light beam from the star to be $c - v$ where $v = 30$ km/h is the speed of the Earth with respect to the same star (we assume that the star does not move with respect to the Sun, so this is also the orbital speed of the Earth). In July you expect to measure the speed of light from the star to be $c + v$, just as for the truck in the example above: The speed of the light beam seen from your frame of reference is supposed to be different depending on whether you move towards it or away from it.

'Michelson-Morley experiment'. The result however, was highly surprising: They measured exactly the same speed of light in both cases. The speed of light seemed to be the same independently of the frame of reference in which it is measured. This has some quite absurd consequences: Imagine that you see the truck driving at the speed of light (or very close to the speed of light, no material particle can ever travel at the speed of light). You are accelerating your car, trying to pass the truck. But no matter at which speed you drive, you see the truck moving with the speed of light with respect to your frame. Even when you reach half the speed of light, you still see the truck moving with velocity $c$. But how is this possible? An observer at rest with respect to the ground measures the truck moving with the speed of light as well, not with the velocity $c + c/2 = 3c/2$ as you would expect given that it moves with velocity $c$ with respect to something moving with velocity $c/2$.
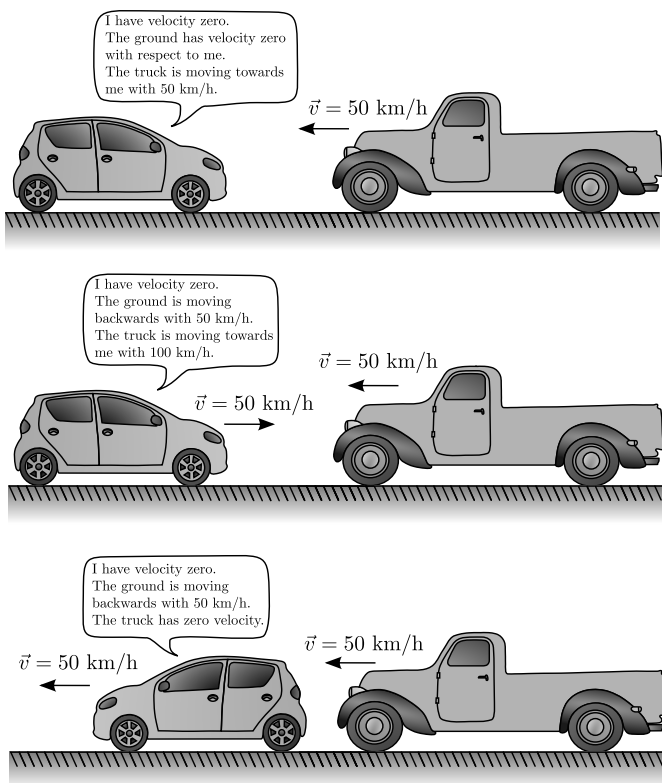


Figure 2: The velocity of the truck seen from the car depends on the velocity of the car.

In 1887 Michelson and Morley performed exactly this experiment which is now famous as the
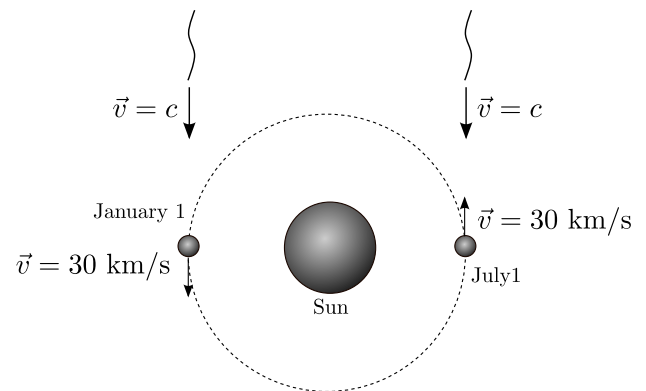


Figure 3: The velocity of the starlight is measured when the Earth has velocity 30 km/s towards and away from the light beam.
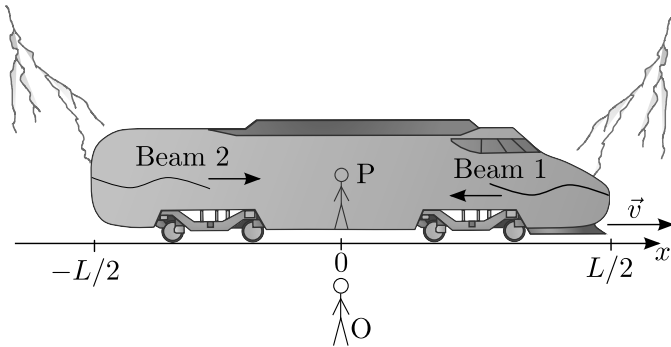
Figure 4: Event A: Lightning strikes the front part of the train. Event B: Lightning strikes the rear part of the train. These two events are observed by observer O on the ground and observer P in the train. The train has length L.

This was one of the first signs showing that something was wrong with classical physics. The fact that the speed of light seemed to be constant in all frames of reference led to several contradictions. We have already seen one example of such a contradiction. We will now look at another one which might shed some light on the underlying reason for these contradictions.

In figure 4 we show the situation. Observer O is standing on the ground (at rest with respect to the ground), observer P is standing in the middle of a train of length $L$ moving with velocity $v$ with respect to the ground. Observer O sees two lightnings striking the front and the rear of the train simultaneously. We call the two events A and B (An event is a point in space and time, a point with a space and time coordinate): Event A is the lightning striking the front, event B is the lightning striking the rear. Events A and B are simultaneous.

The light from these two lightnings start traveling from the front and back end of the train towards observer P. The beam approaching observer P from the front is called beam 1 and the beam approaching from the rear is called beam 2.

Both observers had synchronized their clocks to $t = 0$ at the instant when the lightnings strike the train. Both observers have also defined their own coordinate systems $x$ (observer on the ground) and $x'$ (observer in the train) which is such that the position of observer P is at $x = x' = 0$ in both coordinate systems at the instant $t = 0$ when the lightenings strike. Thus the lightnings

hit the train at the points $x = x' = L/2$ and $x = x' = -L/2$ as seen from both observers. We will now look how each of these observers experience these events:

## From the point of view of observer O standing on the ground:

The frame of reference of observer O on the ground is often referred to as the *laboratory frame*. It is the frame of reference which we consider to be at rest. At what time $t = t_C$ does observer P see beam 1 (we call this event C)? To answer this question, we need to have an expression for the x-coordinate of observer $P$ and the x-coordinate of beam 1 at a given time $t$. Observer P moves with constant velocity $v$ so his position at time $t$ is $x_P = vt$. Beam 1 moves in the negative x-direction with the speed of light $c$ starting from $x_1 = L/2$ at $t = 0$. The expression thus becomes $x_1 = L/2 - ct$. Observer P sees beam 1 when $x_1 = x_P$ at time $t_C$. Equating these two expressions, we find

$$t_C = \frac{L/2}{c + v}. \tag{1}$$

At what time $t = t_D$ does observer P see beam 2 (we call this event D)? Following exactly the same line of thought as above, we find

$$t_D = \frac{L/2}{c - v}. \tag{2}$$

So according to observer O in the laboratory frame, $t_C < t_D$ and observer P should see the light beam from the lightning in front before the light from the back. This sounds reasonable: Observer P is moving towards beam 1 and away from beam 2 and should therefore see beam 1 first.

## From the point of view of observer P standing in the train:

At what time $t = t_C$ does observer P see beam 1? We have just agreed on the fact that the speed of light is independent of the frame of reference. The result is that the speed of light is $c$ also for the observer in the train. Seen from the frame of reference of observer P, observer P himself is at rest and the ground is moving backwards with speed $v$. Thus from this frame of reference, observer P is always standing at the origin $x'_P = 0$ (the coordinate system $x'$ moves with observer P). The expression for $x'_1$ is the same as seen from observer

O:$x_1' = L/2 - ct$ (convince yourself that this is the case!). Again we need to set $x_1' = x_P'$ giving

$$t_C = \frac{L/2}{c}$$

At what time $t = t_D$ does observer P see beam 2? Again we follow the same procedure and obtain

$$t_D = \frac{L/2}{c}$$

As calculated from the frame of reference of observer P, the two beams hit observer P at exactly the same time.

So not only are the exact times $t_C$ and $t_D$ different as calculated from the two frames of reference, but there is also an even stronger contradiction: Observer P should be hit by the two beams simultaneously as calculated from the frame of reference of observer P himself, but as calculated from the laboratory frame, beam 1 hits observer P before beam 2. What does really happen? Do the beams hit observer P simultaneously or not? Well, let's ask observer P himself:

So observer P, two lightnings struck your train simultaneously at the front and rear end. Did you see these two lightnings simultaneously or did you see one flash before the other?
Observer P: Sorry? I think you are not well informed. The two lightnings did not happen simultaneously. There was one lightning which struck the front part and then shortly afterwards there was another one striking the rear. So clearly I saw the flash in the front first.
Observer O: No, no, listen, the lightnings did strike the train simultaneously, there was no doubt about that. But you were moving in the direction of beam 1 and therefore it appeared to you that the front was hit by the lightning first.
Observer P: So you didn't watch very carefully I see. It is impossible that the two lightnings struck at the same time. Look, I was standing exactly in the middle of the train. The speed of light is always the same, no matter from which direction it arrives. Beam 1 and beam 2 had to travel exactly the same distance $L/2$ with exactly the same speed $c$. If the beams were emitted simultaneously I MUST have seen the two flashes at the same time. But I didn't....beam 1 arrived before beam2, and so event A must have happened before event B

So beam 1 did indeed hit observer P before beam 2. And indeed, observer P has got a point: From observer P the two lightnings could not have occurred at the same time. Asking observer O one more time he says that he is absolutely certain that the two lightnings struck simultaneously. Who is right?

We have arrived at one of the main conclusions that Einstein reached when he was discovering the theory of relativity: *simultaneity is relative.* If two events happen at the same time or not depends on who you ask. It depends on your frame of reference. In the example above, the two lightnings were simultaneous for the observer at rest on the ground, but not for the observer moving with velocity $v$. This has nothing to do with the movement of the light beams, it is simply time itself which is different as seen from two different frames of reference. Simultaneity is a relative term in exactly the same way as velocity is: When you say that two events are simultaneous you need to specify that they are simultaneous with respect to some frame of reference.

In order to arrive at the conclusion of the relativity of simultaneity, Einstein excluded an alternative: Couldn't it be that the laws of physics are different in different frames of reference? If the laws of physics in the train were different from those in the laboratory frame, then simultaneity could still be absolute. The problem then is that we need to ask the question 'Physics is different in frames which move with respect to *which* frame of reference?'. In order to ask this question, velocity would need to be absolute. If velocity is relative, then we can just exchange the roles: The observer in the train is at rest and the observer on the ground is moving. Then we would need to change the laws of physics for the observer on the ground. This would lead to contradictions. In order to arrive at the theory of relativity, Einstein postulated the *Principle of Relativity* . The principle of relativity states that all laws of physics, both the mathematical form of these laws as well as the physical constants, are the same in all *free float frames*. In the lectures on general relativity we will come back to a more precise definition of the free float frame. For the moment we will take a free float frame to be a frame which is not

accelerated, i.e. a frame in which we do not experience fictive forces. You can deduce the laws of physics in one free float frame and apply these in any other free float frame. Imagine two space ships, one is moving with the velocity $v = 1/2c$ with respect to the other. If you close all windows in these spaceship there is no way, by performing experiments inside these spaceships, that you can tell which is which. All free float frames are equivalent, there is no way to tell which one is at rest and which one is moving. Each observer in a free float frame can define himself to be at rest.
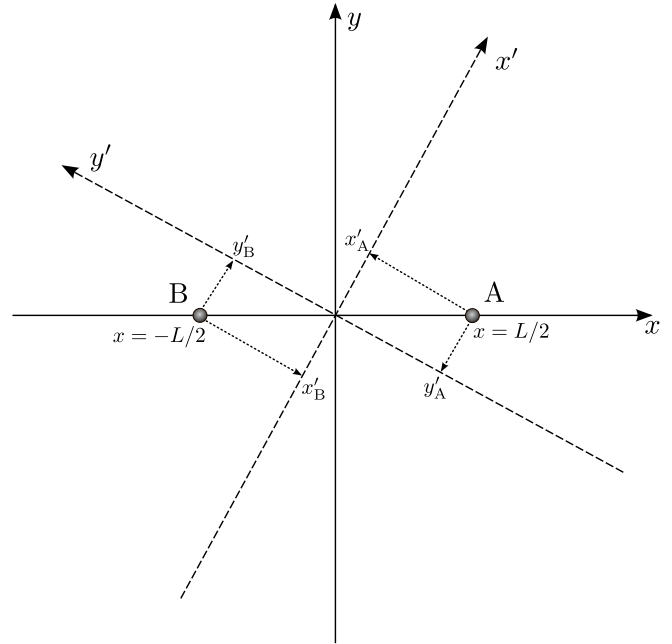


Figure 5: The position of two points A and B measured in two different coordinate systems rotated with respect to each other.

# 2 Invariance of the spacetime interval

We have seen that two events which are simultaneous in one frame of reference are not simultaneous in another frame. We may conclude that time itself is relative. In the same way as we needed two coordinate systems $x$ and $x'$ to specify the position in space relative to two different frames, we need two time coordinates $t$ and $t'$ to specify the time of an event as seen from two different frames. We are used to think of time as a quantity which has the same value for all observers but we now realize that each frame of reference has its own measure of time. Clocks are not running at the same pace in all frames of reference. Observers which are moving with respect to each other will measure different time intervals between the same events. Time is not absolute and for this reason simultaneity is not absolute.

Look at figure 5. It shows two points $A$ and $B$ and two coordinate systems $(x, y)$ and $(x', y')$ rotated with respect to each other. The two points A and B are situated at a distance $\Delta x_{AB} = L$ and at the same y-coordinate $\Delta y_{AB} = 0$ in the $(x, y)$ system. In the rotated $(x', y')$ system however, there is a non-zero difference in the y-coordinate, $\Delta y_{AB} \neq 0$. Now, replace $y$ with $t$. Do you see the analogy with the example of the train above?

If we replace $y$ with $t$ and $y'$ with $t'$, then the two points A and B are the events A and B in *spacetime*. Our diagram is now a *spacetime diagram* showing the position of events in space $x$ and time $t$, rather than a coordinate system showing the position of a point in space $(x, y)$. Consider the two coordinate systems $(x, t)$ and $(x', t')$ as measurements in two different frames of reference, the lab frame and the frame of observer P. We see that in the $(x, t)$ system, the two events are simultaneous $\Delta t_{AB} = 0$ whereas in the $(x', t')$ system, the events take place at two different points in time.

We are now entering deep into the heart of the special theory of relativity: We need to consider time as the *fourth dimension*. And moreover, we need to treat this fourth dimension similar (but not identical) to the three spatial dimen-

sions. That is, we need to talk about distances in space and distances in time. But, you might object, we measure distances in space in meters and time intervals in seconds. Can they really be similar? Yes they can, and you will soon get rid of the bad habit of measuring space and time in different units. From now on you will either measure both space *and* time in meters, or both time *and* space in seconds. By the time you have finished this course you will, without thinking about it, ask the lecturer how many meters the exam lasts or complain to your friends about how small your room in the dormitory is, giving them the size in square seconds.

How do you convert from meters to seconds and vice versa? The conversion factor is given by the universal factor $c$, the speed of light. If you have a time interval measured in seconds, multiply it by $c$ and you have the time interval in meters. If you have a distance in space measured in meters, divide it by $c$ and you obtain the distance measured in seconds:

$$x = ct, \quad t = x/c.$$

From now on we will drop the factor $c$ and suppose that distances in space and time are measured in the same units. When you put numbers in your equations you need to take care that you always add quantities with the same units, if you need to add two quantities with different units, the conversion factor is always a power of $c$.

Measuring time in meters might seem strange, but physically you can think about it this way: Since the conversion factor is the speed of light, a time interval measured in meters is simply the distance that light travels in the given time interval. If the time interval between two events is 2 meters, it means that the time interval between these events equals the time it takes for light to travel 2 meters. We might say that the time interval between these events is 2 meters of light travel time. Similarly for measuring distances in seconds: If the spatial distance between two events is 10 seconds, it means that the distance equals the distance that light travels in 10 seconds. The distance is 10 light seconds. Actually you are already accustomed to measure distances in time units: You say that a star is 4 light years away, meaning

that the distance equals the distance that light travels in four years. Note also one more effect of measuring space and time in the same units: Velocities will be dimensionless. Velocity is simply distance divided by time, if both are measured in meters, velocity becomes dimensionless. We can write this as $v_{\text{dimensionless}} = dx/(cdt) = v/c$ (to convert $dt$ to units of length we need to multiply it by $c$, thus $cdt$). If the velocity $v = dx/dt = c$ is just the speed of light, we get $v_{\text{dimensionless}} = 1$. From now on we will just write $v$ for $v_{\text{dimensionless}}$. Note that some books use $\beta$ to denote dimensionless velocity, here we will use $v$ since we will always use dimensionless velocities when working with the theory of relativity. The absolute value of velocity $v$ is now a factor in the range $v = [0, 1]$ being the velocity relative to the velocity of light.

This was the first step in order to understand the foundations of special relativity. Here comes the second: Let us, for a moment, return to the spatial coordinate systems $(x, y)$ and $(x', y')$ in figure 5. Clearly the coordinates of the points $A$ and $B$ are different in the two coordinate systems. But there is one thing which is identical in all coordinate systems: The distance between points A and B. If we call this distance $\Delta s_{AB}$ we can write this distance in the two coordinate systems as

$$\begin{aligned}
(\Delta s_{AB})^2 &= (\Delta x_{AB})^2 + (\Delta y_{AB})^2 \\
(\Delta s'_{AB})^2 &= (\Delta x'_{AB})^2 + (\Delta y'_{AB})^2
\end{aligned}$$

(check that you understand why!). The distance between A and B has to be equal in the two coordinate systems, so

$$(\Delta s_{AB})^2 = (\Delta s'_{AB})^2.$$

Is this also the case in spacetime? Can we measure intervals between events in spacetime? This is now, at least in theory, possible since we measure space and time separations in the same units. In a spatial $(x, y, z)$ system we know the geometrical relation,

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2,$$

from Euclidean geometry: The square of the distance between two points (called the *line element*) is simply the sum of the squares of the coordinate

distances between these two points. But do the rules of Euclidean geometry apply to spacetime? No, not entirely. The geometry of spacetime is called *Lorentz geometry*. The distance between two events (line element) in Lorentz spacetime $\Delta s^2$, is given by

> **The spacetime interval**
>
> $$(\Delta s)^2 = (\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2).$$

Note the minus sign. This minus sign is the only thing which distinguishes space from time. The square of the spacetime distance between two events equals the square of the time separation between these events *minus* the square of the spatial separations between the events. And in the same way as the distance between two points in space is the same in all coordinate systems, the distance in spacetime, *the spacetime interval* is the same in all frames of reference. We say that the spacetime interval is *invariant*. A quantity is invariant if it has the same value in all frames of reference. We already know another invariant quantity: the speed of light.

So, that was it. We're done. Now you know what the special theory of relativity is all about. Congratulations! You now see that we may write the special theory of relativity in two sentences: Measuring space and time intervals in the same units, you can calculate the spacetime interval between two events using the formula for the line element in Lorentz geometry. This spacetime interval between two events is invariant, it has the same value as measured from all frames of reference. We will now see what this means in practice. But before you continue, take a walk, go for a coffee or simply take half an hour in fresh air. Your brain will need time to get accustomed to this new concept.

## 3 An example

A train is moving along the x-axis of the laboratory frame. The coordinate system of the laboratory frame is $(x, y)$ and of the train, $(x', y')$. In the train a light signal is emitted directly upwards along the y-axis (event A). Three meters above,

it is reflected in a mirror (event B) and finally returns to the point where it was emitted (event C). In the train frame it takes the light beam 3 meters of time to reach the mirror and 3 meters of time to return to the point where it was emitted. The total up-down trip (event A to event C) took 6 meters of time in the frame of the train (light travels with a speed of $v = 1$, one meter per meter of light travel time). From event A to event C, the train had moved 8 meters along the x-axis in the laboratory frame. Because of the movement of the train, the light beam moved in a pattern as shown in figure 6 seen from the lab frame.
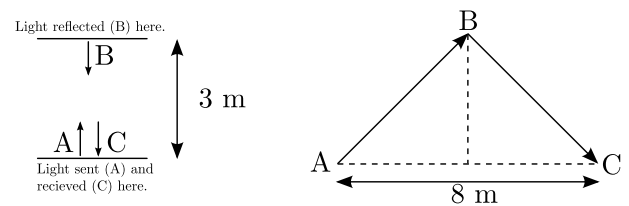


Figure 6: The light emitted (event A) upwards in the train is reflected (event B) and received (event C) at the same place (in the train frame) as it was emitted.

1. *Use the figure to find the total distance d traveled by the light beam in the laboratory frame.* Dividing the triangle into two smaller triangles (see the figure), we find from one triangle that the distance traveled from the emission of the light beam to the mirror is $d/2 = \sqrt{(4 \text{ m})^2 + (3 \text{ m})^2} = 5$ m and similarly for the return path. Thus, the total distance traveled by the light beam from event A to event C is $d = 10$ m.

2. *What was the total time it took for the light beam from event A to event C in the laboratory frame?* We have just seen that in the laboratory frame, the light beam traveled 10 meters from event A to event C. Since light travels at the speed of one meter per meter of time, it took 10 meters of time from event A to event C. In the frame of the train, it took only 6 meters of time.

3. *What is the speed of the train?* The train moved 8 meters in 10 meters of time, so the speed is $v = 8/10 = 4/5$, 4/5 the speed of light.

4. *What is the spacetime interval $\Delta s'$ between event A and event C with respect to the train*

*frame?* In the train frame, event A and event C happened at the same point, so $\Delta x' = 0$. It took 6 meters of time from event A to event C, so $\Delta t' = 6$ m. The spacetime interval is thus $\Delta s' = \sqrt{(6 \text{ m})^2 - 0} = 6$ m. (check that you also got this result!)

5. *What is the spacetime interval $\Delta s$ between event A and event C with respect to the laboratory frame?* In the laboratory frame, the distance between the events were $\Delta x = 8$ m and the time interval was $\Delta t = 10$ m. The spacetime interval is thus $\Delta s = \sqrt{(10 \text{ m})^2 - (8 \text{ m})^2} = 6$ m (check that you also got this result!), exactly the same as $\Delta s'$ in the train frame.

6. *Was there an easier way to answer the previous question?* Oh... uhm, yes, you're right, the spacetime interval is the same in all frames of reference so I should immediately had answered $\Delta s = \Delta s' = 6$ m without any calculation... much easier!

Indeed much easier... remember that this will be very useful when calculating distances and intervals with respect to frames moving close to the speed of light.

# 4 Observer O and P revisited

Armed with the knowledge of the invariance of the spacetime interval we now return to observer O and P in order to sort out exactly what happened for each of the observers. We know that with respect to the laboratory frame, the two lightnings struck simultaneously (events A and B were simultaneous) at points $x = \pm L/2$ at the time $t = 0$ when observer P was at the origin $x_P = 0$. But at what time did the two lightnings strike with respect to observer P in the train? We have learned that with respect to the frame of reference following the train, the events A and B were *not* simultaneous. But in the reference frame of observer P, at what time $t'_A$ and $t'_B$ did the two lightnings strike? The two observers exchange a signal at $t = 0$ such that their clocks are both synchronized to $t = t' = 0$ at the instant when observer P is at the origin in both coordinate systems $x_P = x'_P = 0$. Did event A and B happen

before or after $t' = 0$ on observer P's wristwatch? (It is common to talk about *wristwatches* when referring to the time measured in the rest frame of a moving object, i.e. the time measured by observers moving with the object. This wristwatch time is also called *proper time*).

We know that an event is characterized by a position $x$ and a time $t$ in each of the frames of reference. Let's collect what we know about the position and time of event A, B and the event when observer P passes $x = x' = 0$ which we call event P:

**Event P:**

$$x = 0 \quad t = 0$$
$$x' = 0 \quad t' = 0$$

**Event A:**

$$x = L/2 \quad t = 0$$
$$x' = L_0/2 \quad t' = t'_A$$

**Event B:**

$$x = -L/2 \quad t = 0$$
$$x' = -L_0/2 \quad t' = t'_B$$

Note that the length of the train is $L_0$ for observer P and $L$ for observer O. We have already seen that observers in different frames of reference only agree on the length of the spacetime interval, *not* on lengths in space or intervals in time separately. For this reason, we do expect $L$ and $L_0$ to be different. Look also at figure 5, the distance $\Delta x_{AB}$ between the points A and B differ between the two coordinate systems, in the system $(x, y)$ it is $\Delta x_{AB} = L$, but in the system $(x', y')$ it is $\Delta x'_{AB} = x'_B - x'_A \equiv L'$. The length of the train in the rest frame of the train, $L_0$, is called *the proper length*. We will later come back to why it is given a particular name.

We want to find at which time $t'_A$ and $t'_B$ observed from the wristwatch of observer P, did events A and B happen? Did they happen before or after event P? For observer O all these events were simultaneous at $t = 0$, the moment in which the two observers exchanged a signal to synchronize their clocks. For observer P, could these events

possibly had happened before they happened for observer O? Or did they happen later than for observer O?

In order to solve such problems, we need to take advantage of the fact that we know that the spacetime interval between events is invariant. Let's start with the spacetime interval between events A and B.

**Spacetime interval AB:** From each of the frames of reference it can be written as

$$\Delta s^2_{AB} = \Delta t^2_{AB} - \Delta x^2_{AB},$$
$$\Delta(s'_{AB})^2 = (\Delta t'_{AB})^2 - (\Delta x'_{AB})^2.$$

(note that the $y$ and $z$ coordinates are always 0, so $\Delta y = \Delta y' = 0$ and $\Delta z = \Delta z' = 0$). In order to calculate the spacetime interval, we need the space and time intervals $\Delta x^2_{AB}$, $\Delta t^2_{AB}$, $(\Delta x'_{AB})^2$ and $(\Delta t'_{AB})^2$ separately. In both frames, the spatial distance between the two events equals the length of the train in the given frame of reference. So $\Delta x_{AB} = L$ and $\Delta x'_{AB} = L_0$. For observer O the events were simultaneous $\Delta t_{AB} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AB} = t'_A - t'_B$. Setting the two expressions for the spacetime interval equal we obtain,

$$L^2 = L_0^2 - (t'_A - t'_B)^2. \tag{3}$$

(check that you obtain this as well!). We have arrived at one equation connecting observables in one frame with observables in the other. We need more equations to solve for $t'_A$ and $t'_B$. Let's study the spacetime interval between events A and P.

**Spacetime interval AP:** From each of the frames of reference it can be written as

$$\Delta s^2_{AP} = \Delta t^2_{AP} - \Delta x^2_{AP}$$
$$\Delta(s'_{AP})^2 = (\Delta t'_{AP})^2 - (\Delta x'_{AP})^2$$

In order to calculate the spacetime interval, we need the space and time intervals $\Delta x^2_{AP}$, $\Delta t^2_{AP}$, $(\Delta x'_{AP})^2$ and $(\Delta t'_{AP})^2$ separately. In both frames, the spatial distance between the two events equals half the length of the train in the given frame of reference. So $\Delta x_{AP} = L/2$ and $\Delta x'_{AP} = L_0/2$. For observer O the events were simultaneous $\Delta t_{AP} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AP} = t'_A - 0 = t'_A$. Setting the two expressions for the spacetime interval equal we obtain,

$$(L/2)^2 = (L_0/2)^2 - (t'_A)^2. \tag{4}$$

Note that we have three unknowns, $t'_A$, $t'_B$ and $L$. We need one more equation and therefore one more spacetime interval. The spacetime interval between B and P does not give any additional information, so we need to find one more event in order to find one more spacetime interval. We will use event C, the event that beam 1 hits observer P.

**Spacetime interval CP:** Again, we need

$$\Delta s^2_{CP} = \Delta t^2_{CP} - \Delta x^2_{CP},$$
$$\Delta(s'_{CP})^2 = (\Delta t'_{CP})^2 - (\Delta x'_{CP})^2.$$

In the first section we calculated the time $t_C$ when beam 1 hit observer P in the frame of observer O. The results obtained in the laboratory frame were

correct since the events A and B really were simultaneous in this frame. As we have seen, the results we got for observer P were wrong since we assumed that events A and B were simultaneous in the frame of observer P as well. Now we know that this was not the case. We have $\Delta t_{CP} = t_C - 0 = t_C = L/2/(v+1)$ (from equation 1, note that since we measure time and space in the same units $c = 1$). As event C happens at the position of observer P, we can find the position of event C by taking the position of observer P at time $t_C$ giving $\Delta x_{CP} = v\Delta t_{CP} = vL/2/(v+1)$. In the frame of observer P, event C clearly happened at the same point as event P so $\Delta x'_{CP} = 0$. The time of event C was just the time $t'_A$ of event A plus the time $L/2$ it took for the light to travel the distance $L/2$ giving $\Delta t'_{CP} = t'_A + L_0/2$. Equating the line elements we have

$$\frac{L^2/4}{(v+1)^2}(v^2 - 1) = -(t'_A + L_0/2)^2 \qquad (5)$$

Now we have three equations for the three unknowns. We eliminate $L$ from equation (5) using equation (4). This gives a second order equation in $t'_A$ with two solutions, $t'_A = -L_0/2$ or $t'_A = -vL_0/2$.

The first solution is unphysical: The time for event C is in this case $t'_C = t'_A + L_0/2 = 0$ so observer P sees the lightning at $t' = 0$. Remember that at $t = t' = 0$ observer O and observer P are synchronizing their clocks, so at this moment, and only this moment, their watch show the same time. This means that observer P sees flash A at the same moment as the lightening strikes for observer O. Thus at $t = t' = 0$, the lightning hits the front of the train for observer O, but at the same time he would see the light from the lightening hit observer P. The light from event A would therefore have moved instantaneously from the front of the train to the middle of the train.

Disregarding the unphysical solution we are left with

$$t'_A = -v\frac{L_0}{2}.$$

Thus event A happened for observers in the train before it happened for observers on the ground. Now we can insert this solution for $t'_A$ in equation 4 and obtain $L$,

$$L = L_0\sqrt{1 - v^2} \equiv L_0/\gamma, \qquad (6)$$

with $\gamma \equiv 1/\sqrt{1 - v^2}$. So the length of the train is smaller in the frame of observer O than in the rest frame of the train. We will discuss this result in detail later, first let's find $t'_B$. Substituting for $t'_A$ and $L$ in equation (3) we find

$$t'_B = v\frac{L_0}{2} = -t'_A.$$

So event B happened later for observers in the train than for observers on the ground. To summarize: Event A and B happened simultaneously at $t = t' = 0$ for observers on the ground. For observers in the train event A had already happened when they synchronize the clocks at $t = 0$, but event B happens later for the observers in the train. Note also that the time $t'_A$ and $t'_B$ are symmetric about $t' = 0$. If you look back at figure 5 we see that the analogy with two coordinate systems rotated with respect to each other is quite good: If we replace $y$ by $t$ we see that for the events which were simultaneous $\Delta y_{AB} = 0$ in the $(x, y)$ frame, event A happens before $y = 0$ and event B happens after $y = 0$ in the rotated system $(x', y')$. But we need to be careful not taking the analogy too far: The geometry of the two cases are different. The spatial $(x, y)$ diagram has Euclidean geometry whereas the spacetime diagram $(x, t)$ has Lorentz geometry. We have seen that this simply means that distances are measured differently in the two cases (one has a plus sign the other has a minus sign in the line element).

We have seen that for observer P event A happens before event P when they synchronize their clocks. But does he also see the lightning before event P? As discussed above, this would be unphysical, so this is a good consistency check:

$$t'_C = t'_A + \frac{L_0}{2} = -v\frac{L_0}{2} + L_0/2 = L_0/2(1 - v),$$

which is always positive for $v < 1$. Thus observer P sees the flash after event P. When does observer P see the second flash (event D) measured on the wristwatch of observer P? Again we have $t'_D = t'_B + L_0/2$ giving

$$t'_D = L_0/2(1 + v),$$

so the time interval between event C and D measured on the wristwatch of a passenger on the train is

$$\Delta t' = t'_D - t'_C = vL_0$$

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How long is this time interval as measured on the wristwatch of observer O? We already have $t_C$ and $t_D$ from equations (1) and (2). Using these we get the time interval measured from the ground,

$$\Delta t = v L_0 / \sqrt{1 - v^2}$$

So we can relate a time interval in the rest frame of the train with a time interval on the ground as

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2}} = \gamma \Delta t'. \tag{7}$$

Note that index CD has been skipped here since this result is much more general: It applies to any two events taking place at the position of observer P. This is easy to see. Look at figure 7. We define an observer O which is at rest in the laboratory frame using coordinates $(x, t)$ and an observer P moving with velocity $v$ with respect to observer O. In the frame of reference of observer P we use coordinates $(x', t')$.

We now look at two ticks on the wristwatch of observer P. Observer P himself measures (on his wrist watch) the time between two ticks to be $\Delta t'$ whereas observer O measures the time intervals between these two ticks on P's watch to be $\Delta t$ (measured on observer O's wrist watch). In the coordinate system of observer P, the wristwatch does not move, hence the space interval between the two events (the two ticks) is $\Delta x' = 0$. For observer O, observer P and hence his wristwatch is moving with velocity $v$. So observer O measures a space interval of $\Delta x = v \Delta t$ between the two events. The spacetime interval in these two cases becomes

$$(\Delta s)^2 = \Delta t^2 - \Delta x^2 = \Delta t^2 - (v\Delta t)^2$$
$$= (\Delta t)^2 (1 - v^2)$$
$$(\Delta s')^2 = (\Delta t')^2.$$

Spacetime intervals between events are always equal from all frames of reference so we can equate these two intervals and we obtain equation (7).
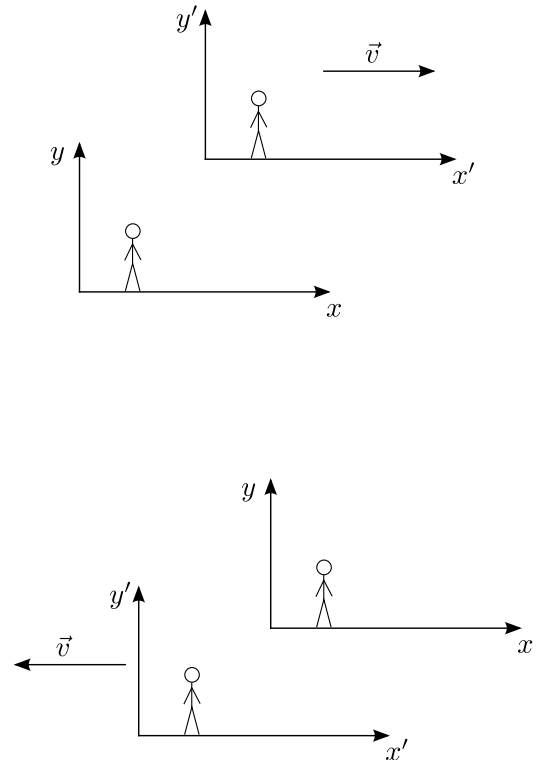


Figure 7: Two reference frames: $(x, y)$ coordinates are used for the system defined to be at rest and $(x', y')$ coordinates are used for the system defined to be moving. In the upper figure, observer O is in the laboratory frame with observer P in the frame moving with velocity $v$. In the lower figure, the two systems have exchanged roles and $v \to -v$. All equations derived in the above system will be valid for the system below by exchanging $v \to -v$.

Going back to the example with the train: If the train moves at the speed $v = 4c/5$ then we have $\Delta t = 5/3 \Delta t' \approx 1.7 \Delta t'$. When observer O on the ground watches the wristwatch of observer P, he notes that it takes 1.7 hours on his own wristwatch before one hours has passed on the wristwatch of observer P. If observer P in the train is jumping up and down every second on his own wristwatch, it takes 1.7 seconds for each jump as seen from the ground. For observers on the ground it looks like everything is in slow-motion inside the train.

How does it look for the observers in the train? Remember that velocity is relative. Being inside the train, we define ourselves as being at rest. From this frame of reference it is the ground which is moving at the speed $-v$. Everything has been exchanged: Since we now define the train to be at rest, the coordinate system $(x, t)$ is now for the train whereas the coordinate system $(x', t')$ is for

the ground which is moving at velocity $-v$ (see figure 7). Note the minus sign: The motion of the ground with respect to the train is in the opposite direction than the motion of the train with respect to the ground.

We can now follow exactly the same calculations as above for two events happening at the position of observer O instead of observer P. For instance we watch two ticks on the clock of observer O. Then we find again formula (7) but with the meaning of $\Delta t$ and $\Delta t'$ interchanges. Assuming again a speed of $v = -4c/5$ (note again the minus sign), observer P sees that it takes 1.7 hours on his wristwatch for one hour to pass on the wristwatch of observer O. It is the opposite result with respect to the above situation. While observers on the ground observe everything in the train in 'slow-motion', the observers on the train observe everything on the ground in 'slow-motion'. This is a consequence of the principle of relativity: There is no way to tell whether it is the train which is moving or the ground which is moving. We can define who is it rest and who is moving, the equations of motion that we obtain will then refer to one observer at rest and one observer in motion. When we change the definition, the roles of the observers in the equation will necessarily also change. Thus, if we define the ground to be at rest and the train to be moving and we deduce that observers on the ground will see the persons in the train in 'slow-motion', the opposite must also be true: If we define the train to be at rest and the ground to be moving, then the observers on the train will observe the observers on the ground in 'slow-motion'. Confused? Welcome to special relativity!

Consider two observers, both with their own wristwatch, one at rest in the laboratory frame (observer O) another moving with velocity $v$ with respect to the laboratory frame (observer P). Going back to equation (7) we now know that if $\Delta t'$ is the interval between two ticks on the wristwatch of observer P, then $\Delta t$ is the time interval between the same two ticks of observer P's watch measured on observer O's wristwatch. Using equation 7 we see that the shortest time interval between two ticks is always the time measured directly in the rest frame of the wristwatch producing the ticks.

Any other observer moving with respect to observer P will measure a longer time interval for the ticks on observer P's wristwatch. This is of course also valid for observer O: The shortest time interval between two ticks on observer O's wristwatch is the time that observer O himself measures. The wristwatch time is called the *proper time* and is denoted $\tau$. It is the shortest interval between these two events that can be measured.

Note that the proper time between two events (two ticks on a wristwatch) also equals the spacetime interval between these events. This is easy to see: consider again the ticks on observer P's wristwatch. In the rest frame of observer P, the wristwatch is not moving and hence the spatial distance between the two events (ticks) is $\Delta x = 0$. The time interval between these two events is just the proper time $\Delta \tau$. Consequently we have for the spacetime interval $\Delta s^2 = (\Delta t')^2 - (\Delta x')^2 = \Delta \tau^2 - 0 = \Delta \tau^2$.

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**Proper time**

$$\Delta s^2 = \Delta \tau^2$$

in the rest frame.

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Now, let's return to another result, the length of the train $L$ as measured by observer O on the ground. Again, the result in equation 6 can be shown in a similar manner to be more general. The length $L_0$ can be the length of any object in the rest frame of this object. We see from equation 6 that any observer which is not at rest with respect to the object will observe the length $L$ which is always smaller than the length $L_0$. The length of an object measured in the rest frame of the object is called the *proper length* of the object. An observer in any other reference frame will measure a smaller length of the object. The proper length $L_0$ is the longest possible length of the object. This also means that an observer in the moving train will measure the shorter length $L$ for another identical train being at rest with respect to the ground (being measured to have length $L_0$ by observers on the ground).

# 5 The Lorentz transformations

Given the spacetime position $(x, t)$ for an event in the laboratory frame, what are the corresponding coordinates $(x', t')$ in a frame moving with velocity $v$ along the x-axis with respect to the laboratory frame? So far we have found expressions to convert time intervals and distances from one frame to the other, but not coordinates. The transformation of spacetime coordinates from one frame to the other is called the Lorentz transformation. In the exercises you will deduce the expressions for the Lorentz transformations. Here we state the results. We start by the equations converting coordinates $(x', y', z', t')$ in the frame moving along the x-axis to coordinates $(x, y, z, t)$ in the laboratory frame,

**The Lorentz transformations**
$$t = v\gamma x' + \gamma t',$$
$$x = \gamma x' + v\gamma t',$$
$$y = y',$$
$$z = z'.$$

To find the inverse transformation, we have seen that we can exchange the roles of the observer at rest and the observer in motion by exchanging the coordinates and let $v \to -v$ (see figure ),

**The Lorentz transformations (cont.)**
$$t' = -v\gamma x + \gamma t,$$
$$x' = \gamma x - v\gamma t,$$
$$y' = y,$$
$$z' = z.$$

Here

$$\gamma = \frac{1}{\sqrt{1 - v^2}}.$$

# 6 List of expressions you should know by now

# 7 Exercises

## Exercise 2A.1

We have seen the effect of Lorentz contraction, namely that a stick of proper length $L_0$ (measured in the rest frame of the stick) moving at a speed $v$ along the x-axis in the laboratory frame, is measured to have a shorter length $L = L_0/\gamma$ in the laboratory frame. But what happens to the size of the stick in $y$ and $z$ directions measured from the laboratory frame? Do we correspondingly measure the stick to become thinner? We will now investigate this:
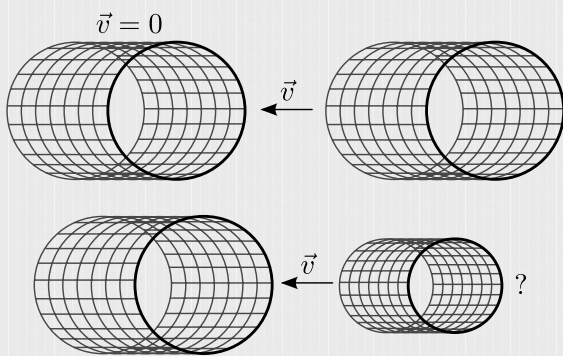


Figure 8: Does a moving cylinder become thinner as well as contracted seen from the laboratory frame? In problem 1 we study this more closely.

To check this possibility, imagine two identical cylinders A and B which are hollow such that if one cylinder becomes smaller (smaller radius) than the other, it might pass inside the larger cylinder (see figure 8). The axis of both cylinders are aligned with the x-axis at $y = z = 0$. Thus, the axis of both cylinders are exactly along the x-axis. Cylinder A is at rest in the laboratory frame, cylinder B is moving with velocity v along the x-axis, approaching cylinder A.

1. We know that the length of cylinder B as measured from the laboratory frame shrinks. Assume that the same effect takes place in the $y$ and $z$ directions such that the radius of cylinder B gets smaller measured in the laboratory frame. What happens when the two cylinders meet?

2. Now, look at exactly the same situation but from the point of view of an observer sitting on cylinder B. What happens when the two cylinders meet?

3. Can you give a good arguments to explain why $y = y'$ and $z = z'$ in the Lorentz transformations? (Note: this transformation is for movements along the x-axis. If there are movements along the y and z axes as well, the Lorentz transformation will look different and much more complicated. This is outside the scope of this course.)

## Exercise 2A.2

You need to have read sections 1 - 4 to be able to do this exercise. When high energy cosmic ray protons collide with atoms in the upper atmosphere, so-called muon particles are produced. These muon particles have a mean life time of about 2 $\mu$s ($2 \times 10^{-6}$ s) after which they decay into other types of particles. They are typically produced about 15 km above the surface of the Earth. We will now study a cosmic ray muon approaching the surface with the velocity of $0.999c$.

1. How long time does it take for a muon to arrive at the surface of the Earth as measured from the Earth frame?

2. Ignore relativistic effects: Do you expect many muons to survive to the surface of the Earth before decaying? (compare with the mean life time)

3. From relativity, we know that from the rest frame of the muon, the time it takes to reach the surface of the Earth is different. We will now use invariance of the spacetime interval to find the time it takes in the frame of the muon to reach the surface of the Earth.

   (a) Find the space and time distances $\Delta x$ and $\Delta t$ in the Earth frame and use these to obtain the spacetime interval $\Delta s$. Give all the answers in seconds.

   (b) What is $\Delta x'$, the spatial distance traveled by the muon in the muon rest frame?

   (c) Use invariance of the line element to obtain the travel time $\Delta t'$ in the muon rest frame. Will we detect muons at the surface of the Earth?

4. The diameter of the galaxy is about 100 000

light years, thus even with the speed of light it would take 100 000 years to pass the galaxy. How long time does it take to transverse the galaxy in the reference frame of a cosmic ray particle traveling at the speed of $v = 0.999999999999c$? (Use again invariance of the spacetime interval). Does this give some hope for future long distance space travel?

## Exercise 2A.3

You need to have read sections 1 - 4 to be able to do this exercise. You have devised a clock which works the following way: It consists of two mirrors a distance $L_0$ apart. A light ray is emitted along the positive x-axis at one of the ends and then reflected back and forth between the two mirrors. Each time it hits one of the mirrors it gives a 'tick'. See figure 9.
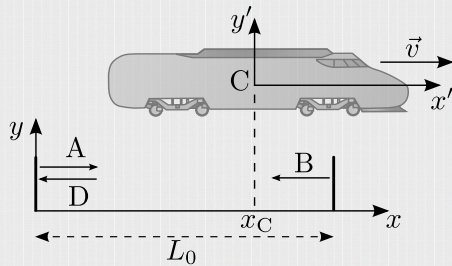


Figure 9: The situation in problem 2A.3: A light beam is emitted when $x = x' = 0$ and $t = t' = 0$ (event A). Then the beam is reflected in the right mirror (event B) and reflected again in the left mirror (event D). This picture is taken from the laboratory frame at event B $t = t_B$ (the position of event A and D are just marked, they are not happening at this moment). Event C happens at the same time as event B in the laboratory frame. The position of event C in the laboratory frame is the position $x = x_C$ of the origin of the train frame.

1. How long does it take between each tick in the reference frame of the clock?

2. Now we observe the clock from a passing train. The clock is at rest in the laboratory frame with coordinates $(x, t)$ and we observe it from the train moving with velocity $v$ along the positive x-axis of the laboratory frame. We use coordinates $(x', t')$ for the train frame (see figure 9). Event A is the emission of light at the left mirror. This is the reference event occurring at $x = x' = 0$

and $t = t' = 0$. Event B is when the light ray hits the opposite mirror. We also introduce event C which takes place at the position of the middle point of the train (where $x' = 0$) at the same time as event B seen from the laboratory frame. We want to find out how long time $\Delta t'_{AB}$ it took for the light beam to reach the right mirror in the train frame. Write a list of events A, B and C and write the position $(x, t)$ and $(x', t')$ in the two frames for all three events. The only unknowns here are $x'_B$, $t'_B$ and $t'_C$. All the other coordinates should be expressed in terms of the known quantities, $L_0$ and $v$.

3. Write the spacetime intervals $\Delta s_{AB}$ and $\Delta s'_{AB}$ between events A and B in the two frames. Show that invariance of the interval gives $x'_B = t'_B$. Could you have guessed this using physical arguments without any calculations?

4. Write the spacetime intervals $\Delta s_{AC}$ and $\Delta s'_{AC}$ between events A and C in the two frames. Show that invariance of the interval gives $t'_C = L_0/\gamma$.

5. Write the spacetime intervals $\Delta s_{BC}$ and $\Delta s'_{BC}$ between events B and C in the two frames. Show that invariance of the interval gives $t'_B = L_0\gamma(1 - v)$.

6. Now define event D which is when the light ray returns to the first mirror at $x = 0$. Use invariance of the spacetime interval for appropriate events to find at what time $t'_D$ event D happened in the train frame.

7. In the frame of the train, how long time did it take from the light was emitted to the first 'tick'? And how long time did it take from the first tick to the second tick? Compare this to the results in the lab frame. Is this a useful clock in the frame of reference of the train?

## Exercise 2A.4

You need to have read sections 1 - 4 to be able to do this exercise. Quasars are one of the most powerful sources of energy in the universe. They are smaller than galaxies, but emit about 100 times as much energy as a normal galaxy. The engine

in a quasar is believed to be a black hole. Jets of plasma are ejected into space from areas close to the black hole.
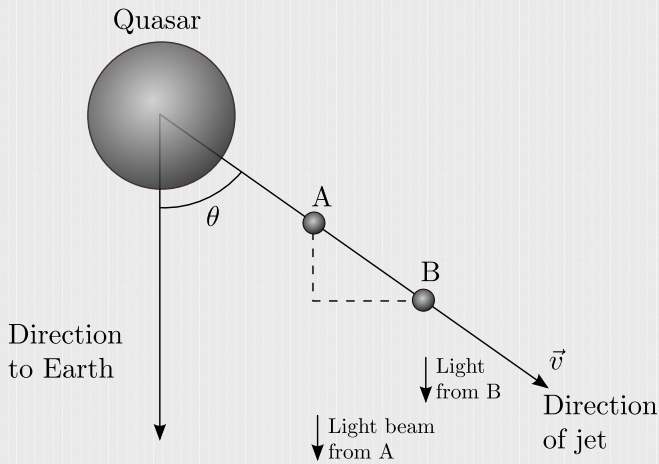


Figure 10: The quasar ejecting matter at an angle $\theta$ with the line of sight. The speed of the ejected matter is $v$. We define two events A and B which are the emission of photons from the ejected matter at the points A and B. At event A, the ejected matter passes point A and emits photons towards Earth. Three years later, the ejected matter passes point B and again emits photons.

1. In a Quasar called 3C273 at a distance of $2.6 \times 10^9$ light years from Earth, such a jet was observed during a period of three years. During this period it was found to have moved an angular distance of $2 \times 10^{-3}$ arc seconds transversally on the sky. Show that the physical speed of the jet was $v = 8.4c$, more than eight times the speed of light.

2. We will now look at the physics of this process in order to understand what is going on. In figure 10 you can see the jet and two events A and B which are the events that photons were emitted as the jet moved through space. The photons emitted in event B were observed three years later than the photons emitted in point A. Here $v$ is the real physical speed of the jet and $\theta$ is the angle between the direction of the jet and the line of sight. Show that the time interval $\Delta t_{\text{observed}}$ between the reception of photons (observations) from these two events is

$$\Delta t_{\text{observed}} = \Delta t (1 - v \cos \theta),$$

where $\Delta t$ is the real time interval (in the Earth frame) between these two events.

**Hint:** No theory of relativity is needed in this calculation, all quantities you need are taken in the same frame of reference.

3. Show that the apparent transversal speed of the jet can be written as

$$v_{\text{observed}} = \frac{v \sin \theta}{1 - v \cos \theta}.$$

4. Assume that $\theta = 45°$. For which range of real speeds $v$ do we observe an apparent speed $v_{\text{observed}}$ which is larger than the speed of light?

5. The theory of relativity says that no signal can travel faster than the speed of light. Is this principle violated?

The effect we have seen here, an apparent speed of an object which exceeds the speed of light, is called *superluminal motion*.

## Exercise 2A.5

You need to have read sections 1 - 4 to be able to do this exercise. In this exercise we will deduce the Lorentz transformations. We start by noting that the transformation equations must be linear in $x$ and $t$. This is because the inverse transformation needs to have the same form as the original transformation by the principle of relativity: We can exchange the definition of who is at rest and who is moving only if the transformation is linear such that if $x \propto x'$ then $x' \propto x$. For instance if we had a coordinate transformation $x \propto (x')^2$, the inverse transformation would read $x' \propto \sqrt{x}$. These two equations would be completely different and the principle of relativity would be violated: The two observers would have completely different equations for transforming from one system to the other. Thus we can write the Lorentz transformations on the form

$$t = f(v)x' + g(v)t', \tag{8}$$
$$x = h(v)x' + k(v)t', \tag{9}$$
$$y = y',$$
$$z = z',$$

where $f(v)$, $g(v)$, $h(v)$ and $k(v)$ are unknown functions of $v$. Note that the motion is along the x-axis, so no transformation is needed for the other two spatial dimensions. And again, by

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the principle of relativity, the inverse transformation must be obtained by exchanging the roles of the observers $(x, y) \leftrightarrow (x', y')$ and the velocity $v \rightarrow -v$ (see again figure 7),

$$t' = f(-v)x + g(-v)t, \qquad (10)$$
$$x' = h(-v)x + k(-v)t, \qquad (11)$$
$$y' = y,$$
$$z' = z.$$

We need to solve for our unknown functions of $v$, namely $f(v)$, $g(v)$, $h(v)$ and $k(v)$.

1. Consider two events A and B. Event A happens at $x = x' = 0$ at $t = t' = 0$. Event B happens at $(x, t)$ in the laboratory frame and at the origin $x' = 0$ at time $t'$ in the moving frame (which moves with velocity $v$ with respect to the laboratory frame). Write the time intervals $\Delta t_{AB}$ and $\Delta t'_{AB}$ in terms of the coordinates $x, t, x', t'$. Then use equation (7) to find a relation between $t$ and $t'$. You see that this relation already resembles one of equations (8)–(11) with one term missing. Look at at your coordinates and compare with the equations (8)–(11) and you will realize that the missing term vanishes. Show that
$$g(v) = \gamma.$$

2. We will still study the same two events. At what position $x$ in the laboratory frame does event B happen? Express the answer in terms of $t$ and $v$. Then use the previous result to elimintate $t$ and write this in terms of $t'$ and $v$. This gives you a relation between $x$ and $t'$. You would need either an $x'$ or $t$ to obtain one of the relations above (equations 8–11), but show that one of these vanishes. Then show that
$$k(v) = v\gamma.$$

3. We will now study two different events A and B. Event A is again $x = x' = 0$ and $t = t' = 0$. But event B now happens at the position $x' = L_0$ in the moving frame and $x = L$ in the laboratory frame. In the laboratory frame, the two events happen at the same time. Use equation 6 to obtain a relation between $x$ and $x'$. Look again at the

Lorentz transformation equations (equations 8–11): Your expression needs either a $t$ or a $t'$ but one of these vanishes. You can thus conclude that
$$h(-v) = \gamma = h(v)$$

4. Now we are only missing $f(v)$ in order to have deduced the full Lorentz transformations. Consider two other events A and B: Event A is again for $x = x' = 0$ at $t = t' = 0$ and event B is at position $(x, t)$ in the laboratory frame and $(x', t')$ in the moving frame. Use equations (8)–(9) to show that the spacetime interval between A and B for the two frames can be written

$$\Delta s^2 = (f(v)x' + \gamma t')^2 - (\gamma x' + v\gamma t')^2$$
$$(\Delta s')^2 = (t')^2 - (x')^2$$

Show that invariance of the spacetime interval gives
$$f(v) = \gamma v.$$

The Lorentz transformations have been deduced.

### Exercise 2A.6

You need to read section 5 before doing this exercise. We will now return to the clock in exercise 2A.3 and solve this using the Lorentz transformations instead of the spacetime interval. We want to find at what time $t'_B$ does the light hit the right mirror and at what time $t'_D$ it has returned to the left mirror. Using the Lorentz transformations we will only need events A, B and D.

1. Again, write up the coordinates $(x, t)$ and $(x', t')$ for these three events. The following are unknown: $x'_B$, $t'_B$, $x'_D$ and $t'_D$.

2. Use the Lorentz transformations to find $t'_B$ and $t'_D$. You do not need to find $x'_B$ and $x'_D$.

3. Use the Lorentz transformations to find the time (in the train frame) of the next two ticks of the clock. Are the intervals consistent with the first two ticks?

### Exercise 2A.7

We will finish this lecture by studying the twin paradox in detail. This long and detailed exercise

is very important to gain some basic understanding for the underlying physics of many of the so-called paradoxes in the theory of relativity. You need to have read sections 1 - 5 to be able to do this exercise.

You are an astronaut traveling to the star Rigel, 800 light years from Earth. You start at $x = x' = 0$ and $t = t' = 0$ where $(x, t)$ are Earth frame coordinates and $(x', t')$ are spaceship coordinates. You travel in your spaceship at a velocity of $v = 0.99995$. We assume that Earth and Rigel do not move with respect to each other and that they therefore are in the same frame of reference.

1. Event A is you departing from Earth. Event B is you arriving at Rigel. In the Earth frame it took $800/0.99995 \approx 800.04$ years to arrive at Rigel. We know that for you it took a factor $\gamma = 1/\sqrt{1 - v^2}$ less ($\Delta t = \gamma \Delta t'$, $\Delta t$ is measured in Earth frame, $\Delta t'$ is measured in spaceship frame). How long time did it take for you (on your wristwatch) to arrive at Rigel?

2. After arriving at Rigel, you make the necessary scientific measurements (which takes very little time and can therefore be ignored) and start the return flight. You fly with exactly the same speed $v = 0.99995$ towards Earth. Event C is when you arrive back on Earth. Use the same arguments (or symmetry arguments) to find the time $\Delta t$ and $\Delta t'$ it took from Rigel and back to Earth in the two frames of reference.

3. If you have done your calculations correct, here is a summary of the situation: In the Earth frame, it took you 1600.08 years to go to Rigel and return. On your wristwatch it took you 16 years to go to Rigel and back. So while hundreds of generations have passed on Earth, you return only 16 years older.

4. We will now make the same calculations again, but just switch the frames: The laboratory frame $(x, t)$ is now the frame of the spaceship and the moving frame $(x', t')$ is the Earth frame. Because of the principle of relativity we are allowed to switch the roles and we should arrive at exactly the same result using the same laws of physics. From this point of view, this is what is happening: You sit in you spaceship which is now the laboratory frame defined to be at rest and at $x = x' = 0$ at $t = t' = 0$ (event A), the Earth starts moving away from you with velocity $v = 0.99995$ and Rigel starts approaching you with the same velocity. After a time $\Delta t$ Rigel arrives at your position (event B). We know from previous calculations that the trip took 8 years in your frame of reference which is now the laboratory frame. Using again that $\Delta t = \gamma \Delta t'$ (and make sure not to confuse $\Delta t$ and $\Delta t'$) show that the clocks on Earth at the moment when Rigel arrives at your position show 0.08 years. Only 0.08 years had passed on Earth during the 8 year (on your watch) trip to Rigel.

5. Now, this might look like a paradox, but we will show further down that it is not. No matter how strange this might sound, it is consistent. The paradox is still to come. After making your investigations of Rigel, Rigel departs and Earth approaches you again at the speed of $v = 0.99995$. Making the same calculations again you will find that it takes the Earth 8 years to return to you. Let's again check carefully how long it takes on the Earth clocks for Earth to return at your position: At the moment you have finished your investigations, the Earth clocks show $t' = 0.08$ years and your clock shows $t = 8$ years. It takes Earth again $\Delta t = 8$ years to arrive at your position. We have as always that $\Delta t = \gamma \Delta t'$. How long did it take for Earth to return to your position measured on Earth clocks?

6. If you made the last calculation correct, this is now the situation: It took you 16 years from Earth departed to Earth returned. However, on Earth clocks it took 0.16 years. So while you are 16 years older, only two months have passed on Earth. Above we found that 1600 years had passed on Earth. Now, this is a paradox!

7. Clearly we made an error somewhere in the calculations. Or maybe we simply forgot some basic principles from special relativity? It appears at first sight that the two

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roles are equal, that we can choose whether we consider the Earth frame as the laboratory frame or the spaceship frame as the laboratory frame. But are the two roles really identical? What is the difference between the two observers, the Earth observer and the spaceship observer?

8. Don't read on until you have found an answer to the previous question. Here comes the solution: The difference is that whereas the Earth observer always stays in the same frame of reference, the spaceship observer changes frames of reference: The spaceship needs to accelerate at Rigel in order to change direction and return to Earth. The Earth does not undergo such an acceleration. The expression $\Delta t = \gamma \Delta t'$ was derived for constant velocity (look back at its derivation). It is not valid when the velocity is changing. In order to solve this problem properly one needs to either use general relativity which deals with accelerations or we can view the acceleration as an infinite number of different free float frames, frames with constant velocity, and apply special relativity to each of these frames. We will not do the exact calculation here, but we will do some considerations giving you some more understanding of what is happening. We will now consider three frames of reference. The Earth frame $(x, t)$, the outgoing spaceship frame $(x', t')$ and the returning spaceship frame $(x'', t'')$. Instead of spaceships we will look at it as elevators going between Earth and Rigel. There are boxes going in both directions. At $x = x' = 0$ and $t = t' = 0$ you jump into one of these boxes leaving for Rigel. There are other observers in other boxes before you and after you. The situation is depicted in figure 11. In the following use the Lorentz transformations to transform between the coordinate systems. We write the distance between Earth and Rigel in the Earth frame as $L_0$. Event A happens at $x_A = x'_A = 0$ and $t_A = t'_A = 0$. Event B is again the moment when you arrive at Rigel. At what time $t_B$ in the Earth frame do you arrive at Rigel? (express the answer in terms of $L_0$ and $v$)



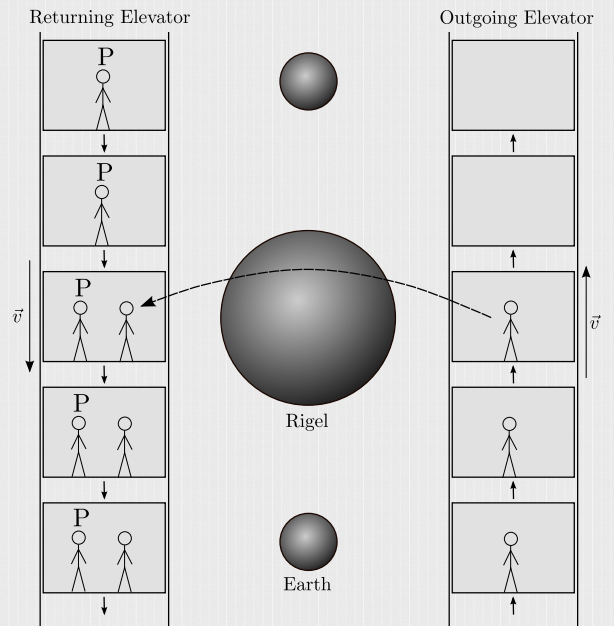Figure 11: The elevator between Earth and Rigel.

9. Use the Lorentz transformations to find $t'_B$, the time on your wristwatch when you arrive at Rigel. Insert numbers and check that you still find that the trip takes 8 years for you.

10. We now define event B'. At the same time as you arrive at Rigel (in your frame of reference which is now the frame of the outgoing elevator), another observer in another box in your elevator (thus another observer in your frame of reference using clocks synchronized with yours) passes the Earth at position $x_{B'} = 0$. Event B' is the event that he looks out and checks what time it is on Earth. So event B' takes place at the position of the Earth, but at the same time as you arrive at Rigel (same time in the outgoing reference frame). Show that the time $t_{B'}$ he reads on the Earth clocks can be written as $t_{B'} = L_0/v - vL_0$. Insert numbers. **Hint:** You first need to find the position $x'_{B'}$ of event B' in the outgoing elevator frame.

11. Insert numbers in your previous result. Explain the result which we found earlier when using the spaceship as the laboratory frame: Namely that when Rigel arrived at the spaceship, we calculated that on the Earth clocks only 0.08 years had passed. Why is this not a surprise? Those who were surprised earlier, do you know understand

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which error you made when you got surprised? Which basic principle of relativity had you forgotten?

12. We learned in the previous question that even if the Earth clocks were observed at the same moment as the spaceship/elevator arrived at Rigel (in the outgoing frame), these two events (the observation of the Earth clocks and the arrival at Rigel) were *not* simultaneous in the Earth frame. For you, sitting in the outgoing elevator, only 0.08 years have passed on Earth when you arrive at Rigel. For observers on the Earth on the other hand, you arrived at Rigel when 800 years had passed. At Rigel you meet a box in the returning elevator. You jump over to the box in the returning elevator at event B where you meet person P who has been traveling in the elevator from far away. Actually, at the same time (in the Earth frame) as you started your journey from Earth, person P started his journey from the other side of Rigel. We call the event when person P started his journey for event D. Event A and event D are simultaneous in the Earth frame. In order for you and person P to meet at event B, person P must have started on a planet a distance $2L_0$ from Earth (a distance $L_0$ from Rigel) as measured in the Earth frame. In that way you both cover a distance $L_0$ with the same speed $v$ and therefore you can both meet at Rigel at time $L_0/v$ as measured on Earth clocks. We call the coordinate system of the returning elevator $(x'', t'')$. The clocks in the system of the returning elevator are set to zero at the moment when person P starts his journey. In the following, we will use spacetime intervals instead of the Lorentz transformation: The reason for this is that the returning elevator is not synchronized with the Earth frame at $x = 0, t = 0$. This was assumed when we deduced the form of the Lorentz transformation which we use in this course. Therefore, we will now again use invariance of the spacetime interval to obtain our answers. We will first check what the wristwatch of person P shows when he meets you at event B. In analogy to your own travel, it should in-

tuitively show the same as your wristwatch: Both of you started at $t = 0$ on Earth clocks as well as on your own wristwatch. Both of you travel a distance $L_0$ (as measured in the Earth frame) at velocity $v$. But we have learned not to trust our intuition when working with relativity, so let's check. We will now consider the spacetime interval $\Delta s_{BD}$ in order to find $t''_B$, the time on the wristwatch of person P at event B. Write down the space and time intervals $\Delta x_{BD}$, $\Delta t_{BD}$, $\Delta x''_{BD}$ and $\Delta t''_{BD}$. Show that invariance of the spacetime interval gives

$$\frac{L_0^2}{v^2} - L_0^2 = (t''_B)^2,$$

which gives $t''_B = L_0/(v\gamma)$, exactly as we thought. Your wristwatches agree at event B. Reassuring to see that our intiution still gives som reasonable results every now and then.

13. We will now try to find out what the time is on Earth for persons in the returning elevator. In the frame of the outgoing elevator, we used a person who was situated in an elevator box at the same position of the Earth and looked out at the clocks on Earth exactly at the same time as event B happened (in the frame of the outgoing elevator). We called this event B' (looking at the clocks on Earth). We found that only 0.08 years had passed on Earth when you arrived at Rigel. We will now make the same check from the returning elevator. A person in an elevator box of the returning elevator being at the position of the Earth exactly at the same time as event B happens (now from the frame of the returning elevator) looks at the clocks on Earth. We call this event B" (the person in the box at the position of the Earth looking at the Earth clocks). We will now try to find out what he saw, i.e. which time $t_{B''}$ he observed on the Earth clocks. For this we will use spacetime interval $\Delta s_{DB''}$. Show that the space and time intervals from each

frame are the following:

$$\Delta x_{DB''} = 2L_0$$
$$\Delta t_{DB''} = t_{B''}$$
$$\Delta x''_{DB''} = L_0/\gamma$$
$$\Delta t''_{DB''} = L_0/(\gamma v)$$

You might be a bit surprised by one of these results, but if you have doubts, do the following: Make one drawing for event D and one for event B". Show the position of the zero-point (the position of person P is the zero point of the $x''$ axis) of each of the x-axes in both plots and find the distances between events. Did it make it clearer?

14. Use invariance of the spacetime interval to show that

$$t_{B''} = \frac{L_0}{v} + L_0 v$$

Setting in for numbers this gives you $t''_{B''} = 1600.00$ years. Surprised? What has happened? You are still at event B, you made a very fast jump so almost no time has passed since you were in the outgoing elevator. But just before the jump, only 0.08 years had passed on Earth since you started your journey. Now, less than the fraction of a second later, 1600 years have passed on Earth. So in the short time that your jump lasted, 1599.92 years passed on Earth! This is were the solution to the twin paradox is hidden: When you jump, you change reference frame: You are accelerated. Special relativity is not valid for accelerated frames (actually one could solve this looking at the acceleration as an infinite sum of reference frames with different constant velocities). When you are accelerated, you experience fictive forces. This does not happen at Earth, the Earth does not experience the same acceleration. This is the reason for the asymmetry: If you speed had been constant, you and Earth could exchange roles and you would get consistent results. But since you are accelerated in the jump while the Earth is not, there is no symmetry here, you and the Earth cannot switch roles.

15. Let's summerize the situation: In your frame, you started your journey at $t = t' = 0$ and arrived at Rigel at $t' = 8$ years. In the Earth frame you arrived at Rigel after 800.04 years of travel. In your frame, the clocks on Earth show 0.08 years when you arrive at Rigel. Only 0.08 years have passed on Earth at the time you arrive at Rigel, seen from your frame. Then you jump to the returning elevator. Your watch still shows $t' = t'' = 8$ years. But now you have switched frame of reference. Now suddenly 1600 years have passed on Earth, clocks on Earth went from 0.08 years to 1600 years during the jump, as seen from your frame. As seen from Earth, the clock showed 800.04 years during your jump.

16. Seen from the Earth, you need 800.04 years to return, so the total time of your travel measured in the frame of reference of the Earth is $t = 1600.08$ years. In your own frame, the return trip took 8 years (by symmetry to the outgoing trip), so the total travel time for yourself is 16 years. But according to your frame of reference, the Earth clocks again aged 0.08 years during your return trip (by symmetry to the outgoing trip). When you were at Rigel, the observer in your frame of reference saw that the Earth clocks showed 1600 years. In your frame, 0.08 years passed on Earth during your return trip. So consistenly you find the Earth clocks to show 1600.08 years when you set your feet on the Earth again. This is also what we find making the calculation in the Earth frame $800.04 \times 2 = 1600.08$. But hundreds of generations have passed, and you have only aged 16 years. But after all these strange findings I'm sure you find this pretty normal by now. Everything clear? Read through one more time.