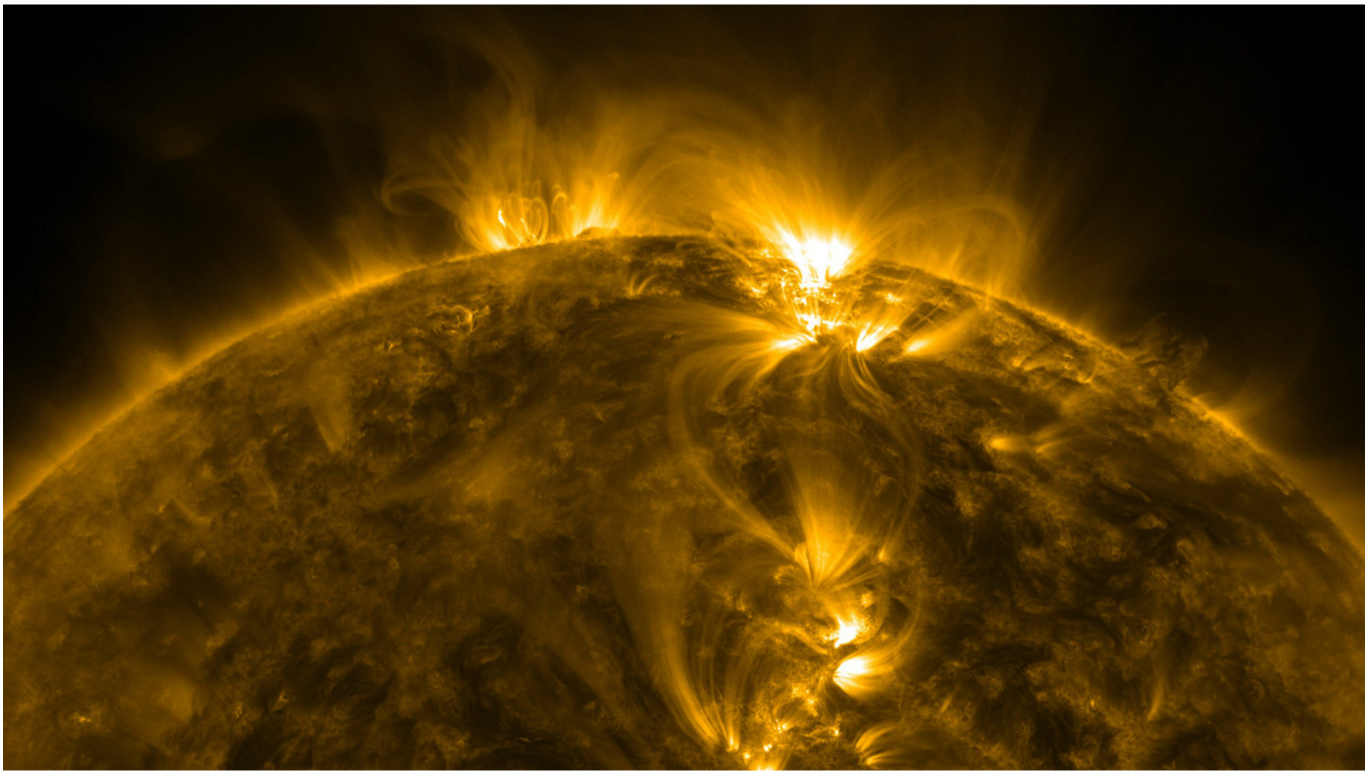
# AST1100 Lecture Notes

## Part 3D
## From the main sequence to the giant stage

### Questions to ponder before the lecture

1. The main sequence stars follow a line in the HR-diagram. Thinking about the meaning of the axes in the HR-diagram, what could this tell us about these stars?
2. If somebody turned off all nucelar reactions in the centre of the Sun now, how long time would it take until you notice? Let's rephrase the question: how does the energy produced in the centre of the Sun reach the surface and how long time does it take? How would you start calculating this?
3. When a star has finished its hydrogen fuel in the core, it expands to become a giant. Why? What makes it expand?



# AST1100 Lecture Notes

## Part 3D
## From the main sequence to the giant stage

### 1 The Hertzsprung-Russell diagram revisited

We have already encountered the Hertzsprung-Russell (HR) diagram, the diagram where stars are plotted according to their temperature and luminosity. There are several versions of this diagram, differing mainly in the units plotted on the axes. The most used units on the x-axis are:

- Temperature
- B-V color index
- spectral classes

We have so far seen temperature on the x-axis. The temperature of a star is directly related to its color and one can therefore also use the $B$–$V$ color index (see the lecture on cosmic distances: $B$–$V$ color index is the difference between the $B$ and $V$ magnitudes: $m_B - m_V$) on the x-axis. There is also another another possibility: *spectral classes*. Stars are classified according to their spectral class which consists of a letter and a number. This historical classification is based on the strength of different spectral lines found in the spectra of the stars. It turned out later that these spectral classes are strongly related to the temperature of the star: The temperature of the star determines the state of the different atoms and therefore the possible spectral lines which can be created.

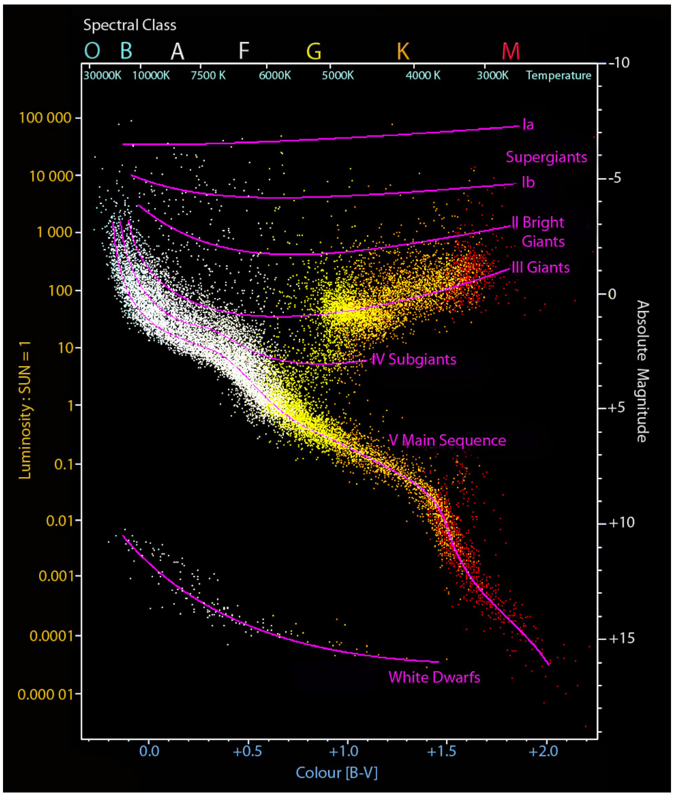



Figure 1: HertzsprungRussell diagram with 22 000 stars plotted from the Hipparcos catalog and 1000 from the Gliese catalog of nearby stars. Stars tend to fall only into certain regions of the diagram. The most predominant is the diagonal, going from the upper-left (hot and bright) to the lower-right (cooler and less bright), called the main sequence. White dwarfs are found in the lower-left, while subgiants, giants, and supergiants are located above the main sequence. The Sun is found on the main sequence at absolute magnitude 4.8 (relative luminosity 1) and BV color index 0.66 (temperature 5780 K, spectral type G2). (Figure:Wikipedia)

The letters used in the spectral classification are, in the order of decreasing temperature, O, B, A, F, G, K, M. The warmest O stars have surface temperatures around 40 000 K, the coldest M stars have surface temperatures down to about 2 500 K. Each of these classes are divided into 10 subclasses using a number from 0 to 9. So the warmest F stars are called F0 and the coldest F stars are called F9.

Normally observational astronomers tend to use either spectral class or color index which are quantities related to the observed properties of the star. Theoretical astrophysicists on the other hand, tend to use temperature which is more important when describing the physics of the star.

Also the y-axis in an HR-diagram have different units. We have already seen luminosity and absolute magnitude which are two closely related quantities. In addition one can use *luminosity classes*. It turns out that stars which have the same spectral class but different luminosities also have some small differences in the spectral lines. These differences have been shown to depend on the luminosity of the star. There are 6 luminosity classes, numbered with Roman numerals from I to VI. The most luminous stars have luminosity class I. Using this classification, the Sun is a G2V star.

Before we start to discuss the diagram in more detail, let us try to understand what it is telling us. We know that the emitted flux at the surface of a star with temperature $T$ can be approximated using the Stefan-Boltzmann law as $F = \sigma T^4$. To obtain the luminosity $L$, we need to integrate this flux over the full area $4\pi R^2$ of the surface of the star giving (why?, check that you understand this!),

$$L = 4\pi R^2 \sigma T^4.$$

Looking at the HR-diagram (see figure 1), we see that there are some spectral classes for which there are stars with many different luminosities. For instance stars with spectral class K0 have a range in luminosity from 0.5 to 1000 solar luminosities. If we fix T in the relation above (remember: fixed T means fixed spectral class) , we see than higher luminosity simply means larger radius. So for a fixed temperature, the higher the star is located in the HR-diagram the larger radius it has. This also means that we can find lines of constant radius in the diagram. Fixing the radius to a constant we get

$$R^2 = \frac{L}{4\pi\sigma T^4} = \text{constant},$$

so that for stars located along lines following $L \propto T^4$ in the diagram, the radius is the same. In figure 2 some of these lines have been plotted. Note that these lines go from the upper left to the lower right, a bit similar to the main sequence. So main sequence stars are stars which have a certain range of radii. The fact that most of the stars are located on the main sequence means that the physics of stars somehow prohibits smaller and larger radii (look at the figure again and check that you understand) . We will come to this in some more detail later.

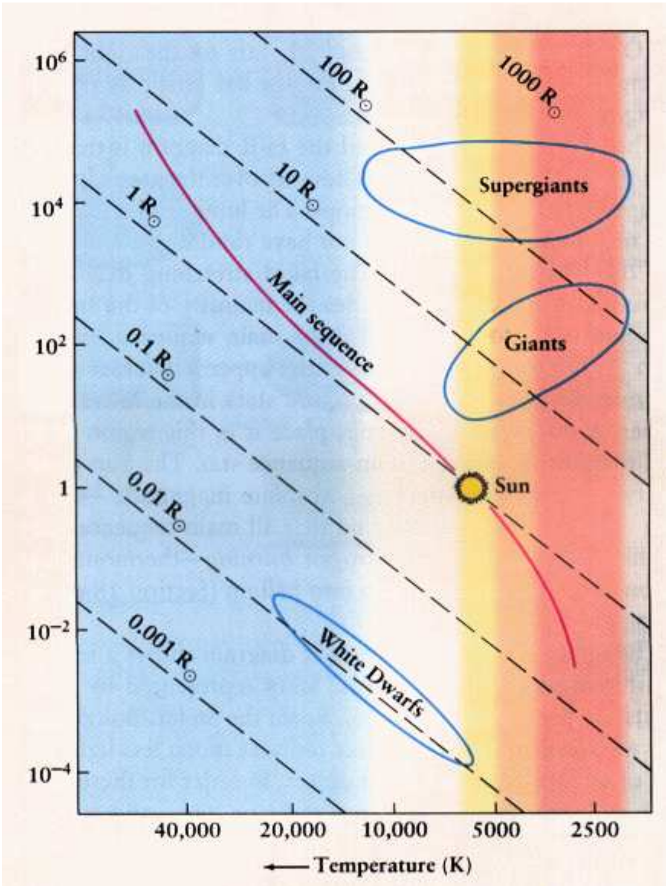



Figure 2: HR-diagram with constant radii lines plotted. From http://astro.wsu.edu/worthey/astro/html/im-Galaxy/

Now it is clear why the stars which are situated above the main sequence are called giants or super giants and the stars well below the main sequence are called dwarfs. Main sequence stars usually have radii in the range $0.1 R_\odot$ to about

$10R_\odot$. Giant stars fall in the range between $10R_\odot$ to about $100R_\odot$ whereas super giants may have radii of several 100 solar radii. The masses of stars range from $0.08M_\odot$ for the least massive stars up to about $100M_\odot$ for the most massive stars. We will later discuss theoretical arguments explaining why there is a lower and an upper limit of star masses.

We already discussed how stars form from a gas cloud. We will now look at the further evolution of stars. Stars start out as huge clouds of gas contracting due to their own gravity. Thus, a star starts out on the far right side of the HR-diagram, with a very low temperature. Then, as it contracts, the radius decreases and the temperature increases. It moves leftwards and finally after nuclear reactions have begun, the star settles on the main sequence. Where it settles on the main sequence depends on the mass of the star. As we will show later, the larger the mass, the higher the luminosity and the higher the surface temperature. So the more massive stars settles on the left side of the main sequence whereas the less massive stars settles on the right side of the main sequence. Stars spend the largest part of their lives on the main sequence. During the time on the main sequence they move little in the HR-diagram. Towards the end of their lives, when the hydrogen in the core has been exhausted, the stars increase their radii several times becoming giants or supergiants. The surface temperature goes down, but due to the enormous increase in radius the luminosity increases. After a short time as a giant, the star dies: Low mass stars die silently, blowing off the outer layers and leaving behind a small white dwarf star in the lower part of the HR-diagram. The more massive stars die violently in a supernova explosion leaving behind a so-called neutron star or a black hole. We will now discuss the physics behind each of these steps in turn.

# 2 Energy transport in stars and the life time on the main sequence

How long does the star remain on the main sequence? It will depend on the available hydrogen in the core. Note that as hydrogen is converted to helium the mean molecular weight $\mu$ increases. We remember that the pressure in an ideal gas can be written as

$$P = \frac{\rho k T}{\mu m_H}.$$

Thus as $\mu$ increases, $P$ decreases provided $\rho$ and $T$ remain approximately constant. The result is that the hydrostatic equilibrium is lost. The battle between the gravitational forces and the pressure forces is won by gravitation and the stellar core starts contracting. The result of the contracting core is that the core density and temperature rise. The nuclear reactions which are more effective at higher temperatures start to be more important. We will now make an estimate of how long time it takes until the hydrogen in the core is exhausted. At this point, the star leaves the main sequence and starts the transition to the giant stage.

Before continuing the discussion on energy production in the core we need to have a quick look at how the energy is transported from the core to the surface. Clearly the photons produced in the nuclear reactions in the core do not stream directly from the core and to the surface. The total luminosity that we observe does not come directly from the nuclear reactions in the core. The photons produced in the nuclear reactions scatter on the nuclei and electrons in the core transferring the energy to the particles in the core. Thus, the high temperature of the stellar core is a result of the energetic photons produced in the nuclear reactions. The high temperature plasma in the core emits thermal radiation. The photons resulting from this thermal radiation constitutes a dense photon gas in the core of the star. How is the energy, that is, the heat of the plasma or the photons in the photon gas, transported to the stellar surface? There are three possible ways to transport energy in a medium:

- By *radiation*: Photons from the photon gas traveling outwards. The photons cannot travel directly from the core, but will be continuously scattered in many different directions by collisions with other particles. After a large number of scatterings and direction changes it will eventually reach the surface and escape.

- By *convection*: Large masses of the hot gas may stream outwards while the cooler gas falls inwards. In this way, the heat and thereby the energy is transfered outwards. Convection is a much more efficient way of energy transport than radiation.

- By *conduction*: Heat is transfered directly outwards by particle collisions.

In stars, mostly the two former mechanisms of energy transport are at play. In solar mass stars, energy is transported from the core by radiation until a distance of about $r = 0.7R_\odot$ where convection starts to be the most important mechanism of energy transport out to the surface.

We will now make a very crude estimate of how long a star remains on the main sequence. In order to do this properly it is necessary to do stellar model building, i.e. solve the coupled set of equations of hydrostatic equilibrium, the equations of energy production and the equations of energy transport. This gives a model of the star in terms of density and temperature as a function of distance from the center. From this model, the proper life time of the star can be calculated. It turns out that the estimates and relations that we now will deduce using some very rough approximations give results close to the results obtained using the full machinery of stellar model building.

The outline of the method is the following: Find an expression for the luminosity of the star. We know that luminosity is energy radiated away per unit of time. If we assume how much energy the star has available to radiate away during its life time, we can divide this energy by the luminosity to find the life time (assuming constant luminosity which is a good assumption during the main sequence phase).

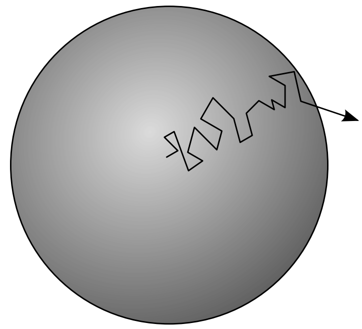


Figure 3: Energy transport by radiation: random walk of the photons from the core of the star to the surface.

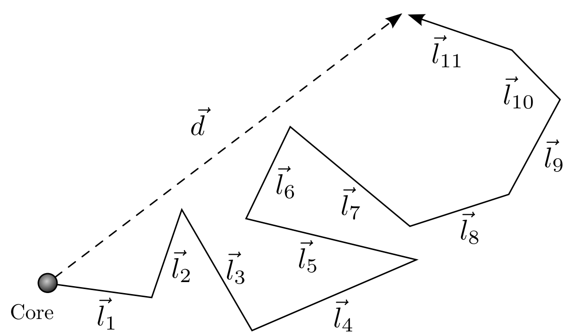


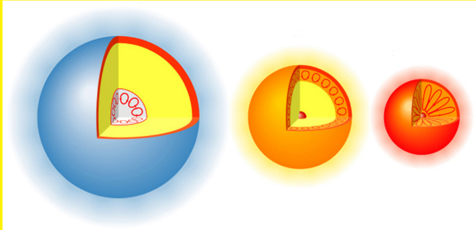

Figure 4: Random walk from the core. The position after $N$ scatterings $\vec{l}_i$ is $\vec{d}$.

We will again consider the photon gas in the stellar core. You will in later courses in thermodynamics show that the energy density, i.e. energy per volume, of a photon gas goes as $\rho_E \propto T^4$ (actually $\rho_E = aT^4$ where $a$ is the radiation constant that we encountered in lecture 1E for the pressure of a photon gas $P = \frac{1}{3}aT^4$). The question is how long time it will take for the photos in the photon gas to reach the surface of the star. We will now assume that the only mechanism for energy transport is by radiation. A photon which starts out in the core will be scattered on particles and continuously change directions until it reaches the surface of the star (see figure 3). We assume that the photon travels a mean free path $\ell$ between each collision. After being scattered $N$ times, the position $\vec{d}$ of the photon (see figure 4) is given by

$$\vec{d} = \sum_{i=1}^{N} \vec{l}_i,$$

where $\vec{l}_i$ is the displacement vector between each scattering $i$ (see again figure 4). The total length

**Fact sheet:** Stars produce energy by fusion in their deep interior because only there are the pressures and temperatures high enough to sustain thermonuclear reactions. However, most of the luminous energy of stars is radiated from the thin region at the surface that we call the photosphere. The two most important ways of transporting energy from the core to the surface in main sequence stars are by radiation and by convection. A low mass main sequence star (middle) will have convection in its outer layers and a radiation zone (yellow area) in the center, like the Sun. If the star is really low mass (right) it will have convection all the way in. A high mass star (left) will have convection only in its core.(Figure: B. Boroson)



$\Delta r$ of the vector $d$ is the total distance the photon has moved from the center. It is given by (check!)

$$\Delta r^2 = \vec{d} \cdot \vec{d} = \sum_{i,j} \vec{l_i} \cdot \vec{l_j} = N\ell^2 + \ell^2 \sum_{i \neq j} \cos\theta_{ij},$$

where $\theta_{ij}$ is the angle between two vectors $\vec{l_i}$ and $\vec{l_j}$. The directions of the scatterings are random, so $\cos\theta_{ij}$ will have values between -1 and 1. After many scatterings, the mean value of this term will approach zero and we have

$$\Delta r = \sqrt{N}\ell,$$

or writing this in terms on number of scatterings $N$ to reach the surface we thus have $N = R^2/\ell^2$ where $R$ is the radius of the star (check!).

The time $\Delta t$ for a photon to reach the surface is then (note that the total distance traveled by the photon is $N\ell$)

$$\Delta t = \frac{N\ell}{c} = \frac{\ell}{c}\frac{R^2}{\ell^2} = \frac{R^2}{\ell c}.$$

If we assume that within a radius $r$ of the star, the temperature $T$ and energy density $\rho_E$ of the photon gas is constant, the total energy content of the photon gas within radius $r$ is

$$E = \frac{4}{3}\pi r^3 \rho_E \propto r^3 T^4,$$

where we used that $\rho_E \propto T^4$. We will now use a very rough model of the star: We assume the density and temperature of the star to be constant everywhere in the star. Then the energy content of the photon gas in the whole star is given by $E \propto R^3T^4$. If we assume that this energy is released within the time $\Delta t$ it takes for the photons in the core to reach the surface, then the luminosity of the star can be written as

$$L \propto \frac{E}{\Delta t} \propto \frac{R^3T^4}{R^2/\ell} \propto RT^4\ell. \tag{1}$$

The mean free path $\ell$ depends on the density of electrons and the different nuclei in the core. If we assume that photons are only scattered on electrons, it can be shown that the mean free path $\ell \propto 1/\rho$ which does seem reasonable: The higher the density the lower the mean free path between each scattering. Since we assume constant density we have $\rho \propto M/R^3$. Inserting this in equation 1 we have

$$L \propto RT^4\ell \propto \frac{RT^4}{\rho} \propto \frac{R^4T^4}{M}. \tag{2}$$

Finally we will use the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho g.$$

If we assume that the pressure can be written as $P \propto r^n$ where $n$ is unknown then

$$\frac{dP}{dr} = nr^{n-1} = \frac{nr^n}{r} = \frac{nP}{r} \propto \frac{P}{r}.$$

The equation of hydrostatic equilibrium then yields

$$\frac{P}{R} \propto \rho g \propto \frac{M}{R^3}\frac{M}{R^2} \propto \frac{M^2}{R^5},$$

or $P \propto M^2/R^4$. We remember that for an ideal gas $P \propto \rho T$. Inserting this in the previous equation gives

$$T \propto \frac{M}{R}.$$

Inserting this in equation 2 we get

$$L \propto \frac{R^4}{M}\left(\frac{M}{R}\right)^4 \propto M^3. \tag{3}$$

The luminosity is proportional to the mass of the star to the third power. A more exact calculation would have shown that

$$L \propto M^{\beta},$$

where $\beta$ is usually between 3 and 4 depending on the exact details of the star. It turns out that most low or medium mass stars have $\beta \approx 4$. This is also supported by observations. Therefore we will in the following use $L \propto M^4$. Having the luminosity of the star, we can easily find the life time. Assume that a fraction $p$ of the mass of the star is converted to energy. Then the total energy radiated away during the lifetime of the star is given by

$$E = pMc^2.$$

If we assume constant luminosity during the lifetime we have

$$L = \frac{pMc^2}{t_{\text{life}}} \propto M^4,$$

giving

$$t_{\text{life}} \propto \frac{1}{M^3}.$$

This can be the total life time of the star, or just the life time on the main sequence (in fact, for most stars the time on the main sequence is so much longer than other stages in a star's life so the time on the main sequence is roughly the same as the life time of the star). If we take $p$ to be the fraction of mass converted to energy during the main sequence, then this is the expression for the time the star spends on the main sequence. We see that the life time of a star is strongly dependent on the mass of the star. The Sun is expected to live for about $10 \times 10^9$ years. A star with half the mass of the Sun will live 8 times longer (which is much longer than the age of the universe). A star with two times the mass of the Sun will live only 1/8 or roughly $10^9$ years. The most massive stars only live for a few million years. We see from equation 3 that this can be explained by the fact that massive stars are much more luminous than less massive stars and therefore burn their fuel much faster. A star with two times the mass of the Sun will burn 16 times (equation 3) as much 'fuel' per time as the Sun, but it only has twice as much 'fuel'. It will therefore die much younger.

As the last expression is just a proportionality, we need to find the constant of proportionality, that is, we need to know the life time and mass of one star in order to use it for other stars. We know these numbers for the Sun and we will now use approximations to calculate this number. One can show that a star will leave the main sequence when about 10% of its hydrogen has been converted to helium. We discussed in the previous lecture that the efficiency of the pp-chain is 0.7%. So the total energy that will be produced of the Sun during its lifetime is therefore $0.1 \times Mc^2 \times 0.007$. Assuming that the solar luminosity $3.7 \times 10^{26}$ W is constant during the time on the main sequence we have

$$\begin{aligned} t_{\odot}^{\text{mainsequence}} &= \\ &\frac{0.1 \times 2 \times 10^{30}\ \text{kg} \times (3 \times 10^8\ \text{m/s})^2 \times 0.007}{3.7 \times 10^{26}\ \text{W}} \\ &\approx 10^{10}\ \text{years}. \end{aligned}$$

We will now try to find a way to estimate the mass of a star. Remember that in the lectures on extrasolar planets, we needed to know the mass of the star by independent measurements in order to be able to estimate the mass of a planet orbiting it. In the above approximation we considered a star with constant density and temperature. The conditions we used are normally valid only for the core of the star. Thus, the approximations we made are more correct in the core of the star. We found that the temperature $T \propto M/R$. For main sequence stars, the core temperature is reasonably constant, there is not a large difference in core temperatures for different main sequence stars. Using this assumption we can write

$$T_c \propto \frac{M}{R} = \text{constant}.$$

We can write this as $R \propto M$. Now, we know that the luminosity of a star can be written in terms of the effective temperature as

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4,$$

where $4\pi R^2$ is the area of the surface and $F = \sigma T_{\text{eff}}^4$ is the flux at the surface. Using $R \propto M$ and $L \propto M^4$ this gives

$$L \propto M^4 \propto R^2 T_{\text{eff}}^4 \propto T_{\text{eff}}^4 M^2,$$

so $M^4 \propto T_{\text{eff}}^4 M^2$ giving

$$M \propto T_{\text{eff}}^2 \quad (4)$$

and we have obtained a way to find the mass of a star from its temperature. In the exercises you will use this expression to find the temperature of stars with different masses.

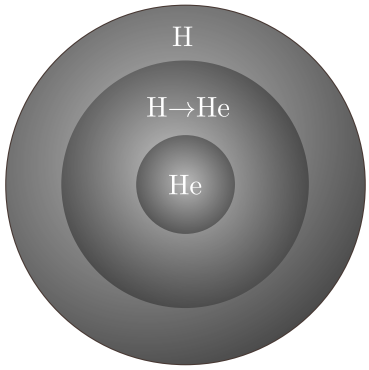



Figure 6: The structure of a subgiant and red giant. The core consists mainly of helium, but the core temperature is not high enough for helium burning. Hydrogen is burning to helium in a shell around the core. For red giants, convection transports material all the way from the core to the surface and the material is mixed (in the figure there is only hydrogen in the outer parts, for red giants the mixing due to convection will also transfer other elements all the way to the surface). The relative sizes of the shells are not to scale, this will depend on the exact evolutionary stage.

## 3 From the main sequence to the giant stage

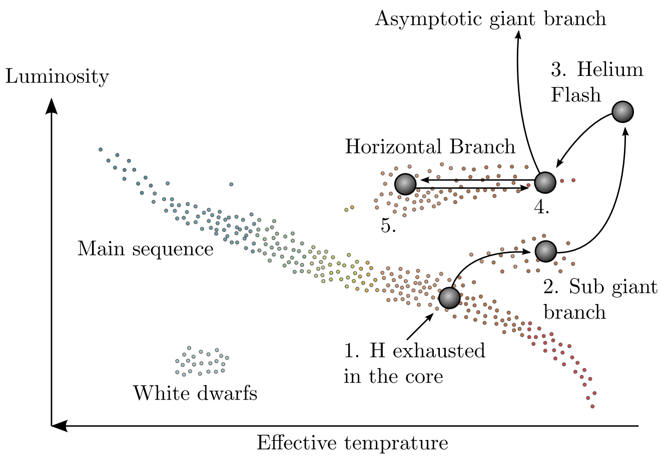



Figure 5: HR-diagram of the evolution of a star from the main sequence to the giant stage.

We will now follow a star during the transition from the main sequence to the giant stage. The exact sequence of events will be slightly different depending on the mass of the star. Here we will only discuss the general features and discuss a few main differences between low and high mass stars. In figure 5 we can follow the evolutionary path of the star in the HR diagram. The theories for stellar evolution are developed using computer models of stars obtained by solving the equations for stellar model building numerically. The chain of arguments that we will use below to describe stellar evolution are obtained by studying the outcome of computer simulations.

When the hydrogen in the core has been exhausted, the forces of pressure are not any longer strong enough to sustain the forces of gravity. The hydrostatic equilibrium is lost and the core starts contracting. During the core contraction, the temperature in and around the core increases. The temperature in the core is still not high enough to 'burn' helium (all energy production is by nuclear fusion, not by 'burning' in the classical sense but it is common practice to use the term 'burning' anyway), but the temperature in a shell around the core now reaches temperatures high enough to start hydrogen burning outside the core. The structure of the star is illustrated in figure 6. Because of the increased outward pressure due to hydrogen burning in the shell, the radius of the star starts increasing significantly. The star has become a *sub giant* of luminosity class IV (see section 1 on the HR diagram and luminosity classes). In figure 5 the star has left the main sequence and is now on the *sub giant branch* between point 1 and 2. The luminosity has been increasing slightly because the energy produced in the shell is higher than the energy previously produced in the core. But because of the increasing radius of the star, the surface temperature is dropping. Thus the star moves to the right and slightly upwards in the HR diagram.

When reaching point 2 in the HR-diagram, the radius of the star has been increasing so much that the surface temperature is close to 2500 K which is a lower possible limit. When reaching this limit, the dominant mechanism of energy transport in the star changes from being radiation to convection. Convection is much more efficient, the energy is released at a much larger rate and the luminosity increases rapidly. The star has now become a *red giant*. At the red giant stage, convection takes place all the way from the core to the surface. Material from the core is moved all the way to the surface. This allows another test of the theories of stellar evolution. By observing the elements on the surface of a red giant we also know the composition of elements in the core. The star is now on the red giant branch in the HR-diagram (figure 5). The structure of the star still resembles that of figure 6. The radius is between 10 and 100 times the original radius at the main sequence and the star has reached luminosity class III.

The next step in the evolution depends on the mass of the star. For stars more massive than $\sim 2M_\odot$, the temperature in the core (which is still contracting) will eventually reach temperatures high enough to start the triple-alpha process burning helium to carbon as well as other chains burning helium to oxygen. In low mass stars, something weird happens before the onset of helium burning. As the core is contracting the density becomes so high that a quantum mechanical effect sets in: there is no more space in the core for more electrons. Remember from part 1G: Quantum physics sets an upper limit on the number of electrons within a certain volume with a certain momentum. This is called electron degeneracy. The core has become *electron degenerate*. Looking back to the lecture on quantum gases you remember that a degenerate gas has a different form of pressure: degeneration pressure. We have already deduced the equation for this pressure and shown that it is independent of temperature. The degeneration pressure is now the outward force which battles the inward gravitational force in the equation of hydrostatic equilibrium. Since the degeneration pressure does not depend on temperature, the core does not expand even when the temperature of the core increases significantly. The degenerate core is close to isothermal and when the temperature is high enough to start helium burning, this happens everywhere in the core at the same time. An enormous amount of energy is released in a very short time causing an explosive onset of the helium burning phase. This is called the *helium flash*. After a few seconds, a large part of the helium in the core has already been burned. The huge amounts of energy released breaks the electron degeneracy in the core and the gas starts to behave normally, i.e. the pressure is again dependent on the temperature allowing the core to expand. The onset of helium burning (which includes the helium flash for low mass stars and a less violent transition for high mass stars) is marked by 3 in figure 5.

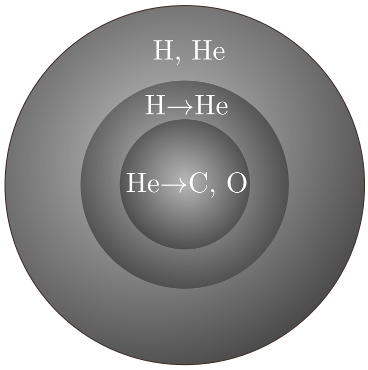



Figure 7: A horizontal branch giant. Helium is burning to carbon and oxygen in the core. Hydrogen is burning to helium in a shell around the core. The relative sizes of the shells are not to scale, this will depend on the exact evolutionary stage.

The final result of the onset of helium burning is therefore the same for both low and high mass stars: The core will finally expand, pushing the hydrogen burning shells outward to larger radii where the gas will cool and the hydrogen burning will therefore cease in large parts of the shell. The energy produced in the helium burning is not enough to substitute the energy production in the shell and the total luminosity of the star will decrease. This is the case also for stars which undergo a helium flash. This is seen in the transition from 3 to 4 in figure 5. The star has now entered the *horizontal branch*. This stage is in a way similar to the main sequence: This is where the star burns its helium to carbon and oxygen in the core. Hydrogen burning is still taking place in parts of the shell. The structure of the star is shown in

**Fact sheet:** The size of the current Sun compared to its estimated size during its red giant phase in the future. The outer atmosphere of a red giant is inflated and tenuous, making the radius immense and the surface temperature low. Prominent bright red giants in the night sky include Aldebaran, Arcturus, and Mira, while the even larger Antares and Betelgeuse are red supergiants. (Figure: Wikipedia)

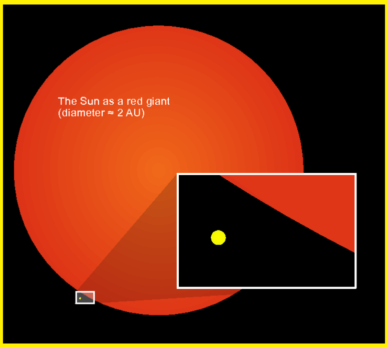



figure 7. Horizontal branch giants are called so because, as we will discuss now, they will move back and forth along a horizontal branch.

After the rapid expansion of the star after the onset of helium burning, the star starts contracting again in order to reach hydrostatic equilibrium. The result is an increasing effective temperature and the star moves to the left along the horizontal branch. After a while on the horizontal branch, the mean molecular weight in the core has increased so much that the forces of pressure in the core are lower than the gravitational forces and the core starts contracting. The temperature of the core increases and the energy released in this process makes the star expand: The effective temperature of the surface is decreasing and the star is moving to the right along the horizontal branch. At this point the helium in the core is exhausted and nuclear energy production ceases. The following scenario resemble the scenario taking place when the hydrogen was exhausted: The core which now mainly consists of carbon and oxygen starts to contract (due to the lack of pressure to sustain the gravitational forces after the energy production ceased). The core contraction heats a shell around the core sufficiently for the ignition of helium burning. Energy is now produced in a helium burning as well as hydrogen burning shell around the core. The radius of the star increases because of the increased pressure. Again we reach a stage of strong convective energy transport which (exactly as on the red giant branch) rises the luminosity. The star now moves to the asymptotic giant branch becoming a bright giant of luminosity class II or even a super giant of luminosity class I. The star now has a radius of up to 1000 times the original radius. The structure of the star is shown in figure 8.

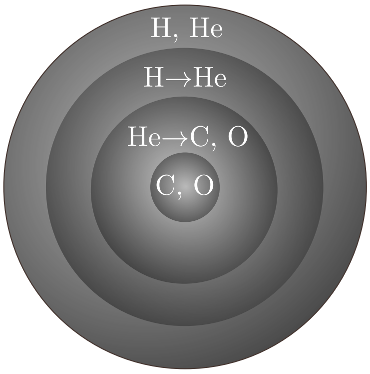



Figure 8: A bright/super giant. The core consists mainly of carbon and oxygen but the temperature is not high enough for these elements to burn. Around the core there is a shell where helium is fused to carbon and oxygen and another shell where hydrogen is fused to helium. In the outer parts the temperature is not high enough for fusion reactions to take place.

Most stars follow an evolution similar to this. The stars with very high mass (more than $\sim 20M_\odot$) do not have a significant convective phase and do therefore not change their luminosity much during their evolution. They will mainly move left and right in the HR-diagram.

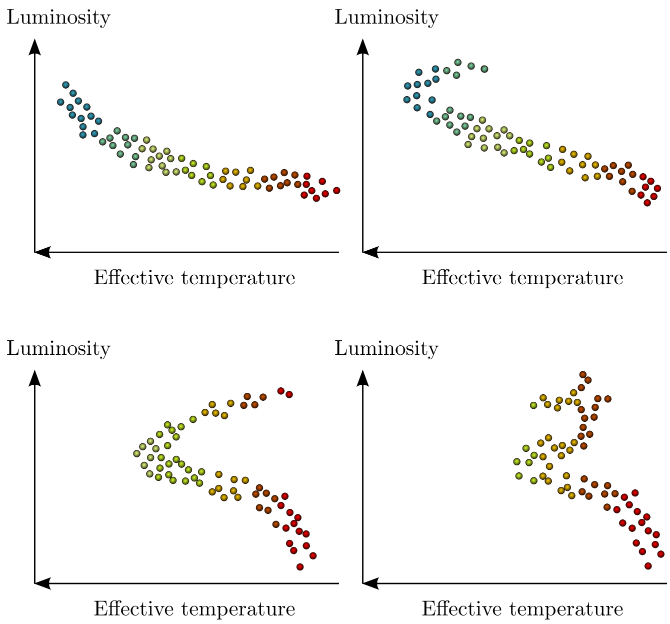



Figure 9: Schematic HR diagrams of open clusters of different ages:

**Upper left:** A cluster still in the process of forming. The less massive stars are still in the contracting phase and have not yet reached the main sequence.
**Upper right:** A cluster with an age of about $10^7$ years. The most massive stars have started to leave the main sequence.
**Lower left:** A cluster of about $10^9$ years. The low mass stars have now reached the main sequence.
**Lower right:** A cluster of about $10^{10}$ years. The medium mass stars have now started to leave the main sequence and we can clearly see the different branches discussed in the text.

Open stellar clusters can be used to test the theories of stellar evolution. An open cluster is a collection of stars which were born roughly at the same time from the same cloud of gas. Observing different open clusters with different ages, we can obtain HR diagrams from different epochs of stellar evolution. We can use observed diagrams to compare with the predicted diagrams obtained using the above arguments. In figure 9 we see a schematic example of HR diagrams taken at different epochs (from clusters with different ages). We see that the most massive stars start to leave the main sequence earlier: This is because the life time of stars is proportional to $t \propto 1/M^3$. The most massive stars exhaust their hydrogen much earlier than less massive stars. As discussed above, the most massive stars do not have a phase with strong convection and do therefore not move vertically up and down but mostly left and right in the diagram. For this reason we do not see the red giant branch and the asymptotic branch for these stars. Only in the HR diagram of the oldest cluster has the intermediate mass stars started to leave the main sequence. For these stars we now clearly see all the different branches. Comparing such theoretical diagrams with diagrams for observed clusters has been one of the most important way to test and understand theories of stellar evolution.

Having reached the asymptotic giant branch, the star has almost ended its life cycle. The final stages will be discussed in more detail in the next lecture. First we will look at a typical feature of giant stars: pulsations.

# 4 OPTIONAL: Stellar pulsations

Some giant stars have been observed to be pulsating. We have already encountered one kind of pulsating stars: the Cepheids. The pulsating stars have been found to be located in narrow vertical bands, so-called instability strips, in the HR-diagram. The Cepehids for instance, are located in a vertical band about 600 K wide around $T_{\text{eff}} \sim 6500$ K. The pulsations start during the core contraction and expansions starting when the star leaves the main sequence. They last only for a limited period when the star passes through an instability strip in the HR diagram. We remember that for Cepheids there is a relation between the pulsation period and the luminosity of the star allowing us to determine the distance to the star (see the lecture on the cosmic distance ladder). The period-luminosity relation for Cepheids can be written in terms of luminosity (instead of absolute magnitude) as

$$\langle L \rangle \propto P^{1.15}, \tag{5}$$

where $\langle L \rangle$ is the mean luminosity and $P$ is the pulsation period. We will now see if we can deduce this relation using physics in the stellar interior.

The pulsations are due to huge density waves, sound waves, traveling through the interior of the star. We can find an approximate expression for the pulsation period of a star by considering the time it takes for a sound wave to go from one end of the star to the other. We will for simplicity consider a star with radius $R$ and constant density $\rho$. The pulsation period $P$ is thus the time it takes for a sound wave to travel a distance $2R$. In thermodynamics you will learn that the sound speed (the so-called adiabatic sound speed) at a given distance $r$ from the center of a star is given by

$$v_s(r) = \sqrt{\frac{\gamma P(r)}{\rho}},$$

where $\gamma$ is a constant depending on the specific heat capacities for the gas. We have assumed constant density and therefore only need to find the pressure as a function of $r$. The equation of hydrostatic equilibrium can give us the pressure. We have

$$\frac{dP}{dr} = -g\rho = -\frac{GM(r)}{r^2}\rho = -\frac{4}{3}G\pi r\rho^2.$$

Integrating this expression from the surface where $P = 0$ and $r = R$ down to a distance $r$ we get

$$P(r) = \frac{2}{3}\pi G\rho^2(R^2 - r^2).$$

We now have the necessary expressions in order to find the pulsation period of a Cepheid. At position $r$, the sound wave travels with velocity $v_s(r)$. It takes time $dt$ to travel a distance $dr$, so

$$dt = \frac{dr}{v_s(r)}.$$

To find the pulsation period, we need to find the total time $P$ it takes for the sound wave to travel a distance $2R$

$$P \approx 2\int_0^R \frac{dr}{v_s(r)} \approx 2\int_0^R \frac{dr}{\sqrt{\frac{2}{3}\gamma\pi G\rho(R^2 - r^2)}}$$
$$= \frac{1}{\sqrt{\frac{2}{3}\gamma\pi G\rho}}\left[-\tan^{-1}\frac{r\sqrt{R^2 - r^2}}{r^2 - R^2}\right]_0^R$$

Taking the limits in this expression, we find

$$P \approx \sqrt{\frac{3\pi}{2\gamma G\rho}} \propto \frac{1}{\sqrt{\rho}} \propto \left(\frac{R^{3/2}}{M^{1/2}}\right).$$

From equation 4 we see that $M^{1/2} \propto T_{\text{eff}}$ but since Cepheids are located along the instability strip in the HR-diagram their effective temperatures are roughly constant. So we have

$$P \propto R^{3/2}.$$

The luminosity of a star can be written as as $L = 4\pi R^2\sigma T_{\text{eff}}^4$. Again we consider $T_{\text{eff}} \approx$ constant so $L \propto R^2$ or $R \propto L^{1/2}$. Inserting this into the previous expression for the pulsation period we have

$$P \propto L^{3/4},$$

or

$$L \propto P^{4/3} \propto P^{1.3}.$$

Comparing to the observed period-luminosity relation (equation 5), this agreement is excellent taking into account the huge simplifications we have made. We have shown that by assuming the pulsations to be caused by sounds waves in the stellar interiors, we obtain a period luminosity relation for Cepheids similar to what we observe.

## 5 Exercises

**Exercise 3D.1** You need to read section 1 before doing this exercise. Look at the HR-diagram in figure 1. Assume that you observe a main sequence star with spectral class G0. The apparent magnitude of the star is $m = 1$.

1. Roughly what luminosity and absolute magnitude would you expect the star to have? (use the diagram)
2. Using this result, can you give a rough approximation of the distance?
3. Looking again at the HR-diagram. Roughly what is the minimum and maximum absolute magnitude you would expect the star to have?
4. What is the range of distances the star could have?

This method for measuring distances is called *spectroscopic parallax* (although it has nothing to do with normal parallax). I have not included this method in the lectures on distance measurements. From the answer to the last question you will understand why it is not a very exact method.

**Exercise 3D.2**

You only need part 1E to do this exercise. We will now assume a very simple model of the Sun in order to show how one can use the equation of hydrostatic equilibrium to understand stellar interiors and the nuclear reactions taking place in the stellar cores. We will assume that the density of the Sun $\rho = \rho_0$ is uniform throughout.

1. Find an expression for the total mass $M(r)$ inside a radius $r$.
2. We will now assume that the only pressure in the Sun is the gas pressure from an ideal gas. We ignore the radiation pressure. Insert this expression for $M(r)$ into the equation of hydrostatic equilibrium and show that it can be written as
$$\frac{dT}{dr} = -\frac{4\pi}{3} G\rho_0 r \frac{\mu m_H}{k}$$
3. Integrate this equation from the core at $r = 0$ to the surface of the Sun at $r = R$ and show that the temperature $T_c$ in the core of the Sun can be written
$$T_C = T(R) + \frac{2\pi}{3} G R^2 \rho_0 \frac{\mu m_H}{k}.$$
4. Assume that the Sun consists entirely of protons with a mass of $1.67 \times 10^{-27}$ kg. Use the solar mass of $2 \times 10^{30}$ kg, the solar radius of 700 000 km and the surface temperature of the Sun $T = 5780$ K to obtain the density $\rho_0$ and thereby the core temperature $T_C$. (By doing this calculation properly taking into account variations of the density with distance from the core, one obtains a core temperature of about 15 million Kelvin)
5. You learned from the lectures on nuclear reactions that hydrogen can fuse to Helium by two different processes, the pp-chain and the CNO-cycle. The pp chain is more efficient at temperatures below 20 million Kelvin whereas the CNO-cycle starts dominating at temperatures above 20 million Kelvin. Use your result for the core temperature of the Sun to decide which of these processes produces most of the energy in the Sun.
6. Write $\rho_0$ in terms of the mass $M$ and the radius $R$ of the Sun. We have seen that the surface temperature of the Sun is much smaller than the core temperature and might therefore be neglected. Show that the core temperature of a star depends on the mass and radius as
$$T_C \propto \frac{M}{R}$$
7. When the Hydrogen in the core of a star has been exhausted, the nuclear fusion processes cease. In this case the pressure forces cannot sustain the force of gravity and the radius of the core starts shrinking. It will continue shrinking until some other force can oppose the force of gravity. If Helium, an element which is now found in large abundances in the core, starts to fuse to heavier elements this would create a photon pressure high enough to sustain gravity. A temperature of at least 100 million degrees Kelvin

is needed in order for this fusion process to start. By how much does the core radius of the Sun need to shrink in order for Helium fusion to start?

8. In the last case, the radiation pressure is giving the dominant contribution to the forces of pressure. Show that in this case, the temperature of the core can be written as

$$T_C = \left(T(R)^4 + \frac{2\pi G}{a}\rho_0^2 R^2\right)^{1/4},$$

again assuming a constant density.

**Exercise 3D.3**

You still only need part 1E to solve this exercise. We will now assume a slightly more realistic model of the Sun. Assume that the density of the Sun as a function of distance $r$ from the core can be written as

$$\rho(r) = \frac{\rho_C}{1 + (r/R)^2},$$

where $\rho_C$ is the density in the core of the Sun and $R$ is the radius at which the density has fallen by a factor 1/2 (check this by inserting $r = R$ in the expression). In this exercise we will use our knowledge about the minimum temperature which is needed to obtain nuclear reactions in order to calculate the density in the solar core.

1. We will now find an expression for the total mass $M(r)$ inside a radius $r$ using this density profile. First write the general integral for $M(r)$ for a general density profile $\rho(r)$. In order to perform this integral we make the substitution $x = r/R$ and integrate over $x$ instead of $r$. Show that $M(r)$ can be written

$$M(r) = 4\pi\rho_C R^3 \int_0^{r/R} dx \frac{x^2}{1 + x^2}$$

2. In order to perform such integrals, the Mathematica package is very useful. Not everybody has access to Mathematica, but a free web interface exists for performing integrals.

   Go to http://integrals.wolfram.com/index.jsp, type `x^2/(1+x^2)` and click "Compute online with *Mathematica*",

   and you get a nice and easy answer. Using this result, together with the assumption of pure ideal gas pressure, show that the equation of hydrostatic equilibrium can now be written

$$\frac{d}{dr}(\rho(r)T(r)) = -\frac{\mu m_H}{k} 4\pi G \rho_C^2 R^3 \times \frac{r/R - \arctan(r/R)}{r^2} \frac{1}{1 + (r/R)^2}.$$

3. We now need to integrate this equation from radius 0 to an arbitrary radius $r$. Again the substitution $x = r/R$ is useful. Show that the equation of hydrostatic equilibrium now reads

$$\rho(r)T(r) - \rho_C T_C = -\frac{\mu m_H}{k} 4\pi G \rho_C^2 R^2 \times \int_0^{r/R} dx \left(\frac{1}{x(1+x^2)} - \frac{\arctan(x)}{x^2(1+x^2)}\right)$$

4. To solve this integral you need to use the 'Integrator' and type the following: `1/(x(1+x^2))` and `arctan(x)/(x^2(1+x^2))`.

   Using these results, show that the core temperature $T_c$ can be written

$$T_C = T(r)/(1+x^2) + \frac{\mu m_H}{k} 4\pi G \rho_C R^2 \times \left(\frac{1}{2}(\arctan x)^2 + \frac{\arctan(x)}{x} - 1\right)$$

5. We will now try to obtain values for the central density $\rho_C$. In order to obtain that, we wish to get rid of $x$ and $r$ from the equation. When $x \to \infty$, that is, when going far from the center, show that the equation reduces to

$$T_C = \frac{\mu m_H}{k} 4\pi G \rho_C R^2 (\frac{\pi^2}{8} - 1)$$

6. Before continuing, we need to find a number for $R$, the distance from the center where the density has fallen by 1/2. Assume that considerations based on hydrodynamics and thermodynamics tell us that the core of the Sun extends out to about $0.2R_\odot$ and that

the density has fallen to 10 percent of the central density at this radius. Using this information, show that

$$R = \frac{0.2R_\odot}{\sqrt{9}} \approx 0.067R_\odot.$$

7. We know that a minimum core temperature of about 15 million degrees is needed in order for thermonuclear fusion to be an efficient source of energy production. What is the minimum density in the center of the Sun? Assume the gas in the Sun to consist entirely of protons. Express the result in units of the mean density $\rho_0 = 1400$ kg/m$^3$ of the Sun. (More accurate calculations show that the core density of the Sun is about 100 times the mean density).

In the last two exercises we have used some very simplified models together with some rough assumptions and observed quantities to obtain knowledge about the density and temperature in the interior of the Sun. These exercises were made to show you the power of the equation of hydrostatic equilibrium: By combing this equation with the knowledge we have about the Sun from observations of its surface together with knowledge about nuclear physics, we are able to deduce several facts about the solar interior. In higher courses in astrophysics, you will also learn that there are more equations than the equation of hydrostatic equilibrium which must be satisfied in the solar interior. Most of these equations come from thermodynamics and fluid dynamics. In the real case, we thus have a set of equations for $T(r)$ and $\rho(r)$ enabling us to do *stellar model building*, without using too many assumptions we can obtain the density and temperature of stars at different distances from the center. These models have been used to obtain the understanding we have today of how stars evolve. Nevertheless many questions are still open and poorly understood. Particularly towards the end of a star's life, the density distribution and nuclear reactions in the stellar interior become very complicated and the equations become difficult to solve. But solving these equations is important in order to understand the details of supernova explosions.

**Exercise 3D.4**

You need to have read section 2 in order to be able to solve this exercise. In the text there is a formula for estimating the effective temperature of a star with a given mass (or estimating the mass of a star with a given effective temperature).

1. Given the effective temperature (5780 K) and mass ($M_\odot$) of the Sun, find the effective temperature of a small star with $M = 0.5M_\odot$, an intermediate mass star $M = 5M_\odot$ and a high mass star $M = 40M_\odot$.
2. The star Regulus in the constellation Leo is a blue main sequence star. It is found to have a peak in the flux at a wavelength of about $\lambda = 240$ nm. Give an estimate of its mass expressed in solar masses.

**Exercise 3D.5**

You need to have read section 2 in order to be able to solve this exercise. In the text we derived that the luminosity of a low/intermediate mass star is proportional to mass to the third power $L \propto M^3$. In this derivation you used the ideal gas law. For high mass stars, the radiation pressure is more important than the ideal gas pressure and the expression for radiation pressure (you need to find it in the text) needs to be used instead of the expression for the ideal gas pressure. Repeat the derivation for the mass-luminosity relation using radiation pressure instead of ideal gas pressure and show that for high mass stars $L \propto M$. How is the relation between the life time and the mass of a star for a high mass star compared to a low mass star?

**Exercise 3D.6**

Read carefully the description for the evolution of a star from the main sequence to the giant stage in section 3. Take an A4-sheet. You are allowed to make some *simple* drawings and write a maximum of 10 words on the sheet. Make the drawings and words such that you can use it to be able to tell someone how a star goes from the main sequence to the giant stage, describing the logic of how the core contracts/expands and how the star moves in the HR-diagram depending on temperature, means of energy transport and nuclear reactions. Bring the sheet to the group and use it (and nothing else) to tell the story of stellar evolution to another student, then exchange your roles.

**Exercise 3D.7**

Look at the HR-diagram for the oldest cluster in figure 9. Can you identify the different branches of stellar evolution?

**Exercise 3D.8**

You need to read all of part 3D in order to be able to solve this exercise. We will now study the phase when the hydrogen in the stellar core has been depleted. The energy production in the core stops and the core starts shrinking. The star reaches the sub giant branch and then the red giant branch while the core keeps shrinking. The core will keep shrinking until the temperature in the core is high enough for helium burning to start. We will try to find out how much the core shrinks before this takes place. For simplicity we will study a star with so high mass that the core does not become degenerate before helium burning sets in. We will assume the core density at the main sequence to be $\rho = 1.7 \times 10^5$ kg/m$^3$.

We imagine the stellar core as an 'independent' sphere of mass $M_C$, radius $R_C$, pressure $P_C$ and temperature $T_C$. We assume the density and temperature to be the same everywhere in the core.

1. Use the equation of hydrostatic equilibrium to show that

$$P_C \propto \frac{M_C^2}{R_C^4}.$$

   This is done in the text, but try to find your own arguments before looking it up.

2. Then combine this with the ideal gas law to show that

$$T_C \propto \frac{M_C}{R_C}.$$

3. We assume that the core temperature of the star on the main sequence was $T_C = 18 \times 10^6$ K. Use the expressions for the nuclear energy production rates from the previous lecture to find out whether it was the pp-chain or the CNO cycle which dominated the energy production in the star while it was on the main sequence. Assume $X_{\rm H} = 0.5$ and $X_{\rm CNO} = 0.01$.

4. Now use the expressions for nuclear energy production to find at which temperature $T$ the energy production rate of the triple-alpha process equals the energy production the star had on the main sequence (using the numbers in the previous question). To calculate the energy production rate from the triple-alpha process you need to find a reasonable number for $X_{\rm He}$ in the core at the onset of helium burning. Give some arguments for how you find this number. You also need a density $\rho$, but since the energy production rate is so much more sensitive to the temperature than to the density you can make the crude approximation that the core density is the same as it was on the main sequence. Use the temperature you find here as the criterion for the onset of helium burning (and therefore the criterion for when the star moves to the horizontal branch in the HR-diagram).

5. Use the equations and numbers we have derived in this exercise to find the radius $R_C$ of the core at the moment when the energy production from helium fusion starts (has become significant). Express the result in terms of solar radii $R_\odot$. At the main sequence, the core radius was $R_C = 0.2R_\odot$. You have now found how much the core needs to contract in order to start helium fusion and therefore to move down to the horizontal branch.

6. When you calculated the temperature for the onset of helium burning you made a very rough approximation: You used the core density which the star had on the main sequence, whereas you should really use the much higher density in the core when the core temperature is high enough for helium burning. Now you have estimated the size of the core radius when helium burning starts.

Use this to obtain the correct density when helium burning starts and go back to find a more correct temperature for the onset of helium burning. Was the error in your first crude estimate of the helium burning temperature large relative to the temperature?