The fundamental wave modes of magneto-hydro-dynamics

The fundamental wavemodes in Magneto Hydro Dynamics (MHD) can be found by linearizing the continuity equation, the momentum equation, the induction equation and the equation of state :

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} &= 0 & (1)\\
\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p - \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0} &= 0 & (2)\\
-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) &= 0 & (3)\\
\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) &= 0 & (4)
\end{aligned}$$

Linearize all equations, that is assume that all parameters can be written as a constant term with subscribt 0 and a small varying term with subscript 1 (example : $\rho = \rho_0 + \rho_1$).

$$\begin{aligned}
\frac{\partial \rho_0 + \rho_1}{\partial t} + (\rho_0 + \rho_1) \nabla \cdot (\mathbf{u_0} + \mathbf{u_1}) &= 0 & (5)\\
(\rho_0 + \rho_1) \frac{\partial \mathbf{u_0} + \mathbf{u_1}}{\partial t} + \nabla p_0 + p_1 - \frac{(\nabla \times \mathbf{B_0} + \mathbf{B_1}) \times (\mathbf{B_0} + \mathbf{B_1})}{\mu_0} &= 0 & (6)\\
-\frac{\partial \mathbf{B_0} + \mathbf{B_1}}{\partial t} + \nabla \times ((\mathbf{u_0} + \mathbf{u_1}) \times (\mathbf{B_0} + \mathbf{B_1})) &= 0 & (7)\\
\frac{\partial}{\partial t}\left(\frac{p_0 + p_1}{(\rho_0 + \rho_1)^\gamma}\right) &= 0 & (8)
\end{aligned}$$

Since all time and space derivatives of the constant (subscript 0) variables are zero, and we can transform this into a co-moving system, that is we move with the velocity $\mathbf{u_0}$, then $\mathbf{u_0} = 0$ we get :

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u_1} &= 0 & (9)\\
\rho_0 \frac{\partial \mathbf{u_1}}{\partial t} + \nabla p_1 - \frac{(\nabla \times \mathbf{B_1}) \times \mathbf{B_0}}{\mu_0} &= 0 & (10)\\
-\frac{\partial \mathbf{B_1}}{\partial t} + \nabla \times (\mathbf{u_1} \times \mathbf{B_0}) &= 0 & (11)\\
(\rho_0 + \rho_1)^{-\gamma} \frac{\partial p_1}{\partial t} + (p_0 + p_1) \frac{\partial (\rho_0 + \rho_1)^{-\gamma}}{\partial t} &= \\
\frac{(\rho_0 + \rho_1)^\gamma \frac{\partial}{\partial t}(p_0 + p_1) - (p_0 + p_1) \frac{\partial}{\partial t}(\rho_0 + \rho_1)^\gamma}{(\rho_0 + \rho_1)^{2\gamma}} &= \\
\frac{1}{p_0 + p_1} \frac{\partial p_1}{\partial t} - \frac{\gamma}{(\rho_0 + \rho_1)^\gamma} (\rho_0 + \rho_1)^{\gamma - 1} \frac{\partial}{\partial t}(\rho_0 + \rho_1) &=
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{p_0+p_1}\frac{\partial p_1}{\partial t}-\frac{\gamma}{\rho_0+\rho_1}\frac{\partial \rho_0+\rho_1}{\partial t} &= \\
(\rho_0+\rho_1)\frac{\partial p_1}{\partial t}-\gamma\,(p_0+p_1)\frac{\partial \rho_1}{\partial t} &= \\
\frac{\partial}{\partial t}\left(\frac{p_1}{p_0}\right)-\frac{\partial}{\partial t}\left(\frac{\gamma\rho_1}{\rho_0}\right) &= 0 \qquad (12)
\end{aligned}
$$

We have here used that eq: 12 is equal to zero, so we can move around the terms by multip[lying and dividing with them. We have furthermore used that since the numbers with subscript 1 is very small, then when two numbers, both with subscripts 1 are multiplied it is even smaller, so we can ignore those.

Now we assume that the variations are of a wave nature, inserting :

$$
\begin{aligned}
\rho_1 &= \rho_A \exp\left(i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right) \qquad (13)\\
p_1 &= p_A \exp\left(i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right) \qquad (14)\\
\mathbf{B}_1 &= \mathbf{B}_A \exp\left(i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right) \qquad (15)
\end{aligned}
$$

and then using Eqs 9, 11 and 12 we get :

$$
\begin{aligned}
-\omega\rho_1+\rho_0\left(\mathbf{k}\cdot\mathbf{u_1}\right) &= 0 \qquad (16)\\
\omega\, B_1-\mathbf{k}\times\left(\mathbf{u_1}\times\mathbf{B_0}\right) &= 0 \qquad (17)\\
-\omega\frac{p_1}{p_0}+\omega\frac{\gamma\rho_1}{\rho_0} &= 0 \qquad (18)
\end{aligned}
$$

Assuming that $\mathbf{B_0}$ is parallel with the z-axis and that $\mathbf{k}$ is perpendicular with y and we use $V_A=\sqrt{\frac{B_0^2}{\mu_0\rho_0}}$ and that $V_S=\sqrt{\frac{\gamma p_0}{\rho_0}}$ then we can find three independent roots of this set of equations with corresponding eigenvectors.

The first and simplest one is the Alfvèn wave which has $\omega = k\,V_A\,\cos\theta$ where θ is the angle between x and $\mathbf{k}$. It furthermore has $\mathbf{k}\cdot\mathbf{u_1}=0$ and $\mathbf{u_1}\cdot\mathbf{B_0}=0$ which means that there is no pertubation in density and and no pertubation in magnetic pressure, i.e. it is a purely magnetic disturbance moving with a speed of $V_A\cos\theta$.

The second and third type is the so called fast and slow magnetosonic waves. Their speed is given by :

$$
V_{\pm}=\sqrt{\frac{1}{2}\left(V_A^2+V_S^2\pm\sqrt{\left(V_A^2+V_S^2\right)^2-4V_A^2V_S^2\cos\theta^2}\right)} \qquad (19)
$$

and they are both able to move across the magnetic field. They are magnetosonic in the sense that it in some sense is a superposition of a sound wave and an Alfvèn wave. They both have thermodynamic pressure and magnetic tension working as a restoring force, and the diffenerence is that the two restoring forces are in phase for a fast magnetosonic wave, while it is in antiphase for a slow magnetosonic wave.

For examples of the three wave types see for instance:

http://solar.physics.montana.edu/magara/Research/Topics/MHD_waves/mhdw.html