UNIVERSITY OF OSLO

AST 4130 Gasdynamics

Second take-home exam

Deadline: Thursday 16/11, 2011 at 16.00

Problem I

A homogeneous plasma is permeated by a constant magnetic field $\boldsymbol{B}_0$. Low frequency, plane MHD waves with wave vector $\boldsymbol{k}$ making an angle θ with the magnetic field is propagating in the plasma. The expression for the velocity perturbation $\boldsymbol{u}_1$ induced in the plasma due to the wave is given by equation (4.70) in the course compendium.

a) Write down equation (4.70), $\mathcal{M} \cdot \boldsymbol{u}_1 = 0$, in matrix form for a cartesian coordinate system in which the z-axis is parallel with $\boldsymbol{B}_0$ and $\boldsymbol{k}$ is lying in the xz-plane, that is, give the explicit form of the matrix $\mathcal{M}$.

b) Check that the dispersion relation for Alfvén waves is one solution of $\det(\mathcal{M}) = 0$. What is the possible polarization of this wave?

c) Show that the phase velocity of the Alfvén wave as a function of the angle θ, form two circles in the xz-plane with centers at $(0, \pm V_A/2)$. For any point on these circles show that the wave vector $\boldsymbol{k}$ divides the angle between the normal $\hat{\boldsymbol{n}}$ to this circle and the corresponding group velocity $\boldsymbol{v}_{gr}$ for the Alfvén wave in two equal halves.

The surface formed by rotating the $v_{ph}(\theta)$-curves around the $\boldsymbol{B}_0$-line is called a phase velocity surface. For the Alfvén wave this is a dumbbell shaped surface. A related surface can be formed by rotating the curve of the refractive index $\mathcal{N} = ck/\omega$ as a function of θ around the same line.

d) What is the form of the refractive index surface $\mathcal{N}(\theta)$ for the Alfvén wave? What relations exist between the group velocity $\boldsymbol{v}_{gr}$, the wave vector $\boldsymbol{k}$ and the normal $\hat{\boldsymbol{n}}$ to the refractive index surface?

In figure 1a the phase velocities for the three MHD waves (slow (S), Alfvén (A) and fast (F) modes) are plotted as a function of θ n the interval $0 < \theta < \pi/2$ for the case $V_A > C_s$. In the figure the polarization for the three modes are indicated, together with the group velocities for the three modes for a given value of θ.

e) Argue why the polarization of the slow and fast modes must both be parallel to the xz-plane.

f) Give a qualitative sketch of the form of the refractive index curves of the slow and fast modes as functions of θ. Does the relations between the directions of the group velocity $\boldsymbol{v}_{gr}$, the wave vector $\boldsymbol{k}$ and the normal $\hat{\boldsymbol{n}}$ of the refractive index curve that you found for the Alfvén wave above seem to hold also for the slow and fast waves?
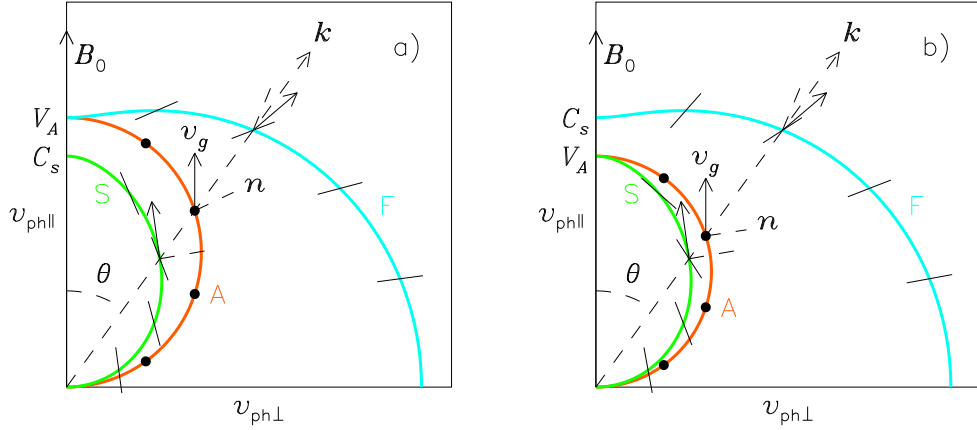
Figure 1: Section through the phase velocity surface for the Alfvén, the fast and the slow magneto-sonic waves for a) $V_a > C_s$ and b) $V_A < C_s$. Group velocities and polarizations for the different modes are indicated. Only one quadrant of the section is plotted.

g) (*Optional*) Are you able to give a mathematical proof that your conclusions under point f) is correct for any wave mode with dispersion relation of the form $\omega = \omega(k, \theta)$?

Problem II

We consider high frequency electromagnetic waves propagating in a cold, magnetized plasma consisting of electrons and one or more species of ions. We consider waves with wave vector $\boldsymbol{k}$ parallel with the magnetic field.

a) What are the dispersion relations valid for this case?

b) At given point, for instance $\boldsymbol{r} = 0$, study the time variation of the electric field $\boldsymbol{E}$ in each of the allowed wave modes.

c) Does these wave modes give rise to any space charge density in the plasma? If so, give the explicit expression for this charge density.

d) For frequencies ω well above all plasma and gyro frequencies in the plasma show that the ions contribute negligibly to the properties of the waves.

We now consider electromagnetic waves propagating along a magnetic field $\boldsymbol{B}_0$ towards an increasing plasma density. We assume the cold plasma model still to be valid.

e) For each of the wave modes considered above discuss what will the happen to the waves as the plasma density increases. Will the waves be absorbed? Will the waves be reflected? If so, where will this take place?