.

UNIVERSITY OF OSLO

AST 4210 Radiation I

First mid-term take-home exam

(to be followed by oral examinations!)

Deadline: Monday 27/9, 2003 at 16.15

Problem 1

In the derivation of the frequency spectrum of the radiation from an accelerated charged particle, $\mathrm{d}I(\omega)/\mathrm{d}\Omega$ we assumed $\mathrm{d}P(t)/\mathrm{d}\Omega$ to be in the form of one short pulse. For radiation from a relativistic electron spiraling in a magnetic field, this assumption is not strictly correct, the radiation received by an observer will be in the form of a series of short pulses, one gyro period apart. In the derivation of $\mathrm{d}I(\omega)/\mathrm{d}\Omega$, instead of making use of the Parseval's theorem for Fourier transforms (table 1.1), we should more properly have made use of the theory for Fourier series for periodic functions as outlined in table 1.2.

Fill out the necessary remaining steps to show that formula (2.48),

$$\frac{\mathrm{d}P_n}{\mathrm{d}\Omega} = \frac{\omega_B^2}{2\pi}\frac{\mathrm{d}I}{\mathrm{d}\Omega}(\omega = n\omega_B),$$

where P_n represents the power in the n.th harmonic, is indeed correct.

Proof: With $T = 2\pi/\omega_B$,

$$\boldsymbol{E}_{\text{one pulse}}(t) = \begin{cases} \boldsymbol{E}(t) & \text{for } t \in [-T/2, T/2] \\ 0 & \text{otherwise} \end{cases}$$

and

$$[\tilde{\boldsymbol{E}}R](\omega) \equiv \frac{1}{2\pi}\int \boldsymbol{E}_{\text{one pulse}}(t)R(t')) \exp(\iota\omega t)\,\mathrm{d}t :$$

$$\frac{1}{T}\int_{-T/2}^{T/2} \boldsymbol{E}^2(t)R^2(t')\,\mathrm{d}t = \cdots$$

$$= \frac{1}{T}\sum_{n,n'=-\infty}^{\infty}\int_{-T/2}^{T/2}\frac{2\pi}{T}[\tilde{\boldsymbol{E}}R]\left(\frac{2\pi n}{T}\right)\exp\left(-\iota\frac{2\pi n}{T}t\right)\cdot\frac{2\pi}{T}[\tilde{\boldsymbol{E}}R]^*\left(\frac{2\pi n'}{T}\right)\exp\left(\iota\frac{2\pi n'}{T}t\right)\,\mathrm{d}t$$

$$= \cdots = \frac{2\pi}{T^2}\cdot 2\pi\sum_{n=-\infty}^{\infty}\left|[\tilde{\boldsymbol{E}}R]\left(\frac{2\pi n}{T}\right)\right|^2$$

and therefore

$$\frac{1}{T}\int_{-T/2}^{T/2}\frac{\mathrm{d}P(t)}{\mathrm{d}\Omega}\,\mathrm{d}t=\sum_{n=1}^{\infty}\frac{\mathrm{d}P_n}{\mathrm{d}\Omega}$$

with

$$\frac{\mathrm{d}P_n}{\mathrm{d}\Omega}=\cdots\,.$$

Problem 2

Free electrons in the solar corona will, when irradiated by the radiation from the Sun, be set into accelerated motion and thus start to radiate. This process represents a scattering mechanism for the solar radiation. We want to estimate the effectiveness of this mechanism. We assume the electron density in the solar corona to be $N(\boldsymbol{r})$. Typical electron temperatures in the corona are 10^6 K. Assume one frequency component of the solar radiation field to be

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_0\cos(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t),$$

with $\boldsymbol{E}_0=E_0\hat{\boldsymbol{\epsilon}}$ and the wave vector $\boldsymbol{k}_0$ pointing radially away from the Sun. We may assume ω_0 to correspond to the H_α-line.

a) What is the energy flux carried by the given radiation field? Can you provide an upper limit estimate for the amplitude E_0?

b) Set up the equation of motions for electrons in the incident radiation field. (Hint: neglect the contribution from the magnetic fields.)

c) What is the typical excursion of an electron during one period of the incident radiation? Can you justify assuming the $\boldsymbol{k}_0\cdot\boldsymbol{r}$-term to be constant when solving for the electron oscillation?

d) What is the expression for the angular distribution of the scattered power $\mathrm{d}P(t)/\mathrm{d}\Omega$ from one electron?

e) What is the fraction of the incident power that is scattered by the electrons per unit volume in the corona and in the direction $\hat{\boldsymbol{n}}$ toward a distant observer?

f) What do you predict with respect to the polarization of the scattered radiation? Discuss.

Problem 3

a) In spherical coordinates show that the quantum mechanical operators $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$, L_z, $\boldsymbol{L}^2$ and $\boldsymbol{p}^2/2m$ take the form

$$\boldsymbol{L}=-\imath\hbar\boldsymbol{r}\times\nabla=-\imath\hbar\left(\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}-\hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right),$$

$$L_z=-\imath\hbar\frac{\partial}{\partial\varphi},$$

$$\boldsymbol{L}^2=-\imath\hbar\boldsymbol{r}\cdot(\nabla\times\boldsymbol{L})=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right],$$

and

$$\frac{\boldsymbol{p}^2}{2m} = -\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\boldsymbol{L}^2}{2mr^2}.$$

b) Now show that the operators H, $\boldsymbol{L}^2$ and L_z all commute,

$$[H, L_z] = 0, \quad [H, \boldsymbol{L}^2] = 0, \quad \text{and} \quad [\boldsymbol{L}^2, L_z] = 0.$$

(Hint: Remember that these operators all operate on scalar wave functions $\Psi(\boldsymbol{r})$.)