Radiation I

Solutions to Selected Problems

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Chapter 4

Spectra of One-Electons Atoms

Quiz 4.5:

$$[\nabla^2, x]\,\psi \equiv \nabla^2(x\psi) - x\nabla^2\psi = \nabla\cdot(\psi\nabla x + x\nabla\psi) - x\nabla^2\psi -$$
$$= 2\frac{\partial}{\partial x}\psi \quad \text{and similarly for } y \text{ and } z.$$

Thus,

$$[H, \boldsymbol{r}] = -\frac{\imath\hbar}{m}\boldsymbol{p}.$$

Similarly,

$$[H, \boldsymbol{p}]\,\psi = [U(r), \boldsymbol{p}]\,\psi = U\boldsymbol{p}\,\psi - \boldsymbol{p}\,(U\psi) = \imath\hbar\nabla U\,\psi.$$

Substituting $Q = \boldsymbol{r}$ and $Q = \boldsymbol{p}$ in (4.12) and noting that the operators $\boldsymbol{r}$ and $\boldsymbol{p}$ are both independent of t (the wave function ψ and therefore $\langle\, \boldsymbol{r}\,\rangle$ and $\langle\, \boldsymbol{p}\,\rangle$ may depend on t!) we find

$$m\frac{\mathrm{d}}{\mathrm{d}t}\langle\, \boldsymbol{r}\,\rangle = \langle\, \boldsymbol{p}\,\rangle \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\, \boldsymbol{p}\,\rangle = -\langle\, \nabla U(r)\,\rangle.$$

That is, the mean values of quantum mechanical operators $\boldsymbol{r}$ and $\boldsymbol{p}$ behaves as the classical quantities $\boldsymbol{r}$ and $\boldsymbol{p}$!

Quiz 4.6:
With $A' \equiv A - \langle\, A\,\rangle$ and similarly for B we may write

$$\begin{aligned}
(\Psi, [A,B]\,\Psi) &= (\Psi, [A',B']\,\Psi) \quad \langle\, A\,\rangle \text{ and } \langle\, B\,\rangle \text{ are numbers} \\
&= (A'\Psi, B'\Psi) - (B'\Psi, A'\Psi) \quad \text{operators } A' \text{ and } B' \text{ are Hermitian} \\
&= (A'\Psi, B'\Psi) - (A'\Psi, B'\Psi)^* \quad \text{from definition of scalar product} \\
&= 2\imath\,\mathrm{Im}\big((A'\Psi, B'\Psi)\big)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{1}{2}\,|\,(\Psi, [A,B]\,\Psi)\,| &\le |\,(A'\Psi, B'\Psi)\,| \quad \text{imaginary part less than absolute value} \\
&\le \langle\, \Delta A\,\rangle\langle\, \Delta B\,\rangle \quad \text{from Schwartz inequality, see section 2.6.2}
\end{aligned}$$

The minimum value of the product of the uncertainties in mean values $\langle\, A\,\rangle$ and $\langle\, B\,\rangle$ is determined by the commutator of the corresponding operators.