Chapter 4

Motion in one dimension

As a professional physicist you will be expected to be able to determine how things move: What is the path of a proton through a curved particle accelerator? What is the motion of a passenger in a car during a collision? How does a blood cell move through the micro-capillaries in your body? Professionally and privately, you will be expected to be able to solve any such problem your friends or your employer may come up with. How can you pull it off?

Fortunately, there is a simple method to determine the motion of an object. Objects move due to the forces acting on them. As soon as you have figured out what forces are acting on them, and you have found a model that predicts the magnitude and direction of the force during the motion, you can find the acceleration of the object. From the acceleration you can determine the motion of the object given its starting position and velocity. You will work through this procedure repeatedly over the next chapters, gradually filling in all the concepts with meaning, until it becomes a natural part of your way of thinking.

In this chapter we concentrate on developing our intuition of motion, on finding methods to formulate mathematical equations that determine the motion, and on developing analytical and numerical methods to solve the equations of motion.

You will learn to describe the motion of an object by its position as a function of time. We introduce the velocity and the acceleration of an object, which are the first and second time-derivatives of the position of the object. We also show how to find expressions for the motion from the velocity or acceleration – finding the equations of motion for the object.

4.1 Description of motion

In a fantastic race in the 100m finals of the 2008 Olympic Games in Beijing, Usain Bolt set a new world record of 9.69 seconds. He even took the time to celebrate his victory over the last 20 meters of the race. But did this affect his winning time? Could he have run even faster?

In order to answer such a question, we need a quantitative description of the race. We already know something: He ran 100 meters in 9.69 seconds. But we want more detail – a finer resolution of the motion. We want to know where he was at any intermediate time from he started until he finished the race.

Motion diagram

The first few seconds of the race are illustrated by the four pictures in figure 4.1. How can we describe the motion of Usain Bolt in lane four? One method is to define his position by the front of his chest. For each image, we draw a dot on the ground directly below his chest, resulting in a sequence of dots along lane four. We can now describe the race by measuring the distance, x, from the starting line to each dot – giving us a sequence of *positions*, x_i, at times t_i, for $i = 0, 1, 2, \ldots$:

i	0	1	2	3	4	5	6
t_i	0.0s	1.0s	2.0s	3.0s	4.0s	5.0s	6.0s
x_i	0.0m	3.4m	11.1m	21.3m	33.2m	45.8m	57.9m

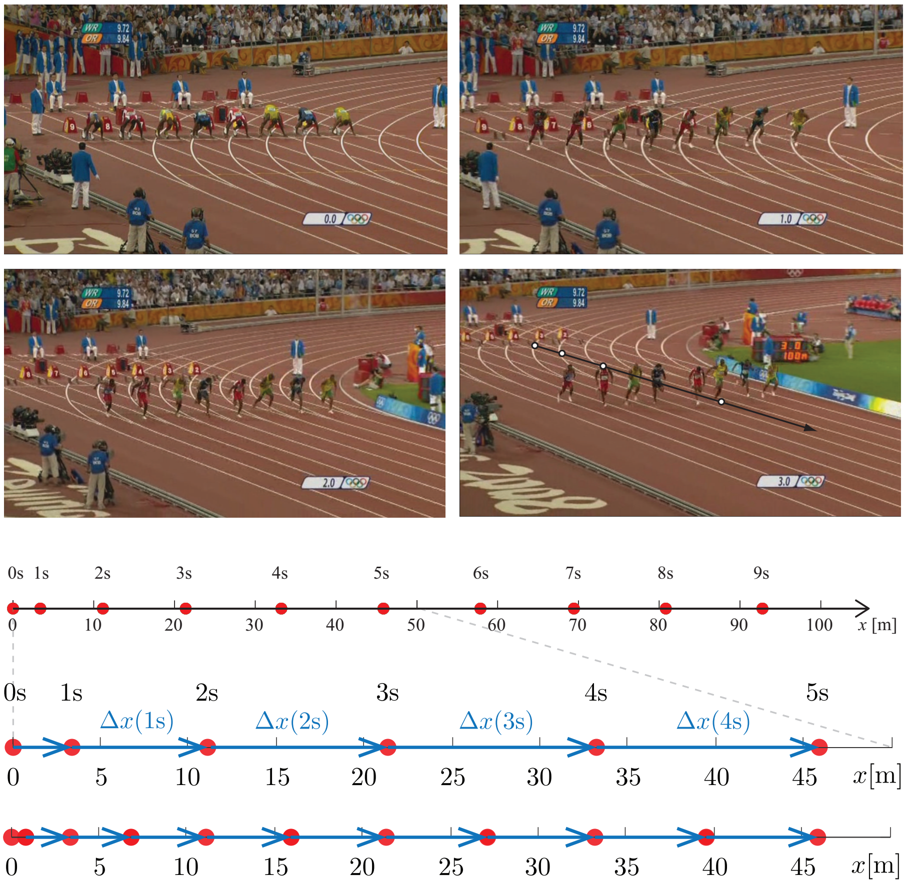


Figure 4.1: *(Top) Pictures from the 100m final in the 2008 Olympic Games in Beijing, showing the position of the runners during the first three seconds. The dots in the 3s image illustrate the position of the runner in lane 4 after 0s, 1s, 2s, and 3s. (Bottom) The position $x(t_i)$ of the runner is shown at 1s and 0.5s intervals. Displacments Δx are drawn in blue.*

We plot a point at the position x_i along the x-axis to illustrate the motion in a *motion diagram* (figure 4.1):

> A **motion diagram** illustrates the motion by a sequence of positions x_i at subsequent times t_i for $i = 0, 1, 2, \ldots$, preferrably at times $t_i = t_0 + i\Delta t$, where Δt is the time interval.

Position and time

From figure 4.1 we see that the runner is at $x(0\text{s}) = 0.0\text{m}$ when $t = 0\text{s}$ and at $x(3s) = 21.3\text{m}$ when $t = 3\text{s}$. Even though we have only measured the position at discrete times t_i, we expect the position of the runner to vary continuously with time, as illustrated by the plot of $x(t)$ in figure 4.2. This is indeed how we characterize motion:

> The motion of an object is described by the **position**, $x(t)$, as a function of time, t, measured in a given reference system.

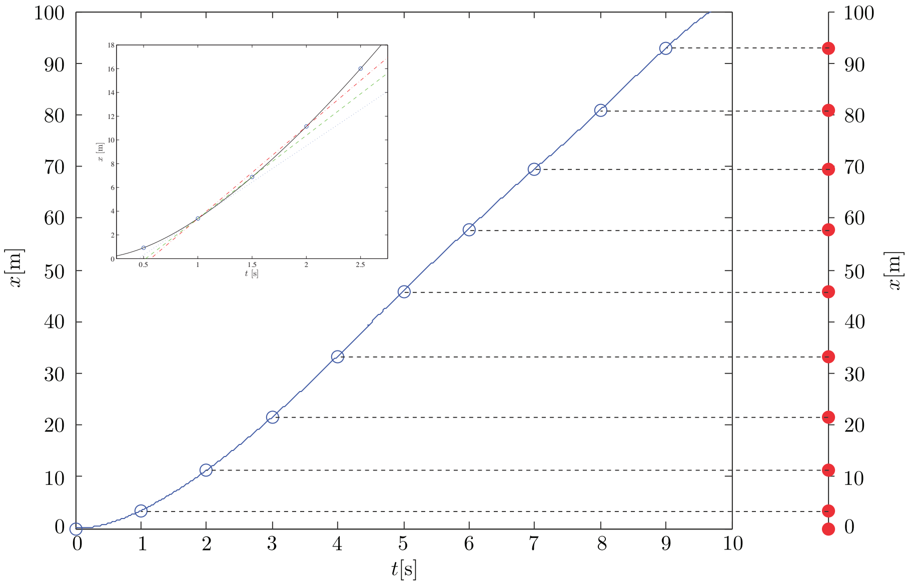


Figure 4.2: *A plot of the position x as a function of time for Usain Bolt. The circles along the curve show the position at time intervals of 1s, corresponding to the positions in the motion diagram. The correspondence between the two representations of the motion is shown by inserting a rotated motion diagram to the right of the plot. (Inset) A magnification of $x(t)$. The average velocities at $t = 1s$ for time intervals $\Delta t = 1s$ and $\Delta t = 0.5s$ are illustrated by the slopes of the red and green lines respectively. The instantaneous velocity is illustrated by the slope of the dotted blue line, which corresponds to the slope of the tangent to the curve at $t = 1s$.*

Reference system and origin

We have chosen to measure the position x along the running lane. We call this direction the x-axis. The position x is measured from the starting line, which we call the origin – the point where x is zero. The choice of an origin and an axis is called a *reference system*. The axis has a direction which tells us in what direction x is increasing – this is indicated by the arrow on the axis. For the race the axis is directed from the starting line toward the finishing line, so that the position of the runner increases during the race.

You are free to choose the axes and the origin of your reference system as you like, but we usually try to choose so that measurements become simple, as we have done here.

4.1.1 Velocity

The motion diagram in figure 4.1 visualizes the change in position over a time interval Δt. The change in position from time $t = 1\text{s}$ to $t = 2\text{s}$ is:

$$x(2\text{s}) - x(1\text{s}) = 11.1\text{m} - 3.4\text{m} = 7.7\text{m} \tag{4.1}$$

We call this change the *displacement*, $\Delta x(1\text{s})$:

> The displacement $\Delta x(t_1)$ over the time interval from $t = t_1$ to $t = t_1 + \Delta t$ is defined as:
>
> $$\Delta x(t_1) = x(t_1 + \Delta t) - x(t_1) \; . \tag{4.2}$$

The displacement is read directly from the motion diagram as the length of the line from $x(1\text{s})$ to $x(2\text{s})$. The displacement has a direction – it is the displacment *from* $x(t_i)$ *to* $x(t_i + \Delta t)$ – and it is therefore drawn as an arrow in figure 4.1.

The first few displacements in figure 4.1 are increasing. This means that he is running faster. But how fast he is running? This cannot be described by displacement alone, because the displacements become smaller when we decrease the time interval as shown in figure 4.1. It is the displacement per time that describes how fast he is running:

The **average velocity** from $t = t_1$ to $t = t_1 + \Delta t$ is:

$$\bar{v}(t_1) = \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} = \frac{\Delta x(t_1)}{\Delta t} . \quad (4.3)$$

The average velocity has units meters per second, m/s.

The average velocities for the runner in figure 4.1 at $t = 1$s and $t = 2$s over the time interval $\Delta t = 1$s are:

$$\bar{v}(1\text{s}) = \frac{7.7\text{m}}{1\text{s}} = 7.7\text{m/s} , \quad (4.4)$$

$$\bar{v}(2\text{s}) = \frac{10.2\text{m}}{1\text{s}} = 10.2\text{m/s} , \quad (4.5)$$

However, if we calculate the average velocity from the bottom-most diagram in figure 4.1, the time interval is $\Delta t = 0.5$s, and the velocities are:

$$\bar{v}(1\text{s}) = \frac{3.5\text{m}}{0.5\text{s}} = 7.0\text{m/s} , \quad (4.6)$$

$$\bar{v}(2\text{s}) = \frac{4.9\text{m}}{1\text{s}} = 9.8\text{m/s} , \quad (4.7)$$

We see that the average velocities depend on the time interval Δt! We can understand this from the inset in figure 4.2. First, we notice that we can read the average velocity $\bar{v}(1\text{s})$ directly from the curve, $x(t)$, as the slope of the curve from the point $x(1\text{s})$ to the point $x(1\text{s} + \Delta t)$. From the figure, we see that $\bar{v}$ changes slightly as we change the time interval from $\Delta t = 1$s to $\Delta t = 0.5$s because the function $x(t)$ is curving. However, we also see that when the time interval Δt becomes smaller and smaller, the average velocity approaches a specific value given as the slope of the curve in the point $t = 1$s. We call the velocity in this limit the *instantaneous velocity* at the time t, $v(t)$:

The **instantaneous velocity** is defined as as the time derivative of the position:

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} . \quad (4.8)$$

In the following, whenever we use the term velocity, we will mean the instantaneous velocity.

Notation for time derivatives

Notice that the notation $x'(t)$ for the derivative that you may be used to from calculus, is not commonly used in physics. This is avoid confusion with x', which is often used to represent a length in a coordinate system called the "marked" coordinate system. The notation $x'(t)$ can therefore be ambiguous: it may be interpreted as either the position x' as a function of time, or as the time derivative of the position x. Instead, we denote the time derivative of a quantity by the placing a dot over it. The velocity is therefore therefore often written as:

$$v(t) = \frac{dx}{dt} = \dot{x} . \quad (4.9)$$

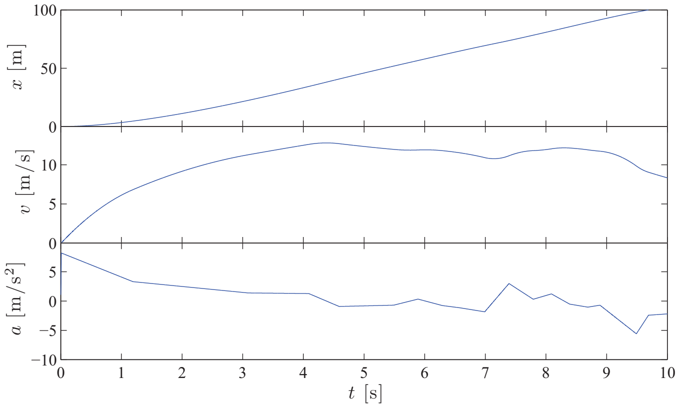


Figure 4.3: *A plot of the position $x(t)$, velocity, $v(t)$, and acceleration, $a(t)$, as a function of time for Usain Bolt.*

Visualizing the velocity $v(t)$

The velocity $v(t)$ represents the slope of the curve, $x(t)$. In many cases it may be useful to visualize the motion by looking at both the plot of $x(t)$ and the plot of $v(t)$, as shown in figure 4.3. In this case, it is evident that the velocity is changing throughout the motion. Initially, the velocity is increasing as the runner sprints out from the starting line. In the middle of the race the velocity is approximately constant, while at the end of the race, the runner is slowing down, and the velocity is dropping.

4.1.2 Acceleration

The velocity may also vary throughout the motion. From figure 4.3 we see that the runner starts at rest and increases his velocity with time. Just as we introduced the velocity to characterize the rate of change of position, we introduce the acceleration to characterize the rate of change of the velocity:

The **average acceleration** over a time interval Δt from t to $t + \Delta t$ is:

$$\bar{a}(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t} \ . \tag{4.10}$$

The instantaneous acceleration is the limit of the average acceleration when the time interval approaches zero:

The **instantaneous acceleration** is defined as:

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \dot{v} \ . \tag{4.11}$$

When we use the term acceleration we mean the instantaneous acceleration.

The acceleration can be found as the slope of the $v(t)$ curve. Figure 4.3 shows a plot of $a(t)$ together with both position $x(t)$ and velocity $v(t)$. Notice that the acceleration curve is “noisy” and consists of clear steps. This is not a physical effect, but rather an effect of how the data was gathered and interpolated. Real data often have noise from various sources – so you should expect noisy curves when you look at real systems. (You can learn more about how this data was measured in boltdatabox).

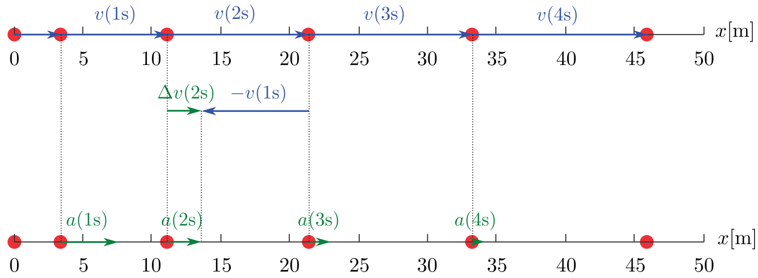


Figure 4.4: *Motion diagram for Usain Bolt. The top figure shows the velocities at time intervals of 1s. The displacements are interpreted as velocities. The top figure shows how the change in velocity at $t = 2s$ is constructed from the velocity at $t = 1s$ and the velocity at $t = 2s$. The resulting difference, $\Delta v(2s)$ is interpreted as the average acceleration. The bottom figure shows the accelerations estimated from the motion diagram.*

Because the velocity is given as the time derivative of the position $x(t)$, we can also write the acceleration as the time derivative of the position $x(t)$ by inserting equation 4.9 into equation 4.11:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2} \ . \tag{4.12}$$

Using the dot-notation, we can write this as:

$$a(t) = \dot{v}(t) = \ddot{x}(t) \ , \tag{4.13}$$

or in shorthand

$$a = \dot{v} = \ddot{x} \ . \tag{4.14}$$

Interpretation of motion diagrams

It is often difficult to obtain a good intuition for acceleration, in particular for two- and three-dimensional motions, but sometimes also for one-dimensional motions. Experience shows that motion diagrams are useful tools for developing a good intuition for accelerations – this is why we include them here.

As long as all the time intervals in a motion diagram are identical, the displacements in the motion diagram may be interpreted as average velocities. In figure 4.4 the displacements and therefore the average velocities, are initially increasing, until at $t = 4$s they are approximately constant. The change in average velocity from $t = 1$s to $t = 2$s is:

$$\Delta\bar{v}(1\text{s}) = \bar{v}(2\text{s}) - \bar{v}(1\text{s}) = 5\text{m/s} \tag{4.15}$$

We introduce the average acceleration as:

$$\bar{a} = \frac{\Delta\bar{v}}{\Delta t} \tag{4.16}$$

The average acceleration can be constructed geometrically from the motion diagram by subtracting two subsequent (average) velocities in the diagram, as illustrated in figure 4.4.

Example 4.1: Motion of a falling tennis ball

This example demonstrates how we can find the velocity and acceleration from the motion diagram of a falling tennis ball, both by hand calculation, using Matlab, and from a mathematical model of the motion.

Motion diagram

The motion of a falling tennis ball were captured with a digital camera. The first few images were combined into one picture as shown in figure 4.5. From the sequence of images, we measure the vertical position of the ball by comparing the height of the center of the ball to the ruler seen in the images. The positions are shown in table 4.1.

i	t_i	y_i	Δy_i	$\bar{v}_i$	$\bar{a}_i$
1	0.0 s	1.60 m	-0.05 m	-0.5 m/s	
2	0.1 s	1.55 m	-0.15 m	-1.5 m/s	-10.0 m/s^2
3	0.2 s	1.40 m	-0.24 m	-2.4 m/s	-9.0 m/s^2
4	0.3 s	1.16 m	-0.34 m	-3.4 m/s	-10.0 m/s^2
5	0.4 s	0.82 m	-0.43 m	-4.3 m/s	-9.0 m/s^2
6	0.5 s	0.39 m			

Table 4.1: *Table with calculated values.*

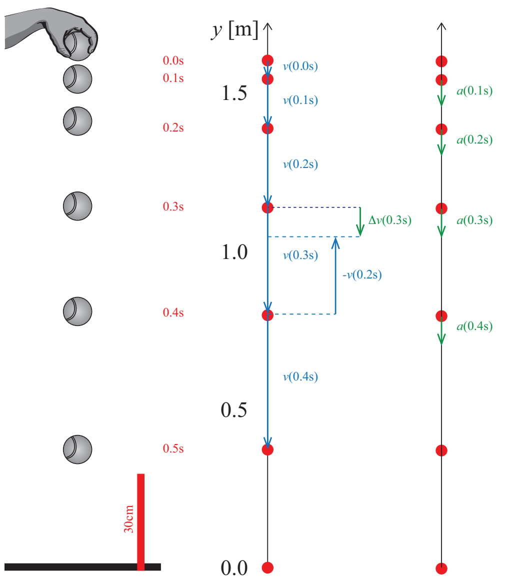


Figure 4.5: *(Left) Digital images from a falling tennis ball – we have made an artistic rendering of the ball for clarity. (Right) Motion diagram for the tennis ball. The left diagram shows the positions and velocities, and the right diagram illustrates the accelerations.*

We draw the motion diagram by marking the positions y_i with dots along the vertically oriented y-axis as illustrated in figure 4.5. We illustrate the velocities by the displacements, which are drawn as arrows from point to point. The average velocities can be calculated from the data: For each i in table 4.1 we calculate the average velocity from t_i to t_{i+1} using:

$$\bar{v}_i = \frac{y_{i+1} - y_i}{\Delta t} . \tag{4.17}$$

The corresponding results are shown in the table. However, we cannot use this method to find a value for $i = 6$, since we do not know y_7. We find that all the velocities are negative. Since we have chosen the positive direction to be up (the arrow on the y-axis points upward) this means that the ball is falling down – as expected.

The velocities are increasing in magnitude since the ball is accelerating downward. We estimate the average accelerations by

$$\bar{a}_i = \frac{\bar{v}_i - \bar{v}_{i-1}}{\Delta t} , \tag{4.18}$$

and the results are shown in table 4.1. For the accelerations, we cannot find a value for $\bar{a}_i$ for $i = 1$ or for $i = 6$, since the velocities are not defined at $i = 0$ or at $i = 6$. If you look at figure 4.5 you can also see how to contruct the accelerations directly from the motion diagram.

The data shows that the acceleration is approximately constant $a \simeq -9.5 \pm 0.5\text{m/s}^2$ throughout the fall. This experiment therefore tells us that a tennis ball falls with a constant acceleration – which is close to what you may recognize as the acceleration of gravity, $g = 9.8\text{m/s}^2$.

Mathematical model

A physicists friend of yours tells you that there is a mathematical model for the motion of a falling tennis ball when there is no air resistance

$$y(t) = y_0 - \frac{1}{2}gt^2 , \tag{4.19}$$

where $g = 9.8\text{m/s}^2$ is a constant and y_0 is the position of the tennis ball at $t = 0$s. Let us see how this model matches up with the observed data.

We calculate the position of the ball for various times. From the experimental data, we see that $y(0\text{s}) = 1.6$m. We use Matlab as a calculator to find the positions for all the times in table 4.1 with a single line of code:

```
g = 9.8;
t = [0.0 0.1 0.2 0.3 0.4 0.5];
y = 1.6 - 0.5*g*t.^2
```

```
y =    1.6000    1.5510    1.4040    1.1590
    0.8160    0.3750
```

Notice that the command `t.^2` tells Matlab to apply the operation for each element in the array `t`, generating an `y`-array of 6 elements. This vectorized notation allows us write the Matlab code in a similar way to the mathematics. We can output the data in a form that looks more like table 4.1:

```
[t;y]'
```

```
ans =
         0    1.6000
    0.1000    1.5510
    0.2000    1.4040
    0.3000    1.1590
    0.4000    0.8160
    0.5000    0.3750
```

where the ' means transpose. Without it, the table would have been oriented differently. Try it!

The resulting values for $y(t)$ are similar to the experimental data, but in the experiment the ball falls a bit slower than in the mathematical model: In the experiment the ball is at $y = 0.39$m at $t = 0.5$s, whereas the mathematical model predicts $y = 0.375$.

We can compare the results better by studying the velocities and accelerations. In the mathematical model, we know $y(t)$, and we can calculate the *instantaneous* velocity and acceleration by applying the definitions directly. The velocity of the ball is defined as:

$$v = \frac{dy}{dt} , \tag{4.20}$$

and if we insert $y(t)$ from equation 4.19 we get

$$v = \frac{d}{dt}\left(y_0 - \frac{1}{2}gt^2\right) = -gt . \tag{4.21}$$

Similarly, the acceleration is defined as

$$a = \frac{dv}{dt} , \tag{4.22}$$

where we insert $v(t)$ from equation 4.21 and get

$$a = -g = -9.8\text{m/s} . \tag{4.23}$$

The acceleration in the mathematical model is a constant. But we cannot really compare with the experimental data, since they have too low precision. We need better data!

High resolution data

To study the process in more detail, the motion of the falling tennis ball was also recorded by a motion detector placed directly above the ball. The detector provides the vertical position y of the ball, but at a much higher time resolution that the images: The detectors measures y at a time interval of $\Delta t = 0.001$s. The data is stored in the file `fallingtennisball02.d`. The first few lines of the file looks like:

```
0.0000000000000000e+00   1.6000000000000001e+00
1.0000000000000020e-03   1.5999950510001959e+00
2.0000000000000044e-03   1.5999803020031378e+00
3.0000000000000070e-03   1.5999557530158828e+00
...                      ...
```

where each line contains the time t_i in seconds and the position y_i in meters (given in scientific notation, but with no unit). We read the data-set from file, using `load`:

```
load -ascii fallingtennisball02.d
t = fallingtennisball02(:,1);
y = fallingtennisball02(:,2);
```

The command `load` generates the array `fallingtennisball10` which has 2 columns. Then, we create variables for the time, `t`, and the position `y`. We see what is in the data-set by plotting the position as a function of time, $y(t)$, using:

```
plot(t,y)
xlabel('t [s]')
ylabel('x [m]')
```

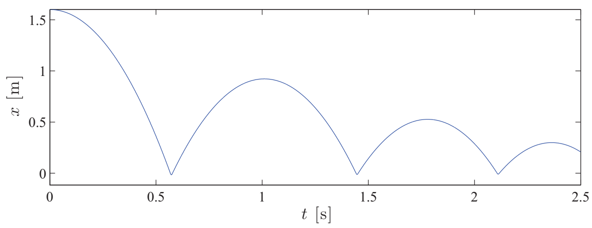


Figure 4.6: *Plot of the position y of the ball as a function of time t.*

What does the resulting plot in as shown in figure 4.6 show? From the plot, we see that the ball falls down, bounces up from the surface to reach a lower height than the first time, and so on. The first 0.5s of the motion resembles what we found by analyzing the images: the position decreases with time. And we see that ball is falling faster with time – it accelerates. But it is difficult to see details of the motion directly from this plot. Could you say if the acceleration is constant or not for the first 0.5 seconds from this plot? To gain more insight, we need to analyze the velocity and acceleration of the ball.

Numerical derivatives

Because we do not know $y(t)$ for all values of t, but only the measured values of $y(t_i)$, we cannot find an exact, analytical expression for the derivative of $y(t)$ as we did when we had a mathematical model. However, we can follow the procedure we used for the image data in equation 4.17: We can approximate the instantaneous velocity by the average velocity from t_i to $t_i + \Delta t$:

$$\frac{dy}{dt} = v(t_i) \simeq \bar{v}(t_i) = \frac{y(t_i + \Delta t) - y(t_i)}{\Delta t} . \tag{4.24}$$

The average velocity is an example of a *numerical derivative* of the position – a numerical method to calculate the derivative. (In numerical methods N.1 you will see that there are many ways to calculate the derivative numerically). This method is easily implemented numerically by directly converting the mathematical formula to Matlab:

```
v(i) = (y(i+1)-y(i))/dt;
```

We need to apply this rule to each element i i from 1 to $n-1$, where n is the number of data points $y(t_i)$. This is done using a `for`-loop:

```
n = length(y);
dt = t(2) - t(1);
v = zeros(n-1,1);
for i = 1:n-1
   v(i) = (y(i+1) - y(i))/dt;
end
```

Here, we find `n`, the number of elements in the `y`-array, and the time difference `dt`, which we calculate from the first two times since the time intervals are regular. We also prepare an empty array `v`, which we will fill with velocities. But why do we only make it $n-1$ elements long? Because the formula `v(i) = (y(i+1) - y(i))/dt` , cannot be applied to the last element in the array, since we would then have no data for $i+1$. (We saw the same in table 4.1). For the same reason, we must stop the loop at $n-1$.

Similarly, we find the acceleration by using the numerical derivative of the velocity:

$$a(t_i) \simeq \bar{a}(t_i) = \frac{v(t_i) - v(t_{i-1})}{\Delta t} . \tag{4.25}$$

We apply this mathematical definition of the derivative directly to the data:

```
a = zeros(n-1,1);
for i = 2:n-1
   a(i) = (v(i) - v(i-1))/dt;
end
```

For the acceleration, the formula `a(i) = (v(i) - v(i-1))/dt` , cannot be applied to the first element in the array, since we have no data for $i = 0$. The loop therefore starts at $i = 2$. (Again, this is the same as in table 4.1).

Plotting

We plot $x(t)$, $v(t)$, $a(t)$ by:

```
subplot(3,1,1)
plot(t,y)
ylabel('y [m]')
subplot(3,1,2)
plot(t(1:n-1),v)
ylabel('v [m/s]')
subplot(3,1,3)
plot(t(2:n-1),a(2:n-1))
```

```
xlabel('t [s]')
ylabel('a [m/s^2]')
```

Here we have used the `subplot` command to generate a set of plots. (Consult Matlab to find out how the plots are numbered using `help subplot`). Notice that the velocity is only defined for i from 1 to $n-1$. We therefore only include the corresponding values of t_i in the plot. Similarly, the acceleration is defined from 2 to $n-1$, and we only plot the corresponding values of t_i.

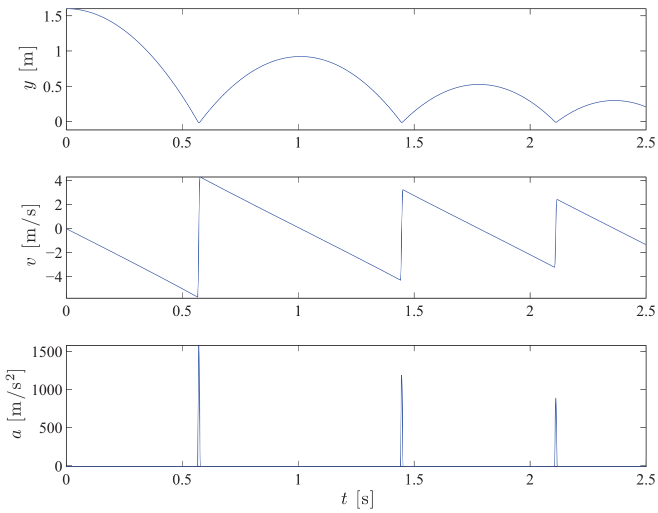


Figure 4.7: *Plot of $x(t)$, $v(t)$, and $a(t)$ for the falling tennis ball.*

Plotting parts of the data

It is difficult to see the acceleration of the ball while it is falling from figure 4.7. How can we plot only the first 0.5 seconds of the motion? We find the value for i where t_i goes from begin smaller than 0.5 to larger than 0.5 using `find`:

```
imax = max(find(t<=0.5));
plot(t(2:imax),a(2:imax));
xlabel('t [s]')
ylabel('a [m/s^2]')
```

and plot $a(t)$ for this range of t-values in figure 4.8. (You could also have made this plot by using the zoom button in the plotting window). The acceleration is clearly *not* a constant in this case. It starts at -9.8m/s^2, but its magnitude becomes smaller with time. (This is due to air resistance).

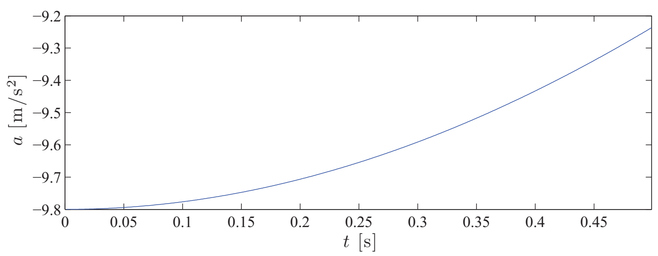


Figure 4.8: *Plot of $a(t)$ for the falling tennis ball in the time interval $t < 0.5$s.*

Comparison with mathematical model

How large are the differences between the experimental data and the mathematical model for motion without air resistance? A good way to compare, is to plot the model in the same plot as the data. The model was:

$$y(t) = y_0 - \frac{1}{2}gt^2 \text{ and } v(t) = -gt \ . \tag{4.26}$$

We implement these formulas directly in the program, and plot both data and model:

```
g = 9.8; % m/s^2
y0 = 1.6; % m
vt = -g*t;
yt = y0 - 0.5*g*t.^2;
subplot(2,1,1)
plot(t(1:imax),y(1:imax),'-r');
hold on
plot(t(1:imax),yt(1:imax),'--b');
hold off
xlabel('t [s]')
ylabel('y [m]')
subplot(2,1,2)
plot(t(1:imax),v(1:imax),'-r');
hold on
plot(t(1:imax),vt(1:imax),'--b');
hold off
xlabel('t [s]')
ylabel('v [m/s]')
```

We use `hold on` to get both plots in the same figure (see figure 4.9). Here we notice that the differences in $y(t)$ and $v(t)$ are more difficult to spot. Using the acceleration for comparions was therefore a better approach to spot the differences. And an approach with a sound, physical basis, since we will later learn that differences in physics appear in differences in the accelerations.

Further work

We leave it to you to look more carefully at what happens during the bounce. Can you zoom in on the relevant area?

(Notice that the data in this example were based on numerical results and not experimental data in order to get clear results. Experimental data will typically contain significant noise, which we did not want to include here. The program used to generate the data-set is `makefallingtennisball.m`).

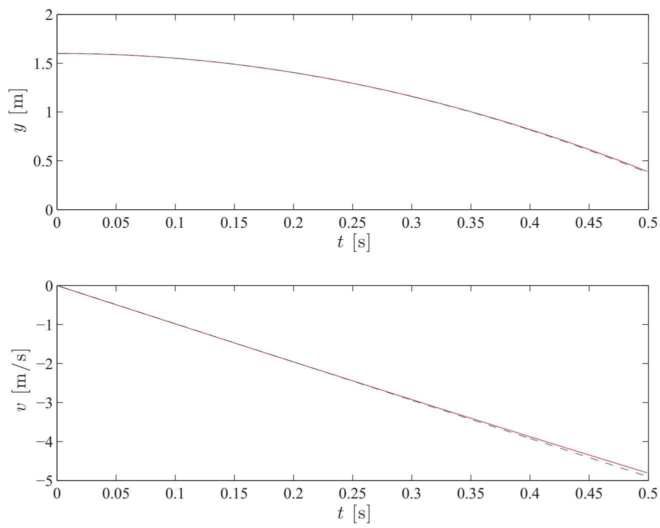
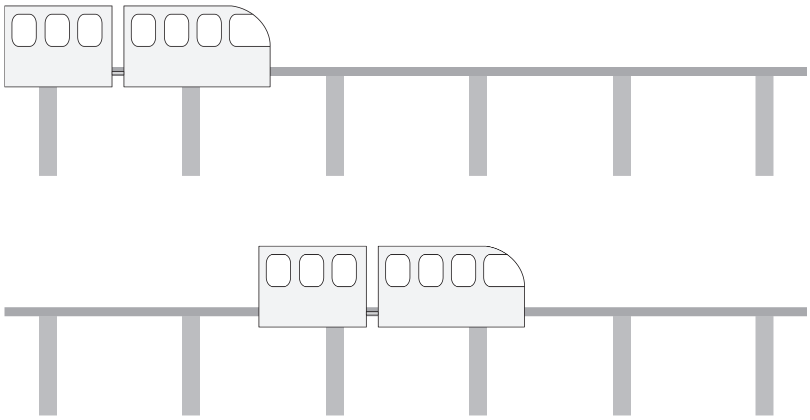


Figure 4.9: *Plot of $y(t)$ and $v(t)$ for the experimental data (red, solid line) and the mathematical model (blue, dashed lined).*



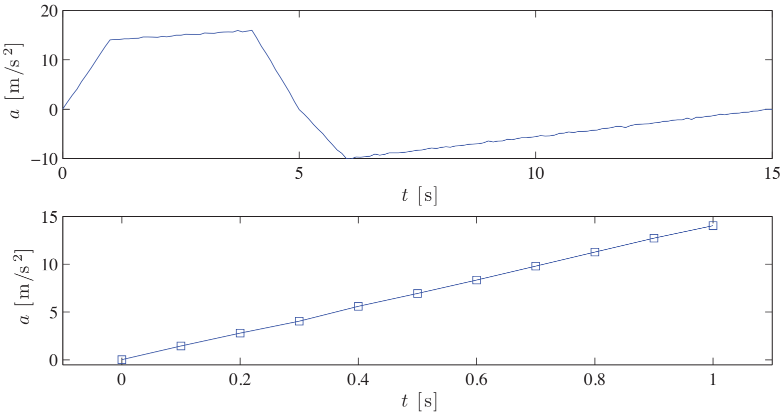


Figure 4.10: *Illustration of the motion of "The Rocket". The accelerations are illustrated for the whole time interval (top figure) and the time-resolution is shown by the squares representing the measurement points (bottom figure).*

4.2 Calculation of motion

Mechanics is about the motion of objects. Usually, we do not know the position as a function of time. Instead, we want to determine the motion based on measurements of the acceleration (or velocity), based on a mathematical expression for the acceleration, or based on a differential equation for the acceleration. We therefore need tools to do the opposite of what we did above: We need tools to find the motion, $x(t)$, from the acceleration, $a(t)$, of an object.

4.2.1 Discrete integration

As lead developer of "The Rocket", a new roller-coaster ride at a major theme-park, you have fitted an accelerometer into a test-cart. The accelerometer records the acceleration of the cart at regular time intervals of 0.1s. How can you use this data to determine the velocity and position of the test cart?

i	0	1	2	3	4	5
t_i	0.0s	0.1s	0.2s	0.3s	0.4s	0.5s
a_i	0.00m/s^2	1.43m/s^2	2.80m/s^2	4.13m/s^2	5.62m/s^2	7.21m/s^2

The problem is how to find the sequence of positions, $x(t_i)$, from the sequence of accelerations, $a(t_i)$? This is the reverse of what we have been doing so far, where we have estimated first the velocities and then the accelerations from the positions using numerical derivatives. Can we simply use the methods we have developed for numerical derivatives "in reverse"? The average acceleration from

$t_1 = 0.0\text{s}$ to $t_2 = 0.1\text{s}$ is

$$\bar{a}(t_i) = \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t} \ . \tag{4.27}$$

(So far this is an *exact* result – we have not done any approximations yet). We can "reverse" equation 4.27 to find an equation for the velocity at the time $t = t_i + \Delta t$:

$$\begin{aligned} \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t} &= \bar{a}(t_i) \\ v(t_i + \Delta t) - v(t_i) &= \Delta t \cdot \bar{a}(t_i) \\ v(t_i + \Delta t) &= v(t_i) + \Delta t \cdot \bar{a}(t_i) \end{aligned} \tag{4.28}$$

This method would allow us to step one step forward in time from the time $t = t_i$ to the time $t = t_i + \Delta t$, if only we knew the average acceleration of the time interval. Unfortunately, the accelerometer does not give the average, but rather the instantaneous acceleration of the cart, $a(t_i)$. Let us ignore this distinction and approximate the average acceleration over the time interval by the instantaneous acceleration at the beginning of the time interval:

$$\bar{a}(t_i) \simeq a(t_i) \ , \tag{4.29}$$

(You can learn more about this approximation and how to improve it in a discussion of numerical integration in numerical methods N.2.) We are now in a position to use equation 4.28 to step forward in small steps of Δt, calculating the changes in the velocities of the cart as we go. However, finding the velocities only takes us part of the way – we also need to determine the positions, $x(t_i)$, of the cart, from the velocities, $v(t_i)$, calculated using equation **??**. This time, we "reverse" the numerical derivative of the position:

$$\begin{aligned} \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t} &= \bar{v}(t_i) \\ x(t_i + \Delta t) - x(t_i) &= \Delta t \cdot \bar{v}(t_i) \\ x(t_i + \Delta t) &= x(t_i) + \Delta t \cdot \bar{v}(t_i) \ . \end{aligned} \tag{4.30}$$

Where we again assume that the average velocity is approximately the same as the velocity we calculated in equation 4.28: $\bar{v}(t_i) \simeq v(t_i)$. We are now ready to use equation 4.28 and equation 4.30 to move forwards in steps of Δt. However, since these methods only give the increments in the velocity and the position, we need to know the first velocity of the cart, $v(t_0) = v_0$ and where the cart starts from, $x(t_0) = x_0$. This is called the initial conditions of the problem. We are now ready to find the velocities and positions, starting at the time $t = t_0 = 0.0\text{s}$:

- At $t = t_0 = 0.0\text{s}$, the velocity and position of the cart is given $v(t_0) = v(0.0\text{s}) = 0.0\text{m/s}$, $x(t_0) = x(0.0\text{s}) = 0.0\text{m}$.
- At $t = t_0 + \Delta t = 0.1\text{s}$, the velocity of the cart is:

  $$v(0.1\text{s}) \simeq v(0.0\text{s}) + \Delta t \cdot a(0.0\text{s}) = 0.5\text{m/s} \ , \tag{4.31}$$

  where the acceleration $a(0.0\text{s}) = 5.0\text{m/s}^2$ is listed in the table figure 4.10. The position of the cart is:

  $$x(0.1\text{s}) \simeq x(0.0\text{s}) + \Delta t \cdot v(0.0\text{s}) = 0.0\text{m} \tag{4.32}$$

- At $t = t_1 + \Delta t = 0.2\text{s}$, the velocity of the cart is:

  $$v(0.2\text{s}) \simeq v(0.1\text{s}) + \Delta t \cdot a(0.1\text{s}) = 0.9\text{m/s};, \tag{4.33}$$

  where the acceleration $a(0.1\text{s}) = 7.0\text{m/s}$ is listed in the table in figure 4.10. The position of the cart is:

  $$x(0.2\text{s}) \simeq x(0.1\text{s}) + \Delta t \cdot v(0.1\text{s}) = 0.05\text{m} \ , \tag{4.34}$$

  where the velocity $v(0.1\text{s}) = 0.5\text{m/s}$ was found in the previous step of the calculation.

This method is called *Euler's method* for numerical integration, and it is sufficiently flexible and robust to solve most problems presented in this book!

> In **Euler's method** we find the position, $x(t_i)$, and velocity, $v(t_i)$, of an object as a function of time by a stepwise summation of the acceleration, $a(t_i)$, and the velocity, $v(t_i)$:
>
> $$\begin{aligned} v(t_0) &= v_0 \\ x(t_0) &= x_0 \\ &\ldots \\ v(t_i + \Delta t) &= v(t_i) + \Delta t \cdot \bar{a}(t_i) \\ x(t_i + \Delta t) &= x(t_i) + \Delta t \cdot v(t_i) \end{aligned} \tag{4.35}$$

We apply this method to find the position and velocities for the motion of "The Rocket". The accelerations for the cart are stored in the file `therocket.dat`, where each line contains a time (in seconds) and an acceleration (in m/s^2):

```
0.0000000e+000   2.7316440e-001
1.0000000e-001   1.4411079e+000
2.0000000e-001   2.6693138e+000
3.0000000e-001   4.2383806e+000
```

We read the data into Matlab, find the time-step Δt from $t_2 - t_1$, and apply Euler's algorithm from equation 4.35 for each i starting from the initial condition $x(t_0) = 0$m and $v(t_0) = 0$m/s using a `for`-loop.

```
temp  = load('therocket.dat');
t = temp(:,1);
a = temp(:,2);
dt = t(2) - t(1);
n = length(t);
v = zeros(n,1);
v(1) = 0.0; % v_0
x = zeros(n,1);
x(1) = 0.0; % x_0
for i = 1:n-1
    v(i+1) = v(i) + a(i)*dt;
    x(i+1) = x(i) + v(i)*dt;
end
```

The resulting position and velocity plots are shown in figure 4.11. (You can learn more about the precision of this method, and more precise methods in numerical methods N.2).

The procedure presented here covers the most important topic in kinematics: How to determine the motion of an object given the acceleration of the object. This is important because you will later learn that the physics of a problem – the interactions between the object and other objects – gives the acceleration of the object. Given the acceleration it will be up to you to determine the motion – and you can do this using the methods provided here: Either by using Euler's method (or more advanced techniques) to solve the problem numerically, or by finding a solution to the problem based on the specialized techniques you have learned in calculus.

4.2.2 Formal integration

A more formal formulation of the problem would be to assume that we know the acceleration $a(t)$ of an object as a function of time. How do we find the position and velocity of the object as a function of time in this case?

Again, we realize that we have already solved the "reverse" problem – we know that the acceleration is the time derivative of the velocity and that the velocity

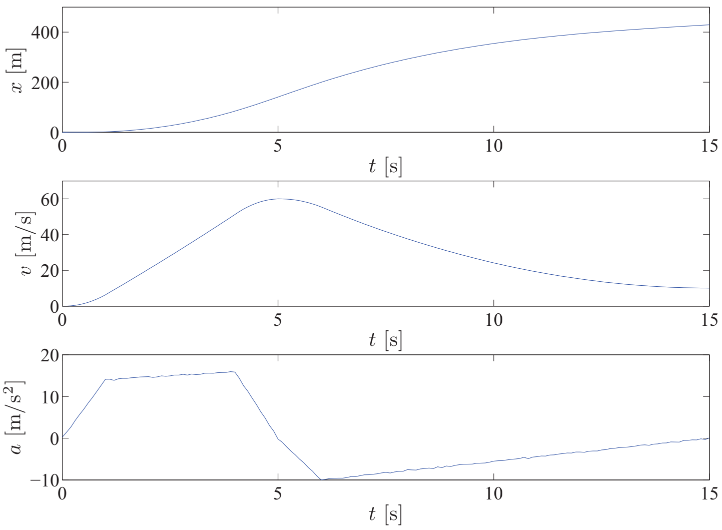


Figure 4.11: *Illustration of the motion of "The Rocket", showing the measured acceleration, and the calculated velocity and position.*

is the time derivative of the position. We find the velocity by integrating the definition of acceleration:

$$a(t) = \frac{dv}{dt} \;\Rightarrow\; \int_{t_0}^{t} a(t)\,dt = \int_{t_0}^{t} \frac{dv}{dt}\,dt = v(t) - v(t_0)\ , \tag{4.36}$$

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)\,dt\ . \tag{4.37}$$

When we know the velocity as a function of time, we can find the position by integrating the velocity, starting from the definition of velocity:

$$v(t) = \frac{dx}{dt} \;\Rightarrow\; \int_{t_0}^{t} v(t)\,dt = \int_{t_0}^{t} \frac{dx}{dt}\,dt = x(t) - x(t_0) \tag{4.38}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)\,dt\ . \tag{4.39}$$

If we insert $v(t)$ from equation 4.37, we get:

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} [v(t_0) + \int_{t_0}^{t} a(t)\,dt]\,dt \\ &= x(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} [\int_{t_0}^{t} a(t)\,dt]\,dt\ . \end{aligned} \tag{4.40}$$

These equations constitute the **integration method** to find the position $x(t)$ and velocity $v(t)$ given the acceleration $a(t)$ of an object:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)\,dt\ , \tag{4.41}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)\,dt = x(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} [\int_{t_0}^{t} a(t)\,dt]\,dt\ . \tag{4.42}$$

There is no need to memorize these equations. They follow from your knowledge of calculus. You only need to remember the definitions of the velocity as the time derivative of the position, and the acceleration as the time derivative of the velocity.

We can apply this method to find the motion for constant acceleration, $a(t) = a_0$, with initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$:

$$v(t) = v(t_0) + \int_{t_0}^{t} a_0 \, dt = v0 + a_0 \, (t - t_0) \ . \tag{4.43}$$

and

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t) \, dt = x_0 + v_0 \, (t - t_0) + \frac{1}{2} a_0 \, (t - t_0)^2 \ . \tag{4.44}$$

4.2.3 Differential equations

Usually, we do not have a set of measurements or a mathematical expression for the acceleration. Instead, we find an expression for the acceleration based on a physical model of the forces acting on the object, and from the forces we find the acceleration. Given this expression for the acceleration, we determine the velocity and position of the object. But this sounds exactly like what we did above? We integrate the acceleration to find the velocity, and then integrate again to find the position. Unfortunately, direct integration only works if the acceleration is *only* a function of time. In most cases, we do not have an expression of the acceleration as a function of time, but instead we know how the acceleration varies with velocity and position. For example, a tiny grain of sand sinking in water has an acceleration on the form:

$$\frac{d^2x}{dt^2} = a = -a_0 - c \cdot v \ , \tag{4.45}$$

where the acceleration depends on the velocity of the grain! And a ball suspended in a vertical spring has an acceleration:

$$\frac{d^2x}{dt^2} = a = -C \cdot x \ , \tag{4.46}$$

that depends on the position of the ball. Such problems cannot be solved by direct integration, because the function $x(t)$ and its derivatives occur on both sides os the equation. Such equations are called differential equations. Analytical solutions of differential equations require some skill and experience, but, fortunately, we can solve them numerically in exactly the same way we did above.

Numerical solution

In most mechanics probems, we want to find the position, $x(t)$, that satisfies an equation on the form:

$$\frac{d^2x}{dt^2} = a\left(t, x, \frac{dx}{dt}\right) \ , \ v(t_0) = v_0 \ , \ x(t_0) = x_0 \ , \tag{4.47}$$

We find the solution by moving forwards in time in small increments Δt. We start from the initial values $x(t_0) = x_0$ and $v(t_0) = v_0$. We find the the velocity and position after a small time-step Δt using Euler's method (equation 4.28):

$$v(t_0 + \Delta t) \simeq v(t_0) + \Delta t \cdot a(t_0, x(t_0), v(t_0)) \ , \tag{4.48}$$

$$x(t_0 + \Delta t) \simeq x(t_0) + \Delta t \cdot v(t_0) \ , \tag{4.49}$$

where $a(t_0, x(t_0), v(t_0))$ is the acceleration we get when we put the values at $t = t_0$ into the expression we have for the acceleration in equation 4.47. We can now continue to step forward in time, finding subsequent values $x(t_i)$ and $v(t_i)$ in steps of Δt. This method is called Euler's method. It is definitely not the best method of integration. It's strength is rather in the simple, intuitive implementation. Surprisingly, changing the step in equation 4.49 to the following:

$$x(t_0 + \Delta t) \simeq x(t_0) + \Delta t \cdot v(t_0 + \Delta t) \ , \tag{4.50}$$

gives significantly better solutions for many problems. This improved method is called Euler-Cromer's method. (You can find a more information about solution methods in numerical methods N.2).

In **Euler-Cromer's method** to solve the (second order) differential equation of motion:

$$\frac{d^2x}{dt^2} = a\left(t, x, \frac{dx}{dt}\right)\ ,\ v(t_0) = v_0\ ,\ x(t_0) = x_0\ , \tag{4.51}$$

we perform the following steps:

$$\begin{aligned} v(t_0) &= v_0 \\ x(t_0) &= x_0 \\ &\dots \\ v(t_i + \Delta t) &\simeq v(t_i) + \Delta t \cdot a(t_i, x(t_i), v(t_i)) \\ x(t_i + \Delta t) &\simeq x(t_i) + \Delta t \cdot v(t_i + \Delta t) \end{aligned} \tag{4.52}$$

Example 4.2: Modeling the motion of a falling tennis ball

This example demonstrates how we can calculate the motion of a falling tennis ball given an expression for the acceleration.

In example 4.1 we studied the motion of a falling tennis ball based on measurements of its motion. However, in physics we do not only want to observe motion, we want to predict it. We do this by first analyzing the problem to find the forces acting on the object, and from the forces we find a mathematical model of the acceleration of the object. (You will learn to do this in the next chapter. For now we will assume that the acceleration is given). From the acceleration, we find the position and velocity by analytical or numerical integration. We call this recipe the structured problem-solving approach.

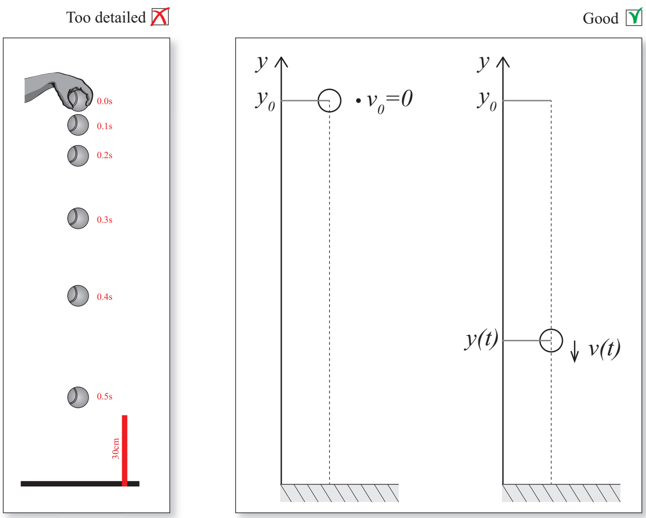


Figure 4.12: (Left) Too detailed illustration. (Right) Correct, simple sketch.

System sketch
Your first step should always be to make a sketch the process. In physics, our sketches are vessels for our thoughts. A good, functional sketch is therefore an important part of solving a problem. While the left part of figure 4.12 has a nice artistic appeal and also illustrates the motion in detail, we do not encourage such detailed sketches. Instead, you should make a sketch that only focuses on the most important features of the process, as in the rightmost figure. Here we illustrate *the object* (the tennis ball), *its surroundings* (most importantly the floor), and *the coordinate system* with a clearly marked axis. We have also illustrated the initial position and velocity of the ball, and its position and velocity at a time t. Drawing a simplified illustration helps you discern the important from the unimportant, and it helps you convert a physical situation into a mathematical problem: The figure shows the axis and the position of the ball, $y(t)$, and nothing else.

Simplified model
From an analysis of the physics in the system, we have found that the acceleration of the ball is a constant:

$$a = -g = -9.8\text{m/s}^2\ . \tag{4.53}$$

(You will learn where this model comes from later. Now we only want to address the consequences of such a model). In addition, we know that the ball starts from rest at the position $y = 2.0$m at the time $t = 0$s:

$$y(0\text{s}) = 2.0\text{m}\ ,\ v(0\text{s}) = 0\text{m/s}\ . \tag{4.54}$$

We have now formulated a mathematical description of the problem we want to solve:

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -g\ ,\ v(0) = v_0\ ,\ y(0) = y_0\ . \tag{4.55}$$

Solving this equation means to find the velocity $v(t)$ and the position $y(t)$ of the ball for any time t. We call this *the modeling step* – finding the mathematical problem to solve – and the next step is *to solve* this problem – to find $v(t)$ and $y(t)$.

Solving the simplified model
Since the acceleration is given and a constant, we can find the velocity by direct integration of the acceleration:

$$\frac{dv}{dt} = -g; \tag{4.56}$$

$$\int_{t_0}^{t} \frac{dv}{dt}\,dt = \int_{t_0}^{t} -g\,dt\ , \tag{4.57}$$

$$v(t) - \underbrace{v(t_0)}_{=0\text{m/s}} = -g\left(t - \underbrace{t_0}_{=0\text{s}}\right)\ , \tag{4.58}$$

which gives

$$v(t) = -gt\ . \tag{4.59}$$

Similarly, we find the position by integrating the velocity:

$$\frac{dy}{dt} = v(t); \tag{4.60}$$

$$\int_{0}^{t} \frac{dy}{dt}\,dt = \int_{0}^{t} v(t)\,dt\ , \tag{4.61}$$

$$y(t) - y(0) = \int_{0}^{t} -gt\,dt = -\frac{1}{2}gt^2\ , \tag{4.62}$$

which gives

$$y(t) = y(0) - \frac{1}{2}gt^2\ . \tag{4.63}$$

Anaysis of the simplified model

This is the complete solution to the problem. We know the position and velocity as a function of time. When you have this solution, you are prepared to answer any question about the motion. For example, you can find out when the ball hits the ground and you can find the velocity of the ball when it hits the ground. How would you do that? You need to translate the question into a mathematical problem. We do this by stating the condition "when the ball hits the ground" in mathematical terms: The ball hits the ground when its position is that of the ground, that is, when $y(t) = 0$m. (Notice, we have ignored the extent of the ball here). We can use our solution in equation 4.63 to find the corresponding time:

$$y(t) = y(0) - \frac{1}{2}gt^2 = 0\text{m} \ \Rightarrow\ t = \sqrt{\frac{2\,y(0)}{g}}\ . \tag{4.64}$$

A more realistic model

Unfortunately, data for the motion of the tennis ball, shown in figure 4.13, show that the ball does not have a constant acceleration.

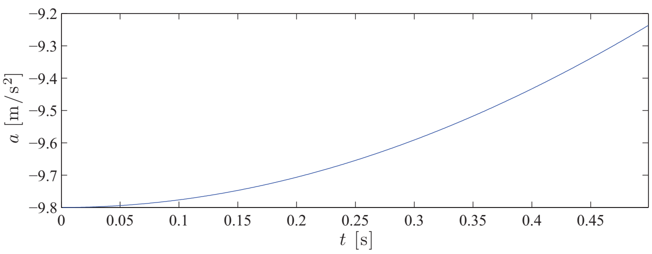


Figure 4.13: *Plot of $a(t)$ for the falling tennis ball. (The plot is the same as in figure 4.8, and is based on a calculatation and not on experimental data).*

This is due to air resistance – an effect not included in the simplified model. Fortunately, we have good models for air resistance. For a falling ball in air, a more realistic model that includes the effect of air resistance is:

$$a = -g - Dv|v|\ , \tag{4.65}$$

where $v = v(t)$ is the velocity of the ball, $g = 9.8\text{m/s}^2$ is the same constant as above, and the constant D depends on details of the ball. For a tennis ball $D = 0.0245\text{m}^-1$ is a reasonable value. (You will learn about the background for this model and how to determine values for D later). We can now formulate a mathematical problem:

$$a = \frac{dv}{dt} = -g - Dv|v|\ , \tag{4.66}$$

with initial conditions $v(0\text{s}) = 0\text{m/s}$ and $y(0\text{s}) = 2.0\text{m}$.

Solution of the realistic model

Our task is to solve this problem – that is to find $v(t)$ and $y(t)$ for the ball. This can be done either numerically or analytically. The numerical solution is straight forward, using the approach we have derived, but the analytical solution requires some knowledge of differential equations.

Numerical solution

We apply Euler-Cromer's method to find the positions and velocities by stepwise integration starting from the initial conditions. The integration step in Euler-Cromer's method is:

$$v(t_i + \Delta t) = v(t_0) + \Delta t \cdot a(t_i, v_i, y_i)\ , \tag{4.67}$$

$$y(t_i + \Delta t) = y(t_0) + \Delta t \cdot v(t_i + \Delta t)\ , \tag{4.68}$$

where we insert the acceleration from equation 4.65:

$$a(t_i, v_i, y_i) = -g - Dv(t_i)|v(t_i)|\ , \tag{4.69}$$

This is implemented in the following. We open a new script file, and start our script by clearing all variables – this is a good habit to ensure that your previous activities do not affect your new calculations:

```
clear all; clf;
```

Then we define the physical constants and values given in the problem: g, D, $y(0)$ and $v(0)$:

```
D = 0.0245; % m^-1
g = 9.8; % m/s^2
y0 = 2.0;
v0 = 0.0;
```

We need to determine for how long we want to calculate the motion – what will be our maximum value of t? There are typically two strategies: We can make an initial guess for the duration of the simulation, or we can determine when the simulation should stop during the simulation. First, we make a guess for the duration of the simulation. Based on the existing data from figure 4.13 we guess that $t = 0.5$s is a reasonable simulation time:

```
time = 0.5;
```

Next, we need to decide the time-step Δt. This needs to be small enough to ensure a good precision of the result, but not too small since this would make the simulation take too long. We try a value of $\Delta t = 0.00001$s:

```
dt = 0.00001;
```

Based on this, we calculate how many simulation steps we need, $n = t/\Delta t$, and generate arrays for the positions, velocities, accelerations and time for the simulation. All values are initially set to zero:

```
% Variables
n = ceil(time/dt);
y = zeros(n,1);
v = zeros(n,1);
a = zeros(n,1);
t = zeros(n,1);
```

Then we set the initial conditions:

```
% Initialize
y(1) = y0;
v(1) = v0;
```

Before, finally, the Euler-Cromer steps are implemented in an integration loop. The whole program is given in the following:

```
clear all; clf;
D = 0.0245; % m^-1
g = 9.8; % m/s^2
y0 = 2.0;
v0 = 0.0;
time = 0.5;
dt = 0.00001;
% Variables
n = ceil(time/dt);
y = zeros(n,1);
v = zeros(n,1);
a = zeros(n,1);
t = zeros(n,1);
% Initialize
y(1) = y0;
v(1) = v0;
% Integration loop
for i = 1:n-1
    a(i) = -g -D*v(i)*abs(v(i));
    v(i+1) = v(i) + a(i)*dt;
    y(i+1) = y(i) + v(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

The resulting plots of $x(t)$, $v(t)$, and $a(t)$ are shown in figure 4.14.

Analysis of realistic model results

We can now use this result to answer questions like how long does it take until the ball hits the ground? Again, we answer the question by translating it into a mathematical question: The ball hits the ground when $y(t) = 0$m. However, in this case, we must find the solution numerically. The simplest approach to this would be to find when y becomes zero during the simulation. It is tempting to do this by checking when $y(t) = 0$m:

```
if (y(i)==0.0)
  t(i)
end
```

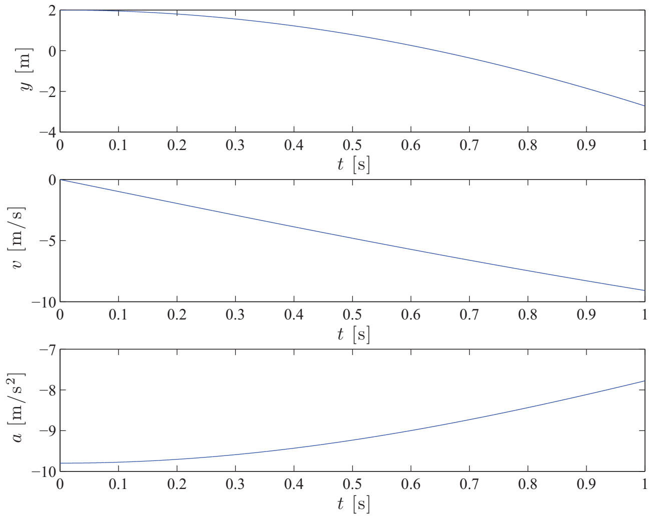


Figure 4.14: Plots of $y(t)$, $v(t)$, and $a(t)$ calculated using the model for air resistance.

But this will not work, because $y(t_i)$ will usually not be zero for any i. Typically, the program will step right past $y = 0$ going from a small positive value at some t_i to a small negative value at t_{i+1}. We should instead find the first time $y(t)$ passes 0, that is, we should find the first t_{i+1} when $y(t_{i+1}) < 0$. Then we know that $y(t) = 0$ somewhere in the interval $t_i < t < t_{i+1}$. We can then estimate a precise value for t using interpolation, or we can simply use the value t_{i+1}, if we find that this gives us sufficient precision. This is implemented in the following modification to the program, where we have also stopped the calculation when the ball hits the ground:

```
for i = 1:n-1
    a(i) = -g -D*v(i)*abs(v(i));
    v(i+1) = v(i) + a(i)*dt;
    y(i+1) = y(i) + v(i+1)*dt;
    if (y(i+1)<0)
        break
    end
    t(i+1) = t(i) + dt;
end
v(i+1)
plot(t(1:i),a(1:i))
xlabel('t [s]');
ylabel('a [m/s^2]');
```

where we have used `break` to stop the loop when the condition is met. Notice that we should now only plot the values up to `i`, because we have not calculated any more values – the values from $i + 2$ to n were set to zero initially for y, v, and a and will make your plot confusing if you include them. (Try it and see).

Test your understanding: What would happen if we considered that the ball had an initial velocity $v_0 = -2v_T$ when it started? Sketch the resulting position, velocity and acceleration as a function of time.

Advanced Analytical solution

The differential equation in equation 4.66 is one of a few equations we can solve analytically as long as the velocity does not change sign. When the ball is falling down, the velocity is negative, and we can replace $|v|$ by $-v$:

$$\frac{dv}{dt} = -g - Dv\,(-v) = -g + D\,v^2 \; . \tag{4.70}$$

This equation can be solved using separation of variables. First, rewrite the equation:

$$\frac{dv}{dt} = -g\left(1 - \frac{D}{g}v^2\right) \; . \tag{4.71}$$

Then, we separate the variables, so that all v's are on the left side and all t's are on the right:

$$\frac{dv}{1 - \frac{D}{g}v^2} = -g\,dt \; . \tag{4.72}$$

The differential equation can now be solved by integrating each side from $v_0 = 0$m/s to v and from $t_0 = 0$s to t:

$$\int_{v_0}^{v} \frac{dv}{1 - \frac{D}{g}v^2} = \int_{t_0}^{t} -g\,dt \; , \tag{4.73}$$

The left-side integral can be solved using your knowledge from calculus or by using a symbolic solver such as Mathematica, giving:

$$\int_0^v \frac{dv}{1 - \frac{D}{g}v^2} = \sqrt{\frac{g}{D}}\tanh^{-1}\left(\sqrt{\frac{D}{g}}v\right) \; . \tag{4.74}$$

We can make this expression simpler to write, by introducing the quantity:

$$v_T = \sqrt{\frac{g}{D}} \; . \tag{4.75}$$

We notice that v_T has dimensions m/s, and we may therefore call it a velocity. Using v_T, we can rewrite the solution to be:

$$\int_0^v \frac{dv}{1 - \frac{D}{g}v^2} = v_T \tanh^{-1}\left(\frac{v}{v_T}\right) , \tag{4.76}$$

which we insert into equation 4.73, getting:

$$v_T \tanh^{-1}\left(\frac{v}{v_T}\right) = -gt . \tag{4.77}$$

We want to find v, and rearrange:

$$\tanh^{-1}\left(\frac{v}{v_T}\right) = -\frac{gt}{v_T} . \tag{4.78}$$

which gives:

$$\frac{v}{v_T} = \tanh\left(-\frac{gt}{v_T}\right) , \tag{4.79}$$

and finally:

$$v = v_T \tanh\left(-\frac{gt}{v_T}\right) . \tag{4.80}$$

We have now found the velocity on the form $v = v(t)$, and we can simply integrate the velocity from t_0 to t to find $y(t)$:

$$y(t) - y(t_0) = \int_{t_0}^t v(t)\,dt = \int_0^t v_T \tanh\left(-\frac{gt}{v_T}\right) dt . \tag{4.81}$$

This integral is solved by a symbolic integrator, such as Mathematica, giving:

$$y(t) = y(0) - v_T^2/g \log\cosh\frac{gt}{v_T} . \tag{4.82}$$

Figure 4.15 shows that the analytical solutions (given by circles) are identical to the numerical solutions (lines).

Analysis of analytical solution

We can now use the analytical solution to solve problems of interest, such as finding out when the ball hits the floor, which occurs at $y(t) = 0$m, that is, when

$$y(0) = v_T^2/g \log\cosh\frac{gt}{v_T} \Rightarrow \frac{gt}{v_T} = \cosh^{-1}\exp\frac{y(0)g}{v_T^2} \tag{4.83}$$

that is:

$$t = \frac{v_T}{g}\cosh^{-1}\exp\frac{y(0)g}{v_T^2} . \tag{4.84}$$

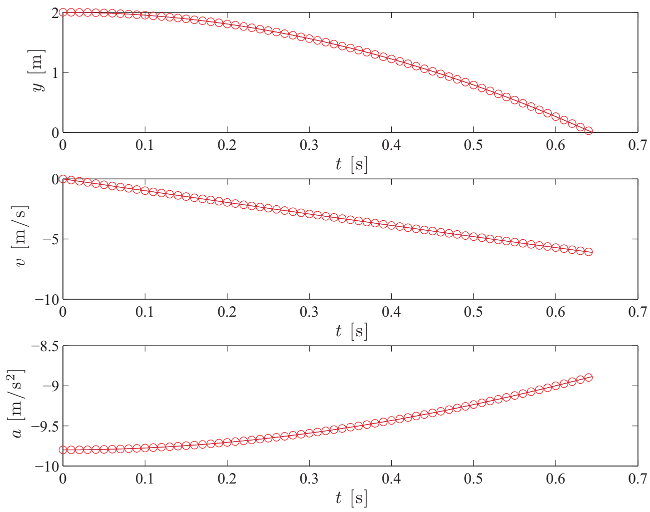


Figure 4.15: *Comparisons of the analytical and numerical solutions to $y(t)$, $v(t)$, and $a(t)$.*

N.1: Numerical derivatives in one dimension

We have found that the time derivative of the position can be approximated by the average velocity

$$\frac{dx}{dt} \simeq \bar{v}(t_i) = \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t} \ , \tag{4.85}$$

and we called this an example of a *numerical derivative*. But how good is this approximation to the derivative, and can we design better numerical methods to find the derivative? These are questions asked in numerical analysis [?]. Here, we show how to develop more precise methods and how to estimate the errors of the methods.

First order method

The average velocity is a *first order* numerical derivative. Its main advantage is its simplicity – it can be directly translated into a numerical algorithm:

```
v(i) = (x(i+1)-x(i))/dt;
```

We can use this method to estimate the derivative from a set of measurements $x(t_i)$, as we did above, or to calculate the derivative of a known function $x(t)$ at a discrete set of t_i. For example, we can estimate the derivative of the function

$$x(t) = \exp(-t^2) \cdot sin(t) \ , \tag{4.86}$$

on the interval from $t = 0$ to $t = 10$ in steps of $\Delta t = 0.01$ by first calculating all the t_i and $x(t_i)$ values, and then applying the first order method:

```
dt = 0.01;
t = (0:dt:10);
x = exp(-t.^2).*sin(t);
v = zeros(length(x),1);
for i = 1:length(x)-1
   v(i) = (x(i+1)-x(i))/dt;
end
```

Here, we use `t=(0:dt:10)` to make an array of t_i values from 0 to 10 in steps of dt.

As the time interval Δt becomes smaller, the approximation in equation 4.85 approaches the exact derivative. We therefore expect the estimate of the derivative to improve as Δt becomes smaller – we expect the error to decrease. But what is the error?

Error estimates

The error is the difference between the estimated value and the exact value of the derivative for a given value of t. For the first order method, we can use Taylor's formula to show that the error, $\epsilon(t_i)$ is:

$$\epsilon(t_i) = \frac{dx}{dt}(t_i) - \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t} = c\Delta t \ . \tag{4.87}$$

where c is some constant. (You can read more about this in section **??** in appendix **??**). This means that the smaller we choose Δt, the smaller the error becomes, unless you choose so small a Δt that you start getting round-off errors due to the finite numerical precision.

We characterize a numerical algorithm by how the error depends on Δt. We say that the first order method is, indeed, first order, because the error is proportional to Δt. If we divide Δt by 2, the error is also reduced by a factor 2. We call a method second order if the error is proportional to Δt^2. This is clearly an improvement, because by reducing Δt by a factor 2, the error is now a factor $2^2 = 4$ smaller. Generally, we say that a method is of order n, if the error is proportional to Δt^n, and we write this as $\epsilon = \mathcal{O}(\Delta t^n)$.

Higher order methods

What does a higher order method for the derivative look like? The simplest example is the centered midpoint method:

$$v(t_i) = \frac{dx}{dt} = \underbrace{\frac{x(t_i + \Delta t) - x(t_i - \Delta t)}{2\Delta t}}_{=v_{\mathrm{m}}(t_i)} + \mathcal{O}(\Delta t^2) \ , \tag{4.88}$$

where the error is of order $\mathcal{O}(\Delta t^2)$. (See section **??** in appendix **??** for a derivation.) Implementation is just as for the first order method, but using this method we cannot evaluate the derivate at either end-points. The following program compares the first and the second order methods, by comparing the numerical derivatives with the exact values (we can find the exact derivative of the given function):

```
dt = 0.01;
t = (0:dt:10);
x = exp(-t.^2).*sin(t);
dxdt = -2*t.*exp(-t.^2).*sin(t) +
    exp(-t.^2).*cos(t);
n = length(x);
v1 = zeros(n,1);
v2 = zeros(n,1);
for i = 1:n-1
   v1(i) = (x(i+1)-x(i))/dt;
end
for i = 2:n-1
   v2(i) = (x(i+1)-x(i-1))/(2*dt);
end
d1 = v1'-dxdt;
d2 = v2'-dxdt;
i = (2:n-1);
plot(t(i),d1(i),'-r',t(i),d2(i),'-b');
```

Figure 4.16 shows that the second order method gives a significantly better fit to the the derivative.

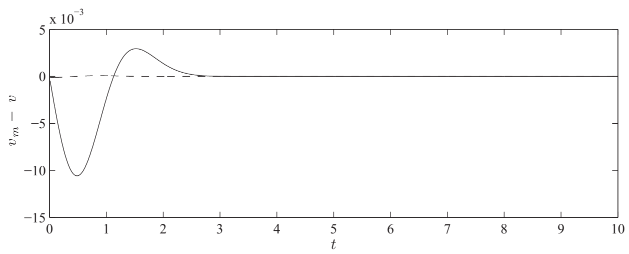


Figure 4.16: *Plot of the difference between the numerical derivative v_n and the exact derivative v for the first order method (solid line), and the second order method (dashed line).*

Skewness in first order method

A particular problem of the first order method in equation 4.85 is that it is skewed – it does not make symmetrical use of $x(t)$ around the point t_i. The centered midpoint method, on the other hand, is symmetrical. However, we may argue that the first order method corresponds to a centered midpoint method applied at the time $t_i' = t_i + \Delta t/2$ with a time-step $\Delta t' = \Delta t/2$, From equation 4.88:

$$v_m(t_i') = \frac{x(t_i' + \Delta t') - x(t_i' - \Delta t')}{2 \cdot \Delta t'} , \tag{4.89}$$

we insert $t_i' = t_i + \Delta t/2$ and $\Delta t' = \Delta t/2$:

$$\begin{aligned} v_m(t_i + \frac{\Delta t}{2}) &= \frac{x(t_i + \frac{\Delta t}{2} + \frac{\Delta t}{2}) - x(t_i + \frac{\Delta t}{2} - \frac{\Delta t}{2})}{2 \cdot \frac{\Delta t}{2}} \\ &= \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t} = \bar{v}(t_i) . \end{aligned} \tag{4.90}$$

The average velocity, $\bar{v}(t_i)$, therefore provides second order accuracy, not at the time $t = t_i$, but at the time, $t = t_i + \Delta t/2$!

Second order derivatives

In the motion diagrams, we characterized the acceleration using the average acceleration:

$$\bar{a}(t_i) = \frac{\bar{v}(t_i) - \bar{v}(t_{i-1})}{\Delta t} . \tag{4.91}$$

Our discussion of skewness of the first order method now makes it clear why we use the velocities at t_i and t_{i-1} to determine the acceleration at t_i. The average velocity $\bar{v}(t_i)$ is really an estimate of the velocity at the time $t_i + \Delta t/2$, while the average velocity $\bar{v}(t_{i-1})$ gives the velocity at $t_i - \Delta t/2$. The average acceleration $\bar{a}(t_i)$ therefore corresponds to:

$$\begin{aligned} \bar{a}(t_i) &= \frac{v_m(t_i + \frac{\Delta t}{2}) - v_m(t_i - \frac{\Delta t}{2})}{\Delta t} \\ &\simeq \frac{v(t_i + \frac{\Delta t}{2}) - v(t_i - \frac{\Delta t}{2})}{\Delta t} \simeq \frac{dv}{dt} . \end{aligned} \tag{4.92}$$

We recognize this expression as similar to the second order centered midpoint method for the numerical derivative of $v(t)$ in the point t_i. (See appendix **??** for details.) Hence, the method introduced for finding the acceleration from the motion diagram provides a reasonable numerical estimate for the derivative of the velocity.

Let us approximate the acceleration directly from the positions, $x(t_i)$, of the object, by inserting the second order numerical estimates $v_m(t_i \pm \Delta t/2)$ into equation 4.92.

$$v_m(t_i + \Delta t/2) = \frac{x(t_{i+1}) - x(t_i)}{\Delta t} , \tag{4.93}$$

and

$$v(t_i - \Delta t/2)_m = \frac{x(t_i) - x(t_{i-1})}{\Delta t} , \tag{4.94}$$

which inserted in equation 4.92 gives:

$$\begin{aligned} \bar{a}(t_i) &= \frac{(x(t_{i+1}) - x(t_i)) - (x(t_i) - x(t_{i-1}))}{\Delta t^2} \\ &= \frac{x(t_{i+1}) - 2x(t_i) + x(t_{i-1})}{\Delta t^2} \simeq \frac{dv}{dt}(t_i) = a(t_i) . \end{aligned} \tag{4.95}$$

The error in this approximation is of order $\mathcal{O}(\Delta t^2)$. (See appendix **??** for details.)

N.2: Numerical solution of the equations of motion

In mechanics we are typically given the acceleration $a(x, v, t)$ as a function of t, or as an expression that may include x, v and t, and we want to find the motion by solving the mathematical problem:

$$\frac{d^2x}{dt^2} = a(x, v, t) \ , \ x(t_0) = x_0 \ , \ v(t_0) = v_0 \ , \tag{4.96}$$

Such problems are often called *initial value problems*. Numerically, we can solve such problems by calculating $v(t_i)$ and $x(t_i)$ in incremental steps, starting from the initial values at t_i. In each step, we find new velocities and positions from the previous time step according to:

$$v(t_i + \Delta t) \simeq v(t_i) + a(x(t_i), v(t_i), t_i) \cdot \Delta t \ , \tag{4.97}$$

$$x(t_i + \Delta t) \simeq x(t_i) + v(t_i) \cdot \Delta t \ . \tag{4.98}$$

This method – Euler's method – is an example of a *numerical method*. But how good is it? To answer that we need to address the properties and precision of numerical methods. Here, we provide a brief mathematical background on numerical methods.

Numerical integration

The problem is simplest to discuss and solve in the case where we know the acceleration, $a = a(t)$. For example, we may know that $a(t) = A_0 \sin \omega t$. In this case, we find the velocity and position by direct integration:

$$v(t) - v(t_0) = \int_{t_0}^{t} \frac{dv}{dt} dt = \int_{t_0}^{t} a(t) dt \ . \tag{4.99}$$

We can do that analytically if we know how to solve the integral of $a(t)$. But what if we do not know how to calculate the integral or if we only know the value of $a(t)$ at certain times, t_i?

Then we solve the integral numerically. We have already seen how we can use Euler's method to solve the integrals iteratively – by moving forward in time in small steps. When we know $a(t)$, we can use equation 4.99 to find the velocity after a small time step Δt:

$$v(t_i + \Delta t) = v(t_i) + \int_{t_i}^{t_i + \Delta t} a(t) dt \ . \tag{4.100}$$

If we cannot determine the integral analytically, we must find approximations for the value of the integral over the time interval Δt. Euler's method for numerical integration represents one of the simplest approximations: We assume that the acceleration is approximately constant throughout the interval from t_i to $t_i + \Delta t$, so that we can replace $a(t)$ with the value, $a(t_i)$, at the beginning of the interval, as illustrated in figure 4.17:

$$\int_{t_i}^{t_i + \Delta t} a(t) dt \simeq \int_{t_i}^{t_i + \Delta t} a(t_i) dt = a(t_i) \Delta t \ . \tag{4.101}$$

We call such as method a single point method, since it uses the acceleration only at a single point, t_i, to estimate the integral over the time-step Δt. However, we know that $a(t)$ varies over the interval Δt. Why choose the value at the beginning of the interval? Figure 4.17 illustrates various methods. For example, a better method would be to choose the average of the value of the beginning and at the end of the interval:

$$\int_{t_i}^{t_i + \Delta t} a(t) dt = \frac{1}{2} \left(a(t_i) + a(t_i + \Delta t) \right) \Delta t \ . \tag{4.102}$$

This method is called the Euler midpoint method or the trapezoidal rule when used for numerical integration, and it is illustrated in figure 4.17b.

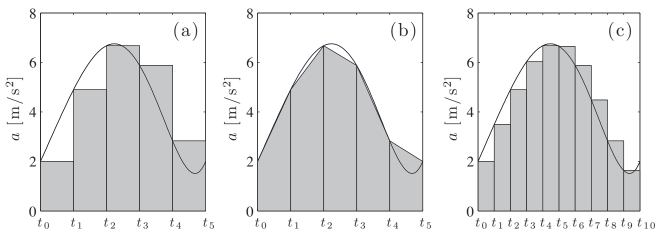


Figure 4.17: *Illustration of Euler's method (a), Euler's midpoit method (b), and Euler's method with a higher time resolution.*

To find the whole integral from t_0 to t using these methods, we divide the interval from t_0 to t into n pieces of size $\Delta t = (t - t_0)/n$, and then sum up the contributions from each of the steps:

$$v(t_1) = v(t_0 + \Delta t) \simeq v(t_0) + \bar{a}(t_0) \cdot \Delta t \ , \tag{4.103}$$

$$v(t_2) = v(t_1 + \Delta t) \simeq v(t_1) + \bar{a}(t_1) \cdot \Delta t \ , \tag{4.104}$$

which gives:

$$v(t) \simeq v(t_0) + \sum_{i=0}^{n-1} \bar{a}(t_i) \Delta t \ , \tag{4.105}$$

where $\bar{a}(t_i) = a(t_i)$ for the single-point method and $\bar{a}(t_i) = (a(t_i) + a(t_i + \Delta t))/2$ for the midpoint method.

Numerically, we can therefore approximate the integral from t_0 to t with:

$$\int_{t_0}^{t} a(t) dt \simeq \sum_{i=0}^{n-1} \bar{a}(t_i) \Delta t \ . \tag{4.106}$$

Error estimates in numerical integration

The error is the difference between the exact result and the numerical result. We characterize numerical methods by how the error depends on Δt. A method is said to be of order n, and we write this as $\mathcal{O}(\Delta t^n)$, if the error is proportional to Δt^n. Higher order methods – that is higher values of n – are better since they converge more rapidly to the correct result as Δt is reduced.

We can show that the single point method has an error of the order $\mathcal{O}(\Delta t)$ and that the trapezoidal rule has an error of the order $\mathcal{O}(\Delta t^2)$:

$$\int_{t_0}^{t} v(t) dt = \sum_{i=0}^{n-1} v(t_i) \Delta t + \mathcal{O}(\Delta t)$$

(Singe value method)

$$\int_{t_0}^{t} v(t) dt = \sum_{i=0}^{n-1} \frac{1}{2} \left(v(t_i) + v(t_{i+1}) \right) \Delta t + \mathcal{O}(\Delta t^2) \tag{4.107}$$

(Trapezoidal rule)

(See appendix **??** for a complete argument for the errors of these methods.)

Numerical solution of differential equations

Generally, the acceleration in equation 4.96 depends on both position and velocity, and we cannot solve the problem by direct integration. Instead, we must solve the differential equation with the given initial conditions. While this can be a daunting task using analytical methods, the numerical approach is general and robust, and almost identical to the approach we used for numerical integration.

We have already introduced Euler's method, which does not change significantly when the acceleration is a function of position or velocity. We find the velocities and positions by stepwise approximative integration starting from the initial conditions:

$$\begin{aligned} v(t_i + \Delta t) &\simeq v(t_i) + a(t_i, x(t_i), v(t_i))\Delta t \\ x(t_i + \Delta t) &\simeq x(t_i) + v(t_i)\Delta t \end{aligned} \tag{4.108}$$

We have already demonstrated how to use this algorithm to find solutions to more complicated problems.

Error estimates for Euler's method

How can we ensure that the numerical method produces the correct solution? We characterize the error in the numerical solution by the deviation from the exact solution. The error after one step in Euler's method is of order $\mathcal{O}(\Delta t^2)$. This can be understood from a simple argument that it is worth following: Euler's method is based on the first order approximation to the derivative: The derivative $v'(t_0)$ of the velocity $v(t)$ is approximately:

$$v'(t_0) = \frac{v(t_0 + \Delta t) - v(t_0)}{\Delta t} + \mathcal{O}(\Delta t) \ . \tag{4.109}$$

We can rewrite this to form a step in Euler's method:

$$\begin{aligned} v(t_0 + \Delta t) &= v(t_0) + v'(t_0)\Delta t + \Delta t\,\mathcal{O}(\Delta t) \\ &= v(t_0) + v'(t_0)\Delta t + \mathcal{O}(\Delta t^2) \end{aligned} \tag{4.110}$$

(Notice that we increase the order $\mathcal{O}(\Delta t)$ when we multiply by Δt.) However, this is only the error after a single step. To find the velocity after a time t we need to perform n such steps, where $n = (t - t_0)/\Delta t$. The error after n steps is n times larger than the error after one step. We therefore expect the error after n steps to be proportional to

$$n \cdot \mathcal{O}(\Delta t^2) = \frac{t - t_0}{\Delta t}\mathcal{O}(\Delta t^2) = \mathcal{O}(\Delta t) \ . \tag{4.111}$$

The error in Euler's method after a time t is therefore proportional to Δt.

Improvements of Euler's method

How can we reduce the error in Euler's method? Let us analyze the method to see how we can improve it. When we calculate a new position, $x(t_i + \Delta t)$, we use the velocity $v(t_i)$ at the beginning of the motion. But the velocity is changing from t_i to $t_i + \Delta t$. Why should we use the velocity at the beginning at the interval? A better choice may be to use the velocity in the middle of the interval at $t = t_i + \Delta t/2 = t_{i+1/2}$. We find the position at a time $t_i + \Delta t$ by

$$x(t_i + \Delta t) \simeq x(t_i) + v(t_{i+1/2})\Delta t \ . \tag{4.112}$$

Similarly, when we find the velocities using Euler's method, we use the acceleration at the beginning of the interval. Since we now need to find the velocities at the times $t_{i+1/2}$, the original Euler scheme corresponds to:

$$v(t_{i-1/2} + \Delta t) = v(t_{i-1/2}) + a(t_{i-1/2})\Delta t \ , \tag{4.113}$$

but also the acceleration is changing throughout the time interval from $t_{i-1/2}$ to $t_{i+1/2}$. Let us instead use the acceleration in the middle of the interval to find the velocities. The middle of the interval from $t_{i-1/2}$ to $t_{i-1/2} + \Delta t$ is t_i. We therefore use $a(t_i)$ when we calculate the new velocity:

$$v(t_{i-1/2} + \Delta t) = v(t_{i-1/2}) + a(t_i)\Delta t \ . \tag{4.114}$$

Only two small problems remain:

First, we have the initial condition $x(t_0) = x_0$ and $v(t_0) = t_0$. What is the initial velocity at the intermediate time $t_{-1/2}$ needed to find the velocity at $t_{1/2}$? Several solutions suggest themselves: We could simply choose $v(t_{-1/2}) = v_0$. This is the choice will will usually make. We could also have used somewhat more of our knowledge of the motion, and put $v(t_{-1/2}) = v_0 - a(t_0)(\Delta t/2)$. The differences between these two choices will not be important for us.

Second, the acceleration may depend on the velocity, $a = a(v, x, t)$, but when we calculate the new velocities, we use $a(t_i) = a(v(t_i), x(t_i), t_i)$. We know $x(t_i)$ since this is found by the scheme, but we do not know $v(t_i)$, since we only evaluate the velocities at $t_{i+1/2}$. Here, we will simply assume that we can use $v(t_i) \simeq v(t_{i-1/2})$ for this calculation.

The method we have developed here is called a leap-frog method, which generally has second order accuracy, that is, the error is $\mathcal{O}(\Delta t^2)$, compared to the first order error in Euler's method. You can find a more complete discussion of the accuracy of this and other related methods, such as the Verlet method, in appendix **??**.

Let us address the implementation of the leap-frog method introduced above. The leap-frog scheme provides the following method for finding the time development of the equations of motion. We start at initial conditions $x(t_0) = x_0$ and $v(t_{-1/2}) \simeq v(t_0) = v_0$, and find subsequent positions and velocities by

$$v_{i+1/2} = v_{i-1/2} + a(v_{i-1/2}, x_i, t_i)\Delta t \ , \tag{4.115}$$

$$x_{i+1} = x_i + v_{i+1/2}\Delta t \ . \tag{4.116}$$

Let us introduce a new quantity, $u_i = v_{i-1/2}$ so that indexing in the algorithm becomes clearer:

$$u_{i+1} = u_i + a(u_i, x_i, t_i)\Delta t \ , \tag{4.117}$$

$$x_{i+1} = x_i + u_{i+1}\Delta t \ . \tag{4.118}$$

where the initial conditions now are x_0 and $u_0 = v_0$. When the algorithm is written in this particular way, we see that it is almost exactly the same as Euler's method, with one small change, we have changed the calculation of the new position to depend on the newly calculated velocity. This method is called Euler-Cromer's or Euler-Richardson's method, and it is surprisingly robust and stable. However, when we interpret the results we should remember that the velocities really are calculated at intermediate times. In practice, we will usually use Euler-Cromer's method to solve the equations of motion and interpret the velocities $v(t_{i-1/2})$ to be a good approximation of $v(t_i)$.

Euler-Cromer's method

While Euler's method is attractive because of its transparency and simple implementation, there are cases when the errors produced by this method become significant. For example, Euler's method is well know to break down for periodic motion, such as for harmonic oscillations. The seemingly insignificant modification to Euler's method in the form of Euler-Cromer's method makes the model second order, and it is also robust enough to address oscillations.

Euler-Cromer's method consist of the following iterative scheme to find the velocity and position from the acceleration:

$$v(t_i + \Delta t) \simeq v(t_i) + a(t_i, v_i, x_i)\Delta t \ , \tag{4.119}$$

$$x(t_i + \Delta t) \simeq x(t_i) + v(t_i + \Delta t)\Delta t \ . \tag{4.120}$$

In this text we prefer to use Euler-Cromer's method to solve most problems because the implementation of the model is transparent, and it produces sufficiently accurate results for almost all applications. However, you should know that in your professional career, after finishing this book, there is no good excuse for not using a much more accurate method, such as the fourth-order Runge-Kutta method.

Using general "Solvers"

Every time we have solved a problem numerically, we have written the solution algorithm explicitly into the program. Frequently, this has been the Euler or Euler-Cromer solver. However, the methods used to solve the problem numerically have essentially been the same each time – these methods are general. For a particular problem, the acceleration is a particular function of time, position and velocity, and we have a particular set of initial conditions. That is, we know that

$$a = a(t, x, v) \ , \tag{4.121}$$

where we know the functional form of the acceleration. Euler-Cromer's method is then implemented by the following iterations:

$$v(t_i + \Delta t) = v(t_i) + \Delta t a(t_i, x(t_i), v(t_i)) \ , \tag{4.122}$$

$$x(t_i + \Delta x) = x(t_i) + \Delta t v(t_i + \Delta t) \ , \tag{4.123}$$

starting from the initial conditions, $x(t_0) = x_0$ and $v(t_0) = v_0$. This is the part of the structured problem solving method that we call the "Solver" – it consists of finding the position, $x(t_i)$, and velocity, $v(t_i)$, given the initial conditions and the functional form of the acceleration. We can therefore make this part of the "Solver" into a general program – a function – that we simply call with the functional form of the acceleration and the initial conditions as the input. This also means that we separate the "Solver" from the formulation of the mathematical program, and from the discussion of the results – and we can therefore also use more advanced solvers than we have so far introduced.

In order to use a "Solver" as part of our program, we need to write a function that returns the acceleration for a given time, position, and velocity. Let us use the example from example **??**, where the acceleration is:

$$a = -A \cos \frac{2\pi x}{b} - c \cdot v \ , \tag{4.124}$$

How can we write a function returning the acceleration, given values for t, x, and v?

In Matlab a function needs to be placed in a separate file with the same name as the name of the function. However, the file needs to have the extension `.m` for Matlab to recognize it as a function. This file needs to be present in the current working directory or in the search path used by Matlab. The function `surfacc` is implemented in the following way:

```
function a = surfacc(t,x,v)
A = 3.0e9; %
b = 1.0e-9; % m
a = -A*sin(2.0*pi*x/b);
return
```

In addition, we need the "Solver" function. This is a general function that should work for any problem. The problem specific parts are the function that returns the acceleration, and the initial conditions for the problem. This "Solver" returns the time, t_i, position, $x(t_i)$, velocity, $v(t_i)$, and acceleration, $a(t_i)$ at the discrete times t_i. For this, we also need to specify the beginning, t_0, and end, t_1, for the solution, and the time interval, Δt.

An example of a "Solver" using Euler-Cromer's method is presented in the following program. Again, notice that in Matlab this function should be saved in a file called `eulercromer.m`, and this file needs to be present in the current working directory or in the search path used by Matlab.

```
function [x,v,t] =
    eulercromer(accel,time,t0,dt,x0,v0)
% Define variables
n = (time - t0)/dt;
t = zeros(n,1);
x = zeros(n,1);
v = zeros(n,1);
% Initial conditions
v(1) = v0;
x(1) = x0;
t(1) = t0;
% Calculation loop
for i = 1:n-1
    a = feval(accel,t(i),x(i),v(i));
  v(i+1) = v(i) + a*dt;
  x(i+1) = x(i) + v(i+1)*dt;
  t(i+1) = t(i) + dt;
end
return
```

In Matlab we cannot pass a function name directly into a function. Instead, we pass a text string with the name of the function, and then execute the function with this name, as illustrated in the code example above.

When we use a general "Solver" to find the solution to the problem, the numerical approach comes very close to the structured problem solving method: We define the acceleration, introduce initial conditions, and call the "Solver" to find the motion. This is illustrated in the following short code that corresponds to the program used in example **??**:

```
t0 = 0.0; % s
t1 = 10.0e-9; % s
dt = t1/10000.0;
x0 = 0.0; % m
v0 = 0.7; % m/s
[t,x,v,a] = eulercromer('surfacc',x0,v0,t0,t1,dt);
%
subplot(3,1,1)
plot(t,x)
xlabel('t [s]')
ylabel('x [m]')
subplot(3,1,2)
plot(t,v)
xlabel('t [s]')
ylabel('v [m/s]')
subplot(3,1,3)
plot(t,a)
xlabel('t [s]')
ylabel('a [m/s^2]')
```

While this general procedure of using a "Solver" is very attractive, we will not use this method in this text because we believe it is important to be completely confident with the use of the numerical methods before we introduce an additional level of abstraction by using the "Solver". However, the aim is for you – the student – to reach such a level that the use of a "Solver" becomes natural.

Runge-Kutta methods

While Euler's method has a very simple implementation, and therefore is a useful starting point when finding coarse solutions to problem, it does not always produce sufficiently good results. We address the mathematical theory for various methods, including Euler's method, in detail in appendix ??. In this book we will in many cases use Euler's or Euler-Cromer's methods, because the implementation is so simple, and the results are sufficiently good, but we will also encounter problems when Euler's or Euler-Cromers's methods clearly are not sufficiently precise. In this case we will use higher order integration methods. However, the tool of choice in our toolbox of methods for solving equations of motion depends on the type of problem you will be solving: If you solve for the motion of a single object, the method of choice is the fourth order Runge-Kutta method, but if you solve for the motion of many objects at the same time, such as when you determine the motion of atoms or molecules in molecular dynamics simulations, the method of choice is the Velocity-Verlet method. You can read more about these methods in appendix ??.

In your work as a professional physicist we strongly urge you to consider the fourth-order Runge-Kutta and the Velocity-Verlet as your standard workhorses for solving differential equations. These methods will in most cases provide a decent trade-off between computational efficiency and accuracy.

Summary – Chapter 4

Motion

The motion of an object is described by:

- the position, $x(t)$, as a function of time, measured in a specified coordinate system
- the velocity $v(t) = \frac{dx}{dt}$
- the acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Structured problem-solving approach

Identify		Model		Solve		Analyse
What object is moving? How is the position, $x(t)$, measured? (Origin and axes of coordinate system). Find initial conditions, $x(t_0)$ and $v(t_0)$.	→	Find the forces acting on the object. Introduce models for the forces. Apply Newton's second law of motion to find the acceleration, $a = a(x, v, t)$.	→	Solve the equation: $\frac{d^2x}{dt^2} = a(x, v, t)$, with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$ using analytical or numerical techniques. The solution gives the position and velocity as a function of time, $x(t)$, and $v(t)$.	→	Check validity of $x(t)$ and $v(t)$. Use $x(t)$ and $v(t)$ the answer questions posed. Evaluate the answers.

Solution methods

In the "Solver" we solve the equation:

$$\frac{d^2x}{dt^2} = a(t, x, \frac{dx}{dt}) \ .$$

with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$.

Numerical solution

Numerically, we solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$v(t_i + \Delta t) = v(t_i) + \Delta t \cdot a(x(t_i), v(t_i), t_i) \ ,$$

$$x(t_i + \Delta t) = x(t_i) + \Delta t \cdot v(t_i + \Delta t) \ .$$

Analytical solution

When the acceleration, $a = a(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)dt \ ,$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)dt \ ,$$

A typical example is motion with constant acceleration.

When the acceleration has a general form, $a = a(t, x, v)$, we need to solve the differential equation. In this case, there are no general approaches that always work. Instead, you must rely on your experience and your knowledge of calculus.

Exercises – Chapter 4

Discussion questions

4.1: Pedometer. Can you use the accelerometer in your phone as a pedometer? Explain.
4.2: Error in speedometer. If your speedometer overestimates your velocity by 10%, how will that affect your measurement of your cars acceleration?
4.3: Speed of the clouds. Is it possible to use your camera to measure the speed of the clouds? What would you need to know to do that?
4.4: The slow trip. Is is possible to go for a trip (in one dimension) where the total displacement is zero, but your average velocity is non-zero?
4.5: Driving backwards. You drive in a train that is subject to constant acceleration. Can the train reverse its direction of motion?
4.6: No motion. Is is possible to envision a motion where you for a period have no displacement, but non-zero velocity? (You may use an $x(t)$ plot for illustration).
4.7: Non-falling ball. You throw a ball downwards from a high building. Can you think of a situation where the ball would have an acceleration upwards? What would happen?
4.8: Travels by sea. A boat is sailing north. Is it possible for the boat to have a velocity toward the north, but still have an acceleration toward the south?
4.9: Acceleration during throw. You throw a ball upwards as far as you can. The ball reaches its maximum height far above you. When was the magnitude of the acceleration the largest? While in your hand while throwing it or during its subsequent motion through the air?
4.10: Passing objects. A disgruntled physics student drops his pc from a window onto the ground. (You should not try this at home). At the same time as she lets the pc go, another student throws a ball upwards. The ball reaches its maximum position at the exact height where the pc was released. At what height does the pc and the ball pass each other? At the midpoint, above the midpoint or below the midpoint? Do they have the same magnitudes of their velocities at this point?

Discrete motion

4.1: Space shuttle launch. When the space shuttle is lifting off, the vertical positions for the first 10 seconds in 1 second intervals are given as

t [s]	y [m]	t [s]	y [m]
0	0	5	375
1	15	6	540
2	60	7	735
3	135	8	960
4	240	9	1215

(a) Draw the motion diagram and the displacements for this motion.

(b) Use the motion diagram to find the average velocity as a function of time after lift-off.

(c) Use the motion diagram to find the average acceleration as a function of time after lift-off.

4.2: Capturing the motion of a falling ball. We use an ultra-sonic motion detector to measure the vertical position of a small ball. We throw the ball upwards, and measure the position until it hits the ground. You find the measured data in the file ballmotion.d. Each line in the file consists of a time, t_i, measured in seconds, and a distance, x_i, measured in meters.

(a) Plot the position as a function of time for the ball.

(b) How long time does it take until the ball hits the ground?

(c) Plot the average velocity as a function of time for the ball.

(d) What is the maximum and minumum velocity of the ball?

(e) What is the initial velocity – the velocity of the ball at the start of the motion?

(f) Plot the average acceleration as a function of time for the ball.

(g) When is the maximum and minimum accelerations? Does this correspond with your physical intuition?

4.3: Motion graphs. A car is driving along a straight road. Sketch the position and velocity as a function of time for the car if:

(a) The car drives with constant velocity.

(b) The car accelerates with a constant acceleration.

(c) The car brakes with a constant acceleration.

4.4: Random walker. Figure 4.18 shows the motion of a tiny grain of dust bouncing randomly around in an air chamber.

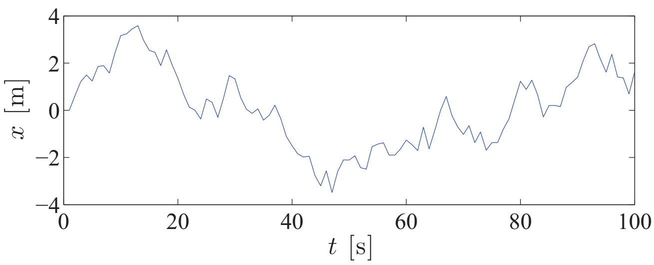


Figure 4.18: Random motion of a grain of dust.

(a) When is the grain to the left of the origin?

(b) When is the grain to the right of the origin?

(c) Is the grain ever exactly at the origin?

4.5: Motion diagram for a car. Figure 4.19 shows the motion diagram for a car driving along a straight road.

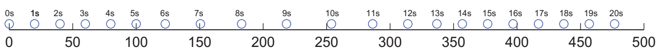


Figure 4.19: Motion diagram for a car.

(a) Describe the motion of the car.

(b) Sketch the position as a function of time.

(c) Estimate the velocity of the car throughout the motion.

(d) Estimate the acceleration of the car throughout the motion.

4.6: Discover the motion. Figure 4.20 shows the motion diagram for a motion.

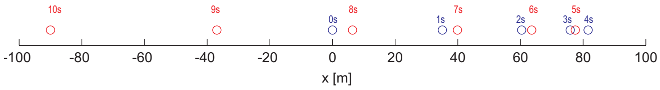


Figure 4.20: *Can you describe the motion?*

(a) Describe the motion qualitatively.

(b) Suggest a process that leads to this motion diagram.

Continuous motion

4.7: The fastest indian. In the film "The World's Fastest Indian" Anthony Hopkins plays Burt Munro who reaches a velocity of 201 mph in his 1920 Indian motorcycle.

(a) At this velocity, how far does the Indian travel in 10s?

(b) How long time does the Indian need to travel 1km?

4.8: Meeting trains. A freight train travels from Oslo to Drammen at a velocity of 50km/h. An express train travels from Drammen to Oslo at 200 km/h. Assume that the trains leave at the same time. The distance from Oslo to Drammen along the railway track is 50km. You can assume the motion to a long a line.

(a) When do the trains meet?

(b) How far from Oslo do the trains meet?

4.9: Catching up. Your roommate sets off early to school, walking leasurly at 0.5m/s. Thirty minutes after she left, you realize that she forgot her lecture notes. You decide to run after her to give her the notes. You run at a healthy 3 m/s.

(a) What is her position when you start running?

(b) What is your position when $t < t_1$?

(c) Sketch the position of you and your roommate as functions of time and indicate in the figure where you catch up with her.

(d) How long time does it take until you catch up with her?

(e) How far has she come when you catch up with her? We find the position from $x(t)$ for $t = 2160$s for either person:

$$x_A(t) = 0.5\text{m/s}\ 2160\text{s} = 1080\text{m} .$$

Now you have developed a strategy to solve such a problem, let us make the problem more complicated and see if you still can use your strategy.

First, let us assume that you start off at $v_0 = 5$m/s, but then you tire gradually, so that your speed drops off with distance, reducing your speed by 1m/s for every hundred meters you run, until you reach a speed of $v_1 = 2$m/s, which you can keep for a long time.

(f) Show that your velocity as a function of position can be written as:

$$v(x) = \begin{cases} v_0 - b\,x & v < v_1 = 2\text{m/s} \\ v_1 & \textit{otherwise} \end{cases} \tag{4.125}$$

where $b = 1$m/s/100m.

(g) Plot or sketch $v(x)$.

(h) If you know your position and velocity at a time t, how can you find the position and velocity at $t + \Delta t$, a small time-step later?

(i) Write a program to find your position as a function of time. (Remember that you first start running at the time $t = t_1 = 1800$s. Before this you are standing still.)

(j) Validate your program by setting $b = 0$ and comparing the calculated $x(t)$ with the exact result, $x_e(t) = v_0(t - t_1)$ when $t > t_1$.

(k) How can you use this result to find where you catch up with your roomate?

(l) Where do you catch up with your roommate?

(m) What parts of your solution strategy are general, that is, what parts of your strategy does not change if we change how either person moves?

4.10: Electron in electric field. An electron is shot through a box containing a constant electric field, getting accelerated in the process. The acceleration inside the box is $a = 2000\text{m/s}^2$. The width of the box is 1m and the electron enters the box with a velocity of 100m/s. What is the velocity of the electron when it exits the box?

4.11: Archery. As an expert archer you are able to fire off an arrow with a maximum velocity of 50m/s when you pull the string a length of 70cm. If you assume that the acceleration of the arrow is constant from you release the arrow until it leaves the bow, what is the acceleration of the arrow?

4.12: Collision. A car travelling at 36km/h crashes into a mountain side. The crunch-zone of the car deforms in the collision, so that the car effectively stops over a distance of 1m. Let us assume that the acceleration is constant during the collision, what is the acceleration of the car during the collision? Compare with the acceleration of gravity, which is $g = 9.8\text{m/s}^2$.

4.13: Braking distance. When you brake your car with your brand new tyres, your acceleration is 5m/s^2.

(a) Find an expression for the distance you need to stop the car as a function of the starting velocity.

With your old tires, the acceleration is only two thirds of the acceleration with the new tyres.

(b) How does this affect the braking distance?

4.14: Motion with constant acceleration. An object starts at $x = x_0$ with a velocity $v = v_0$ at the time $t = t_0$ and moves with a constant acceleration a_0. Show that the velocity v when the object has moved to a position x is $v^2 - v_0^2 = 2a_0(x - x_0)$.

4.15: Position plots. The position $x(t)$ of a particle moving along the x-axis is given in figure 4.21.

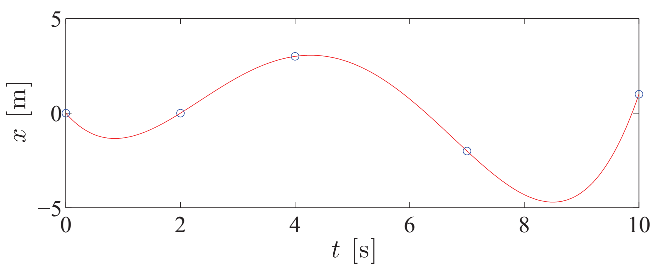


Figure 4.21: *The position of a particle moving along the x-axis.*

(a) Indicate in the figure where the velocity of the particle is positive, negative, and zero?

(b) Indicate in the figure where the velocity is maximal and minimal.

(c) Indicate in the figure where the acceleration is positive, negative, and zero?

4.16: Velocity plots. The velocity $v(t)$ of a particle moving along the x-axis is given in figure 4.22.

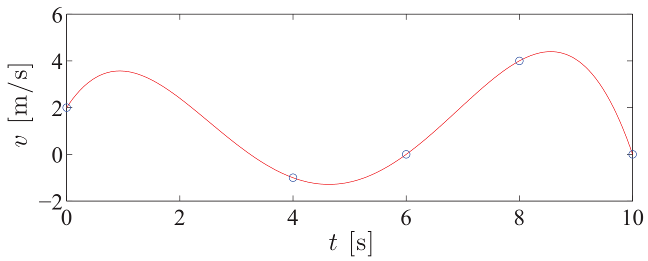


Figure 4.22: *The velocity of a particle moving along the x-axis.*

(a) Indicate in the figure where the velocity of the particle is positive, negative, and zero?

(b) Indicate in the figure where the velocity is maximal and minimal.

(c) Indicate in the figure where the acceleration is positive, negative, and zero?

(d) Indicate in the figure where the acceleration is maximal and minimal.

4.17: Velocity plots. The velocity $v(t)$ of a particle moving along the x-axis is given in figure 4.23.

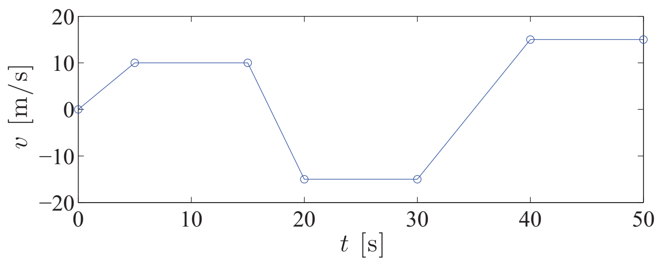


Figure 4.23: *The velocity of a particle moving along the x-axis.*

(a) Indicate in the figure where the velocity of the particle is positive, negative, and zero?

(b) Indicate in the figure where the particle speeds up and slows down.

(c) Indicate in the figure where the particle is stationary – that is, where it does not move.

(d) Indicate in the figure where acceleration is the largest and the smallest.

(e) Sketch the position as a function of time, $x(t)$.

4.18: A swimming bacterium. When the heliobacter bacteria swims, it is driven by the rotational motion of its tiny tail. It swims almost at a constant velocity, with small fluctuations due to variations in the rotational motion. As a simple model for the motion, we assume that the bacteria starts with the velocity $v = 10\mu$m/s at the time $t = 0$s, and is then subject to the acceleration, $a(t) = a_0 \sin(2\pi t/T)$, where $a_0 = 1\mu\text{m/s}^2$, and $T = 1$ms.

(a) Find the velocity of the bacterium as a function of time.

(b) Find the position of the bacterium as a function of time.

(c) Find the average velocity of the bacterium after a time $t = 10T$.

4.19: Resistance. *(This problem requires some knowledge of statistics).* An electron is moving with a constant acceleration, a_0, through a conductor. However, there are many small irregularities in the conductor – called scattering centers. If the electron hits a scattering center it stops, that is, its velocity immediately becomes zero. The scatting centers have a constant density. The probability for the electron to hit a scattering center when it moves a distance Δx is $P = \Delta x/b$, where b is a length describing the typical length between two scattering centers. Assume that the electron starts from rest. (For simplicity, we measure lengths in nm and time in ns, and you can assume that $b = 1$ nm and that $a_0 = 1\text{nm/ns}^2$).

First, we address the case without scattering.

(a) Write a program to find the motion of the electron using Euler-Cromer's method to find the velocity and position from the acceleration. Plot the position, $x(t)$, and velocity, $v(t)$, of the electron as functions of time and compare with the exact result.

During the time interval Δt, the electron moves from $x(t)$ to $x(t + \Delta t)$. The probability for the electron to stop during this interval is $P = (x(t + \Delta t) - x(t))/b$.

(b) Explain why the following method models a collision:

```
dx = x(i+1) - x(i);
p = dx/b;
if (rand(1,1)<p)
   v(i+1) = 0.0;
end
```

where `rand(1,1)` produces a random number uniformly distributed between 0 and 1.

(c) Rewrite your program to include the effect of collisions using the algorithm described above. Plot the position, $x(t)$, and the velocity, $v(t)$, as functions of time. What do you see? Comment

(d) Find the average velocity v_{avg} for the electron.

The following parts are difficult:

(e) How does v_{avg} depend on a_0 and b? Can you make a theory that gives the value of v_{avg}?

(f) (Requires knowledge of statistics). What is the probability density for the distance, X, between two collisions?

4.20: Ball on vibrating surface. A ball is falling vertically through air over a vibrating surface. The position of the surface is $x_w(t) = A\cos\omega t$, where $A = 1$cm and ω is called the angular frequency of the vibrations. The ball starts from a position $x = 10$cm at $t = 0$s. The acceleration of the ball is given as:

$$a(x, v, t) = \begin{cases} -g & x > x_w \\ -g - C(x - x_w) & x \le x_w \end{cases} . \tag{4.126}$$

where $g = 9.81\text{m/s}^2$ and $C = 10000.0\text{s}^{-2}$.

(a) Write down the equation you need to solve to find the motion of the ball. Include initial conditions for the ball.

(b) Write down the algorithm to find the position and velocity at $t_{i+1} = t_i + \Delta t$ given the position and velocity at t_i. Use Euler-Cromer's scheme.

(c) Write a program to find the position and velocity of the ball as a function of time.

(d) Check your program by comparing the initial motion of the ball with the exact solution when the acceleration is constant. Plot the results.

(e) Check your program by first studing the behavior when the vibrating surface is stationary, that is, when $A = 0$m and $x_w = 0$m. Plot the resulting behavior. Ensure that your timestep is small enough, $\Delta t = 10^{-5}$s. What happens if you increase the timestep to $\Delta t = 0.02$s?

(f) Finally, use your program to model the motion of the ball when the surface is vibrating. Use $A = 0.01$m, $\omega = 10\text{s}^{-1}$, and simulate 5s of motion. Plot the results. What is happening?

(g) What happens if you increase the vibrational frequency to $\omega = 30\text{s}^{-1}$?. Plot the results. Can you explain the difference from $\omega = 10\text{s}^{-1}$?.

Project 4.1: Sliding on snow

In this project we address the motion of an object sliding on a slippery surface – such as a ski sliding in a snowy track. You will learn how to find the equation of motion for sliding systems both analytically and numerically, and to interpret the results.

We start by studying a simplyfied situation called frictional motion: A block is sliding on a surface as illustrated in figure 4.24, moving with a velocity v in the positive x-direction. The forces from the interactions with the surface results in an acceleration:

$$a = \begin{cases} -\mu(|v|)g & v > 0 \\ 0 & v = 0 \\ \mu(|v|)g & v < 0 \end{cases} , \qquad (4.127)$$

where $g = 9.8\text{m/s}^2$ is the acceleration of gravity. Let us first assume that $\mu(v) = \mu = 0.1$ for the surface. That is, we assume that the coefficient of friction does not depend on the velocity of the block. We give the block a push and release it with a velocity of 5m/s.

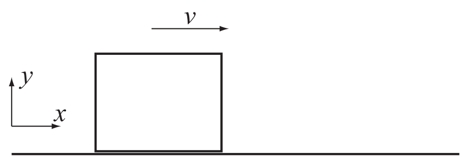


Figure 4.24: *A block moving on a slippery surface.*

(a) Find the the velocity, $v(t)$, of the block.

(b) How long time does it take until the block stops?

(c) Write a program where you find $v(t)$ numerically using Euler's or Euler-Cromer's method. (Hint: You can find a program example in the textbook.) Use the program to plot $v(t)$ and compare with your analytical solution. Use a timestep of $\Delta t = 0.01$.

The description of friction provided above is too simplified. The coefficient of friction is generally not independent of velocity. For dry friction, the coefficient of friction can in some cases be approximated by the following formula:

$$\mu(v) = \mu_d + \frac{\mu_s - \mu_d}{1 + v/v^*} , \qquad (4.128)$$

where $\mu_d = 0.1$ often is called the dynamic coefficient of friction, $\mu_s = 0.2$ is called the static coefficent of friction, and $v^* = 0.5\text{m/s}$ is a characteristic velocity for the contact between the block and the surface.

(d) Show that the acceleration of the block is:

$$a(v) = -\mu_d g - g\frac{\mu_s - \mu_d}{1 + v/v^*} , \qquad (4.129)$$

for $v > 0$.

(e) Use your program to find $v(t)$ for the more realistic model, with the same starting velocity, and compare with your previous results. Are your results reasonable? Explain.

The model we have presented so far is only relevant at small velocities. At higher velocities the snow or ice melts, and the coefficient of friction displays a different dependency on velocity:

$$\mu(v) = \mu_m \left(\frac{v}{v_m}\right)^{-\frac{1}{2}} \text{ when } v > v_m , \qquad (4.130)$$

where v_m is the velocity where melting becomes important. For lower velocities the model presented above with static and dynamic friction is still valid.

(f) Show that

$$\mu_m = \mu_d + \frac{\mu_s - \mu_d}{1 + v_m/v^*} , \qquad (4.131)$$

in order for the coefficient of friction to be continuous at $v = v_m$.

(g) Modify your program to find the time development of v for the block when $v_m = 1.5\text{m/s}$. Compare with the two other models above: The model without velocity dependence and the model for dry friction. Comment on the results.

(h) The process may become more clearer if you plot the acceleration for all the three models in the same plot. Modify your program to plot $a(t)$, plot the results, and comment on the results. What would happen if the initial velocity was much higher or much lower than 5m/s?