Project 16.2: Snow crystal

In this project you will apply your knowledge of linear and angular momentum to study the aggregation of small droplets of ice to form large grains of snow.

As snow crystals form in clouds they start falling through the cloud. Due to air resistance, larger particles fall faster than smaller particles. A large particle will therefore overtake smaller particles. When a smaller particle is overtaken, it will stick to the larger particle, adding further to the size. This process forms aggregate snowflakes, which is one of the most common types of snowflakes.[4]. This mechanism is often called differential sedimentation, and is a process important for pattern formation in many natural systems, and it is also a process important for many industrial processes. An example of a complex aggregate formed by a related aggregation process called Diffusion Limited Aggregation in figure 16.39 shows the complex geometries typically found in aggregate grains.

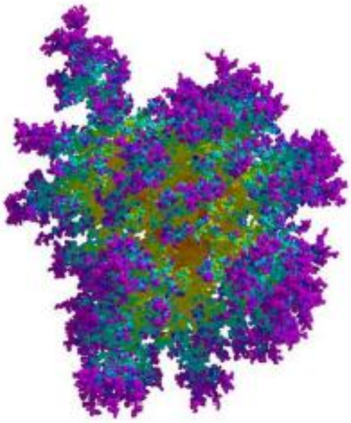

Figure 16.39: *Image of a (fractal) cluster formed by diffusion limited aggregation of 10000 particles. (Goold, 2004).*

In this project, we will study the aggregation process in detail. We will study an approximately spherical grain of ice of mass M and radius R, hitting and sticking to an identical grain of ice.

First, let us address why large particles fall faster than small particles. The mass of an ice grain of radius R and mass density ρ_m is

$$M = \rho_m \frac{4\pi}{3} R^3 \,. \tag{16.197}$$

We will assume that air resistance can be modelled using the approximation:

$$\vec{F}_v = -k_v \vec{v} \,, \tag{16.198}$$

where

$$k_v \simeq 20.4 R \eta \tag{16.199}$$

is a constant depending on the viscocity η of the fluid.

(a) Find the forces acting on an ice grain with radius R, and write down Newton's second law of motion for the grain.

(b) Show that the acceleration of the grain is

$$\vec{a} = \vec{g} - \frac{20.4\eta}{\rho_m \frac{4\pi}{3} R^2} \vec{v} \,, \tag{16.200}$$

where $\vec{g} = -g\hat{j}$ and g is the acceleration of gravity. Can you now explain why larger grains fall faster than smaller grains?

We will now study a collision between two identical ice grains. One grain is at rest relative to the reference system and the other grain has a velocity v_0 downwards. When the two grains collide, they stick together at the point of contact, and remain stuck together. We call this combination of two grains a compound grain.

(c) The moment of inertia of one ice grain around its center is $I_c = \frac{2}{5}MR^2$. Show that the moment of inertia, I, around the center of mass for a compound grain consisting of two grains sticking together is:

$$I = \frac{14}{5} M R^2 \tag{16.201}$$

First, we consider a linear collision where the upper grain hits the lower grain directly in the center, as illustrated in figure 16.40a. We assume the collision to be instantaneous, so you can ignore the effect of air resistance and gravity during the collision.

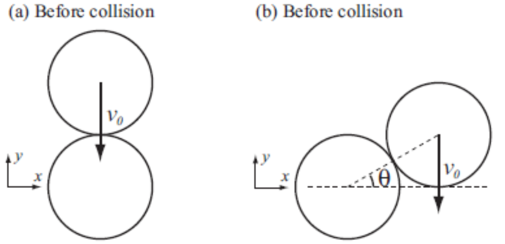

Figure 16.40: *Illustration of a collision between two identical ice grains. The lower grain is not moving, and the top grain is moving downwards with a velocity $\vec{v}_0$ as illustrated. In (a) the top grain hits at the top of the stationary grain, and in (b) the top grain hits the lower grain when the angle between the line connecting the centers of the grains and the horizontal is θ.*

(d) What is the velocity, $\vec{v}_1$, of the center of mass the compound grain after the collision?

(e) What is the angular velocity, $\vec{\omega}_1$, around the center of mass of the compound grain after the collision?

Let us now consider the more general case illustrated in figure 16.40b. When the two grains touch, the line between the centers of the two grains forms the angle θ with the horizontal. The upper grain still has the initial velocity v_0 downwards before the collision, and the lower grain is at rest.

(f) What is the velocity, $\vec{v}_1$, of the center of mass of the compound grain after the collision?

(g) What is the angular velocity, $\vec{\omega}_1$, around the center of mass of the compound grain after the collision?

(h) What is the loss of energy in the collision?

Let us now address the motion of the compound grain after the collision. Initially, it is rotating with the angular velocity ω_1.

(i) If we ignore air resistance, find $\omega(t)$ as a function of time for the subsequent motion.

In the following we will not ignore air resistance, but rather develop a simplified model for the air resistance. In order to determine the force acting on the compound object due to air resistance, we either need to perform experiments on such objects, or we can use numerical simulations of the fluid flow around the object to determine the forces.

Here, we will use a strong simplification: We assume that we may consider the compound object to consist of two separate spheres. The force on each of the spheres due to air resistance is described by equation 16.198, where the corresponding velocity, v, in equation 16.198 is the velocity of the center of the sphere, and the force acts in the center of the sphere.

The compound object has velocity $\vec{v}_{cm}$ and angular velocity $\vec{\omega}$.

(j) Argue that the velocities, $\vec{v}_A$ and $\vec{v}_B$, of each of the ice grains A and B are

$$\vec{v}_A = \vec{v}_{cm} + \vec{\omega} \times \vec{r} \,, \tag{16.202}$$

and

$$\vec{v}_B = \vec{v}_{cm} - \vec{\omega} \times \vec{r} \tag{16.203}$$

where $\vec{r}$ describes the position of grain A relative to the center of mass of the compound grain.

(k) Show that the net force on the center of mass of the compound object is:

$$\sum \vec{F} = 2M\vec{g} - 2k_v \vec{v}_{cm} \,, \tag{16.204}$$

where $\vec{g} = -g\hat{j}$ and g is the acceleration of gravity.

(l) Show that the torque around the center of mass of the compound object due to air resistance is:

$$\vec{\tau} = -2k_v \vec{\omega} R^2 \,. \tag{16.205}$$

(Hint: Use Lagrange's formula).

(m) Show that the angular acceleration α of the compound object around its center of mass can be written as

$$\alpha = \frac{d\omega}{dt} = -\frac{1}{t_0}\omega \tag{16.206}$$

and find the characteristic time t_0.

(n) Describe (with words) the motion of the compound object.

(o) (Optional) Sketch the time development of the velocity v_{cm} and the angular velocity ω of the compound object, and discuss how the behavior would change if you changed the radius, R, of the grains.

(p) (Optional) How would our argument change if we instead studied large particles, where the air resistance force depends on the square of the velocity?

Final comment: Notice that the result above for the net force on the compound grain indicates that small and large grains have the same acceleration, which is not consistent with our initial result. This is due to our (incorrect) simplification of adding the air resistance force for each of the grains together to get the air resistance force for the compound grain. For a real ice crystal formed by aggregation, the dependence of the air resistance on the size of the compound grain is more complicated, and will also depend on the complex geometry attained by a compound grain after a few hundred collisions with smaller grains.