Fys-Mek1110 – 2013 – Oblig 3

Ball in a spring

In this project you will study an advanced model of a pendulum. The pendulum consists of a ball in a massless rope moving in a vertical plane. The ball has mass m. You can neglect air resistance. We describe the position of the ball by the position vector, $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. In this project we will introduce a *model* for the pendulum by assuming that the rope can be modelled as a spring with a spring constant k and an equilibrium length L_0.

Figure 0.1: Illustration of a pendulum consisting of a ball of mass m attached to rope of length L_0. The other end of the rope is attached at the point O.

(a) Identify the forces and draw a free-body diagram of the ball.

(b) Show that the net external force acting on the ball can be written as:

$$\sum_j \vec{F} = -mg\hat{\jmath} - k\left(r - L_0\right)\frac{\vec{r}}{r}\,, \tag{1}$$

where $r = |\vec{r}|$ is the length of the (stretched) rope, and the origin of the coordinate system is chosen to be the attachment point, O, of the rope.

(c) Rewrite the expression of the external force on component form by writing the force components, F_x, and F_y, as functions of the components x and y of the position vector, $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$.

In this project, we will not assume that the ball is following a particular path, such as a circle, but we will instead use Newton's second law to determine the motion of the ball from the forces acting on it. Using our model, we can measure the tension in the rope, as well as the motion of the ball, and analyze these to learn about the motion.

(d) For a pendulum, it is customary to describe the position of the pendulum by its angle θ with the vertical. Does the angle θ give a sufficient description of the position of the ball in this case? Explain your answer.

(e) If the ball is at rest at $\theta = 0$ with no velocity ($\vec{v} = \vec{0}$) and no acceleration, what is the position of the ball? What happens if you increase the value of k for the rope?

We will now study a specific pendulum, consisting of a ball with a mass of 0.1kg, and a rope of equilibrium length $L_0 = 1$m with a spring constant $k = 200$N/m, which corresponds to a rather elastic rope. Initially, you can as0sume that the ball starts with zero velocity at an angle $\theta = 30°$ at a distance L_0 from the origin. We want to study the motion of the ball by integrating the equations of motion numerically.

(f) Find an expression for the acceleration, $\vec{a}$, of the ball. You should write it both on vector form, where there acceleration vector is a function of the position vector $\vec{r}$ and its length, r, and on component form, where the components a_x and a_y are functions of the x and y components of the position vector.

(g) What is the mathematical initial value problem you need to solve in order to find the motion of the ball? Include both the differential equation you need to solve and the initial conditions in your answer.

(h) How can you solve this problem numerically? Write down a set of equations that find the position and velocity at a time $t + \Delta t$ given the position and velocity at t using Euler-Cromer's method. Insert your expression for the acceleration from above. Mark the terms in your equations that vary in time.

(i) Write a program that "solves" the problem by finding the motion of the ball. The program should plot the position of the ball in the xy-plane for the first 10s of the motion. Hint 1: You may write the mathematical expression almost directly into your program if you use a vector notation and vector operations in your code. Hint 2: Remember that $r = r(t) = |\vec{r}(t)|$ varies in time! Hint 3: Do *not* use $\theta(t)$ to describe the position of the ball. Describe the motion using $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ and use your results from above for the acceleration.

(j) Use the program to find the behavior for the given initial conditions using a time-step of $\Delta t = 0.001$. Plot the resulting motion. Describe what you see.

(k) What happens if you increase Δt to $\Delta t = 0.01$ and $\Delta t = 0.1$? Can you explain this? (Optional: Test what happens if you use Euler's method with $\Delta t = 0.001$ instead of Euler-Cromer's method.)

(l) Rerun the program with $k = 20$ and $k = 2000$. Describe the motion in these cases and compare with $k = 200$ case. Are your results reasonable? Based on this, can you suggest how to use this method to model a pendulum in a stiff rope? What do you think would be the limitation of this approach? (Test what happens if you use $k = 2\ 10^6$ in your program).

(m) Rewrite your program to ensure that the rope tension is zero if the spring is compressed, because the rope cannot sustain compression. Use this program to determine the motion with the initial conditions $\vec{v}_0 = 6.0\hat{\imath}$m/s and $\vec{r}_0 = -L_0\hat{\jmath}$. What happens? Explore various initial conditions and explain what you observe.

End of Oblig 3