Project 15.1: Micro-electromechanical system

In this project you will learn about the moment of inertia, and the potential and kinetic energy of a rotating object, and we will use this to study a simple electromechanical system, similar to a micro-mirror used in most modern projectors.

Modern production techniques for microscopic systems allows us to construct small mechanical elements made of silicon. For example, we can construct small silicon cantilevers with dimensions down to a few micrometers. In this project we will address the motion of a thin, microscopic beam using energy techniques.

First, let us consider a small, square mechanical element of dimensions $L \times L \times h$ and mass M. We assume that the thickness h is so small that we can neglect the finite thickness of the square. The square is attached with a hinge at one of the ends of length L. The hinge follows the y-axis as show in figure 15.20.

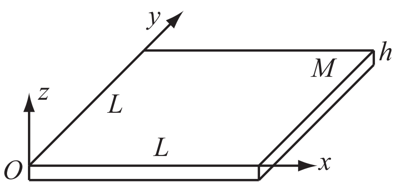

Figure 15.20: Illustration of a square mechanical element. Each side has a length L, and the mass of the square is M. The thickness h is small compared to L.

(a) Find the position $\vec{R} = X\hat{\imath} + Y\hat{\jmath}$ of the center of mass of the square. The origin is in one of the corners of the square.

(b) Show by integration that the moment of inertia, $I_{cm,y}$, for rotations around an axis parallel with the y-axis going through the center of mass is

$$I_{cm,y} = \frac{1}{12} M L^2 \; . \qquad (15.91)$$

(c) Find the moment of inertia, I_y, for rotations around the y-axis.

The micromechanical cantilever we want to study is a bit more complicated than a single square. We can construct the cantilever from four identical squares, each with dimensions $L \times L$ and masses M, as illustrated in figure 15.21. The squares are rigidly attached to each other, so that they move as a single body. At the edge, along the y-axis, the cantilever is attached with a hinge so that the cantilever can rotate freely about this axis.

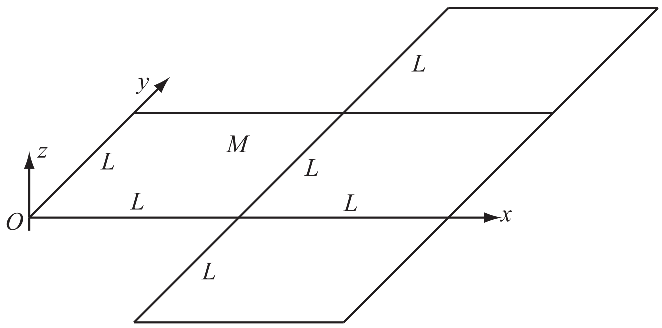

Figure 15.21: Illustration of a cantilever constructed from four squares.

(d) Show that the center of mass of the cantilever is:

$$\vec{R} = X\hat{\imath} + Y\hat{\jmath} = \frac{5}{4} L\hat{\imath} + \frac{L}{2}\hat{\jmath} \; . \qquad (15.92)$$

(e) Show that the moment of inertia, I_y for the whole object is

$$I_y = \frac{22}{3} M L^2 \; . \qquad (15.93)$$

Hint 1: You can calculate the moment of inertia for each part of the object independently and sum the results. (This is called the superposition principle). Hint 2: Use the parallel-axis theorem to find the moment of inertial around the y-axis for each part of the object.

Even though we are considering a microscopic system, where the effect of gravity typically will be negligible, let us first consider the motion of the cantilever when it is affected by gravity. For a real microscopic cantilever the effects of electrostatic forces is typically more important. Often such interactions results in a constant electrostatic force on the cantilever. The results from studying the behavior of the cantilever when affected by gravity as we do here can therefore easily be translated into the behavior of a cantilever affected by electrostatic forces.

The gravitational force acts in the negative z-direction. As a result of gravity, the cantilever rotates and angle θ around the y-axis as illustrated in figure 15.22.

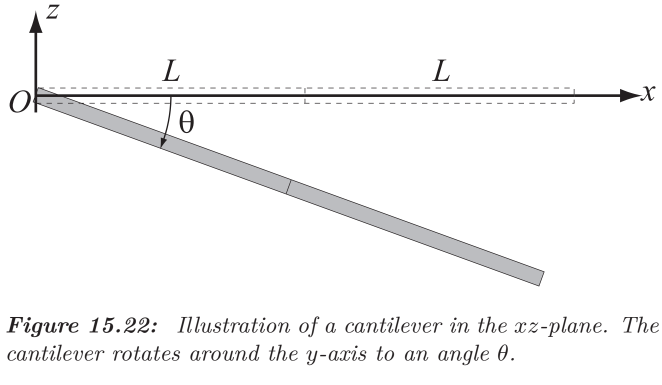

Figure 15.22: Illustration of a cantilever in the xz-plane. The cantilever rotates around the y-axis to an angle θ.

(f) Show that the potential energy of the cantilever due to the gravitational force is:

$$U_G = -5MLg\sin\theta\ . \tag{15.94}$$

Where the potential energy is zero when the cantilever is horizontal.

(g) Assume that the cantilever starts with an initial angular velocity $\omega_0 = 0$ when $\theta = 0$. Find the angular velocity of the cantilever, $\omega(\theta)$, when it has reached the angle θ.

In the following, we will no longer assume that the cantilever rotates freely around the y-axis. Instead, we will assume that it bends around a hinge along the y-axis, and that the bending is like the bending of an elastic body. This means that there is an potential energy associated with the bending. The potential energy of the cantilever when it is bent an angle θ due to the stiffness of the hinge is

$$U_h = \frac{1}{2}\kappa\theta^2 \tag{15.95}$$

where κ is a constant that depends on the material properties (and the size) of the hinge.

(h) Again, assume that the cantilever starts with an initial angular velocity $\omega_0 = 0$ when $\theta = 0$. Find the angular velocity of the cantilever, $\omega(\theta)$, when it has reached the angle θ.

(i) Describe the motion of the cantilever.

The rest of the project is optional.

For small θ we can approximate $\sin\theta \simeq \theta$. We will use this approximation in the following.

(j) Find the maximum angle θ of the cantilever when it is released as described above.

(k) Draw an energy diagram in the form of the total potential energy of the cantilever as a function of θ. Can you find any equilibrium points for the cantilever?

Small cantilevers are used for many technological applications. For example, projectors using the DLP technlogy consists of a vast number of micromirrors – small cantilevers that reflect light. When an electrical field is applied to the cantilever – the cantilever is affected by an electrostatic force. We can describe this in the same way as we described gravity above, but the electrical field can be turned on or off. As a result, the cantilever can be bent, and the light is reflected in a different direction. You can look at other interesting applications by searching for MEMS in your favourite search engine.