Chapter 17

Fictive forces

You now have now learned to apply Newton's second law to find the acceleration from the forces acting on an object. This method has proved to be a powerful tool that allows us to calculate the motion of an object or of a system of several objects.

However, Newton's second law has a significant limitation – it is only valid in inertial systems. What do we do if we are in an accelerated system and want to describe the motion relative to this system? The simplest alternative may be to describe the motion in an inertial system – where Newton's laws of motion are valid – and then map the motion back onto the accelerated system. Such a method will always work, but it is not always practical. For example, observations made on the surface of the Earth are done in an accelerated coordinate system because the Earth rotates. Consequently, all systems following the Earth's rotation will be accelerated. In this case it is often practical to describe the motion in the accelerated systems, but to do so we need to introduce a set of fictive forces in order to be able to apply Newton's laws of motion.

17.1 Example: Forces on the bus

Let us start from a situation from your everyday experience: You are sitting in a bus and a baby stroller is standing on the floor in front of you. Suddenly, the stroller starts moving toward you. What is causing the acceleration of the stroller? We know that according to Newton's second law, an acceleration is caused by a net force acting on an object. However, there is another alternative: If the bus is braking, the bus will be accelerated relative to the ball. For a person on the bus, in the accelerated coordinate system, it will appear that a force is acting on the ball. Let us describe the situation more precisely by introducing two coordinate systems, the system S is at rest relative to the road, and the system S' follows the bus as shown in figure 17.1. We can then relate the position of the stroller relative to the ground, $\vec{r}$, to the position relative to the bus, $\vec{r'}$, by:

$$\vec{r} = \vec{R} + \vec{r'} \ , \tag{17.1}$$

where $\vec{R}$ describes the position of the origin in S' relative to S. We can relate the accelerations by taking the time derivative of equation 17.1 twice:

$$\ddot{\vec{r}} = \vec{a} = \ddot{\vec{R}} + \ddot{\vec{r'}} = \vec{A} + \ddot{\vec{r'}} = \vec{A} + \vec{a}' \ , \tag{17.2}$$

where $\vec{a}$ is the acceleration of the stroller relative to the ground, $\vec{A}$ is the acceleration of the bus relative to the ground, and $\vec{a'}$ is the acceleration of the stroller relative to the bus.

A person on the bus observes $\vec{a'}$, and she will try to relate this acceleration to the sum of external forces acting on the stroller. We know that Newton's laws of motion are valid in the inertial system S, where the acceleration is given by the sum of external forces:

$$\sum \vec{F} = m\vec{a} \ . \tag{17.3}$$

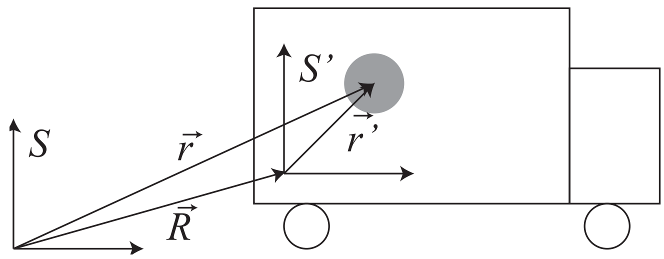

Figure 17.1: A ball lying on the floor in a bus is observed in an inertial system S on the ground, and in a system S' following the bus. The position of the ball is $\vec{r}$ in S and $\vec{r'}$ in S'. The position of the bus in S is $\vec{R}$.

We insert the expression for $\vec{a}$ from equation 17.2 and get

$$\sum \vec{F} = m\vec{A} + m\vec{a'} \ , \tag{17.4}$$

which gives

$$m\vec{a'} = \sum \vec{F} - m\vec{A} = \sum \vec{F} + \vec{F}_A \ , \tag{17.5}$$

where $\vec{F}_A = -m\vec{A}$. If the person on the bus wants to use Newton's second law to describe the motion, she has to add the force $\vec{F}_A$ to the sum of external forces. This force is an exampel of a fictive force, which we introduce in order to be able to use Newton's second law in an accelerated system. It is not a force acting from one object on another object. The fictive force affects the stroller, but it is not the interaction with another object that causes the force. The fictive force has no reaction.

17.2 Mapping between inertial and rotating systems

17.2.1 Horizontal motion on the Pole

Let us start our exploration of rotating systems by addressing the motion of a pendulum on the North Pole. You have built a tall tent and hang the pendulum from the top of the tent. The pendulum is started by releasing the pendulum with an initial velocity towards the center of the tent. Seen from an inertial system outside the Earth, the pendulum will swing in a plane given by the initial velocity and the initial position of the pendulum ball. However, the Earth will rotate underneath the pendulum. For an observer standing on Earth's surface, following the rotation, it will appear as if the plane of the pendulum is rotating. Because the pendulum is very small compared to the radius of the Earth, we can consider the Earth as approximately flat around the pole, and it is sufficient to describe the motion of the pendulum in two dimensions.

How can we relate the motion of the pendulum in the inertial system S to the motion as observed in the system S' following the motion of the Earth? The position of the pendulum in the inertial system S is:

$$\vec{r} = x\hat{i} + y\hat{j} \ . \tag{17.6}$$

The system S' follows the rotation of the Earth. We describe the position of the pendulum relative to the Earth by giving the coordinates in a coordinate system that follows the rotation of the Earth. We use the unit vectors $\hat{i}'$ and $\hat{j}'$ to describe the system S' following the rotation of the Earth. After a time t, the Earth has rotated an angle $\theta = \omega t$, and the unit vectors in S' have rotated correspondingly relative to S as illustrated in figure 17.2.

Let us determine the position of the pendulum in the Earth's system S':

$$\vec{r'}(t) = x'(t)\hat{i}' + y'(t)\hat{j}' \ , \tag{17.7}$$

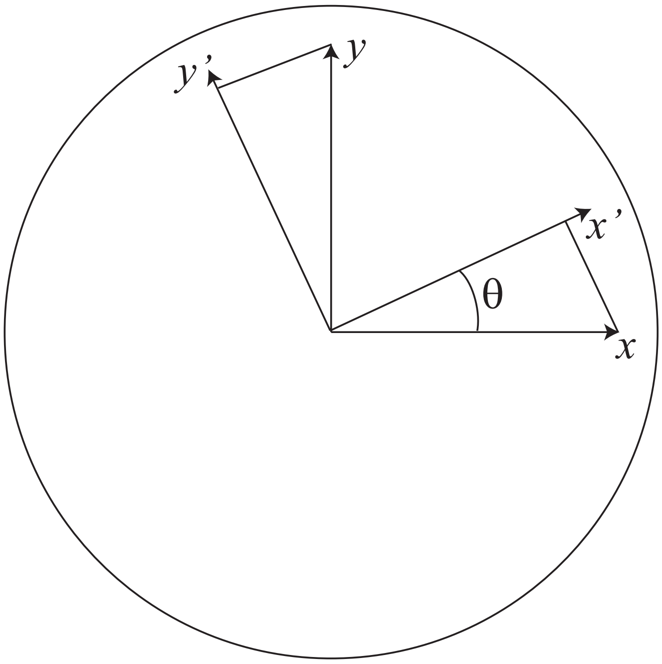

Figure 17.2: *An illustration of the motion of a coordinate system following the motion of the Earth on the North Pole as seen from above. After a time t, the Earth has rotated an angle $\theta = \omega t$.*

where the unit vectors $\hat{\imath}'$ og $\hat{\jmath}'$ are rotated relative to $\hat{\imath}$ and $\hat{\jmath}$ as shown in figure 17.2. We find that we can relate the unit vector $\hat{\imath}$ and $\hat{\jmath}$ to the unit vectors $\hat{\imath}'$ and $\hat{\jmath}'$:

$$\hat{\imath} = \cos(\theta)\hat{\imath}' - \sin(\theta)\hat{\jmath}' \ , \tag{17.8}$$

$$\hat{\jmath} = \sin(\theta)\hat{\imath}' + \cos(\theta)\hat{\jmath}' \ . \tag{17.9}$$

where $\theta = \omega t$ and ω is the angular velocity of the Earth.

The position of the pendulum in the Earth's system is therefore

$$\begin{aligned} \vec{r} &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= x(t)\left(\cos(\theta)\hat{\imath}' - \sin(\theta)\hat{\jmath}'\right) + y(t)\left(\sin(\theta)\hat{\imath}' + \cos(\theta)\hat{\jmath}'\right) \\ &= \left(x(t)\cos\omega t + y(t)\sin\omega t\right)\hat{\imath}' + \left(-x(t)\sin\omega t + y(t)\cos\omega t\right)\hat{\jmath}' \end{aligned} \tag{17.10}$$

Let us assume that we release the pendulum along the x-direction at $t = 0$. The position of the pendulum in the inertial system is:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = A\sin\Omega t\hat{\imath} \ , \tag{17.11}$$

where $\Omega = 2\pi/T$ is related to the period T of the pendulum. This means that $y(t) = 0$, and we find that the position of the pendulum in the Earth's system is:

$$\vec{r} = \vec{r}' = x(t)\cos\omega t\ \hat{\imath}' - x(t)\sin\omega t\ \hat{\jmath}' = A\sin\Omega t\cos\omega t\hat{\imath}' - A\sin\Omega t\sin\omega t\hat{\jmath}' \ . \tag{17.12}$$

which simply means that the pendulum plane measured on the Earth is rotating slowly as the Earth rotates.

A person on the Earth surface will therefore observe that the pendulum plane is rotating. If she wants to use Newton's laws of motion to describe the motion observed on the Earth, it is necessary to introduce a force causing this rotation of the rotation plane. This force is called the Coriolis force, and we will derive an exact form for it below.

17.2.2 Vertical motion at the equator

Before we address the general case, let us address another simplified case: an object falling off a high tower at the equator. Let us assume that an aspiring

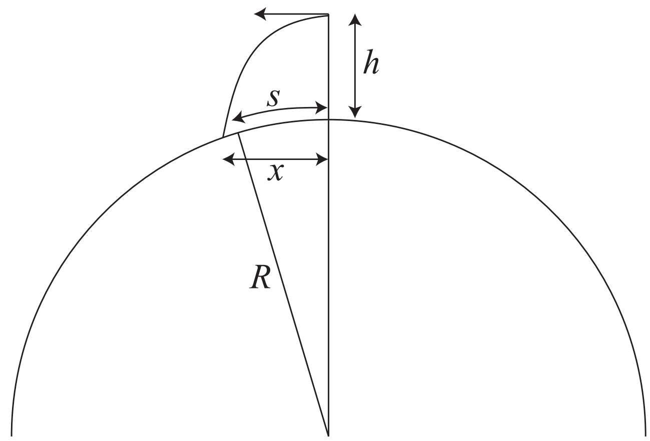

Figure 17.3: *Illustration of an object falling from a high tower at the equator. The height of the tower is h, the radius of the Earth is R, and the angular velocity of the Earth is ω. In an interial system, the path will be that of projectile motion, while on the surface of the Earth, the path will bend as illustrated with the dashed line.*

Galileo builds a very tall tower at the equator and releases a heavy object from the tower. How will the path of the object appear to a person standing on the surface of the Earth?

For a person in an inertial system that does not follow the rotation of the Earth, the object has a finite initial velocity. The object follows the rotation of the Earth when it is released and it therefore has the initial velocity $v_0 = \omega r$, where ω is the angular velocity of the earth, and r is the distance from the center of the Earth to the initial position of the object, as shown in figure 17.3. The distance from the center of the Earth to the object is $r = R + h$ where R is the radius of the Earth and h is the height of the tower. Neglecting air resistance, the only force acting on the object is gravity. The object therefore follows a parabolic path characteristic of projectile motion, and stops where the path meets the surface of the Earth. Because of the curvature of the Earth, the object falls a height which is larger than h. If the object hits the surface of the Earth after a time t, the object has moved a horizontal distance:

$$s = v_0 t \; . \tag{17.13}$$

The distance from the position of the tower on the surface of the Earth to the landing point will be somewhat longer than s, but we will approximate this length by s. During the time t, the Earth has also rotated an angle $\theta = \omega t$, and the tower has moved a distance:

$$s_t = \omega t R \; . \tag{17.14}$$

The object has therefore hit the ground at a distance

$$\Delta s = s - s_t = \omega t(R + h) - \omega t R = \omega t h \; , \tag{17.15}$$

from the tower. This means that the object hits the ground in front of the tower! For a 1km high tower, the fall time is approximately the same as for a fall of 1km:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2km}{9.81ms^{-2}}} \simeq 14.3s \; . \tag{17.16}$$

$$\Delta s = \frac{2\pi}{243600s} 14.3s1000m = 1.04m \; . \tag{17.17}$$

This is therefore a measureable effect for a 1km high tower, but it is a small effect. It is also interesting to notice that to first order, the distance does not depend on the radius of the Earth. The effect increases with the fall time. For falls over short distances the fall time is very small compared with the rotation time of the Earth, and the effect is small.

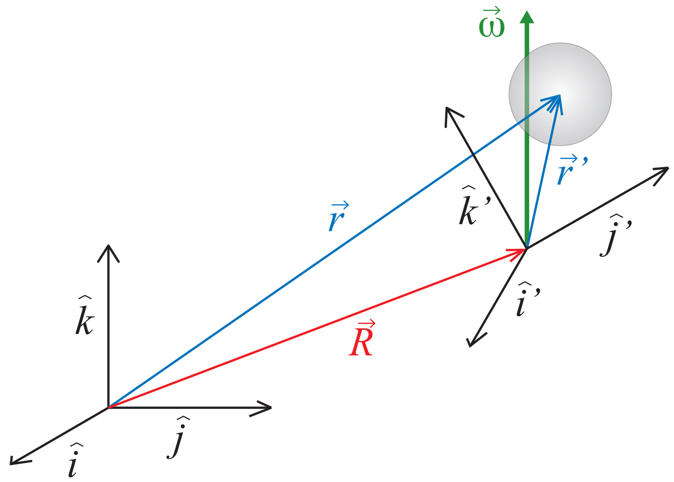

Figure 17.4: *The coordinate system S is an inertial system. The system S' is both translated and rotated relative to system S. The system S' rotates with an angular velocity $\vec{\omega}$. Consequently, the unit vectors in the system S' also rotate, and they change with time. The vector $\vec{R}$ points to the origin of the system S' measured in S.*

For a person on the Earth's surface the path bends. If he want to explain the motion using Newton's laws of motion in the Earth's system, it is necessary to introduce fictive forces. We will now introduce a general derivation and a general expression for the forces acting on an object in a rotated coordinate system.

17.3 Rotating coordinate systems

Let us introduce a general, mathematical description of an accelerated and rotated coordinate system, so that we may relate the acceleration in the accelerated system to real and fictive forces. The system S is an inertial system, and the system S' is both translated and rotated relative to S as illustrated in figure 17.4. The position of the origin of system S' in system S is given by $\vec{R}$. The rotating system S' momentarily rotates with the angular velocity $\vec{\omega}$. We want to describe the motion of an object of mass m with position $\vec{r}$ in S and the position $\vec{r'}$ in S', as shown in figure 17.4.

The coordinate system in the inertial system is described by the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$. The unit vectors in system S' are $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$ and rotate with the system. These unit vectors are therefore time-dependent. The position of an object is described in each system by:

$$\vec{r} = \vec{R} + \vec{r'} \ . \tag{17.18}$$

We can decompose the vector $\vec{r'}$ both using the unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ of S and the unit vectors $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$ of S'. When decomposed in the rotated coordinate system, the position of the object is:

$$\vec{r'} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}' \ . \tag{17.19}$$

This expression defines the coordinates x', y', og z', which is the position of the object measured in the rotating coordinate system.

First, we find the velocity of the object as measured in both systems by taking the time derivative of equation 17.18:

$$\frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r'}}{dt} \ , \tag{17.20}$$

where $\vec{V} = d\vec{R}/dt$ is the (linear) velocity of system S' relative to systme S. If we express $\vec{r'}$ using the unit vectors $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$, we also need to take into consideration the derivatives of the unit vector when performing the time derivative.

$$\frac{d\vec{r'}}{dt} = \frac{dx'}{dt}\hat{\imath}' + \frac{dy'}{dt}\hat{\jmath}' + \frac{dz'}{dt}\hat{k}' + x'\frac{d\hat{\imath}'}{dt} + y'\frac{d\hat{\jmath}'}{dt} + z'\frac{d\hat{k}'}{dt} \ . \tag{17.21}$$

The system S' is rotating around the axis $\vec{\omega}$. The velocity of a point $\hat{i}'$ is therefore given as:

$$\frac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}' \ , \tag{17.22}$$

and we have similar expressions for $\hat{j}'$ og $\hat{k}'$:

$$\frac{d\hat{j}'}{dt} = \vec{\omega} \times \hat{j}' \ \ , \frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}' \ \ . \tag{17.23}$$

We recognize the first part of equation 17.21 as the velocity, $\vec{v'}$, of the object measured in S':

$$\vec{v'} = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}' \ . \tag{17.24}$$

Equation 17.21 can therefore be simplified to:

$$\frac{d\vec{r'}}{dt} = \vec{v'} + \vec{\omega} \times \vec{r'} \ . \tag{17.25}$$

And we write the velocity of the object as:

$$\vec{v} = \vec{V} + \vec{v'} + \vec{\omega} \times \vec{r'} \ . \tag{17.26}$$

Where $\vec{v'}$ and $\vec{r'}$ are the velocity and position of the object in S' respectively.

The acceleration is found by taking one more time derivative of equation 17.26:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{V}}{dt} + \frac{d}{dt}\vec{v'} + \frac{d}{dt}(\vec{\omega} \times \vec{r'}) \ , \tag{17.27}$$

We recognize

$$\frac{d\vec{V}}{dt} = \vec{A} \ , \tag{17.28}$$

as the acceleration of system S' relative to S. This is the same term we used to study fictive forces in the bus.

Let us address each term individually:

$$\frac{d}{dt}\vec{v'} = \frac{d}{dt}(\frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}') \ . \tag{17.29}$$

Here, we use the notation $\dot{x}' = dx'/dt$ for the time derivative:

$$\begin{aligned} \frac{d}{dt}\vec{v'} &= \frac{d}{dt}\left(\dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'\right) \\ &= \left(\ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}'\right) + \left(\dot{x}'\frac{d\hat{i}'}{dt} + \dot{y}'\frac{d\hat{j}'}{dt} + \dot{z}'\frac{d\hat{k}'}{dt}\right) \end{aligned} \tag{17.30}$$

We use the result from equation 17.22:

$$\dot{x}'\frac{d\hat{i}'}{dt} + \dot{y}'\frac{d\hat{j}'}{dt} + \dot{z}'\frac{d\hat{k}'}{dt} = \dot{x}'(\vec{\omega} \times \hat{i}') + \dot{y}'(\vec{\omega} \times \hat{j}') + \dot{z}'(\vec{\omega} \times \hat{k}') = (\vec{\omega} \times \vec{v'}) \ . \tag{17.31}$$

We also recognize:

$$(\ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}') = \vec{a'} \ . \tag{17.32}$$

Only the last term in equation 17.27 remains:

$$\frac{d}{dt}(\vec{\omega} \times \vec{r'}) = \frac{d\vec{\omega}}{dt} \times \vec{r'} + \vec{\omega} \times \frac{d\vec{r'}}{dt} \ . \tag{17.33}$$

We have already found an expression for $d\vec{r'}/dt$ in equation 17.25. We can therefore simplify equation 17.27 to:

$$\vec{a} = \vec{A} + \vec{a'} + \vec{\omega} \times \vec{v'} + \frac{d\vec{\omega}}{dt} \times \vec{r'} + \vec{\omega} \times (\vec{v'} + \vec{\omega} \times \vec{r'}) \ , \tag{17.34}$$

which gives the following expression for the relation between the accelerations in the two systems S and S':

$$\vec{a} = \vec{A} + \vec{a'} + \frac{d\vec{\omega}}{dt} \times \vec{r'} + 2\vec{\omega} \times \vec{v'} + \vec{\omega} \times (\vec{\omega} \times \vec{r'}) \ . \tag{17.35}$$

We use this result to find the fictive forces in the rotated system S'. Newton's second law is valid in the inertial system S. Therefore, the sum of the external forces is related to the acceleration by:

$$\sum \vec{F} = m\vec{a} = m[\vec{A} + \vec{a'} + \frac{d\vec{\omega}}{dt} \times \vec{r'} + 2\vec{\omega} \times \vec{v'} + \vec{\omega} \times (\vec{\omega} \times \vec{r'})] \ . \tag{17.36}$$

Which can be written:

$$\sum \vec{F} - m\vec{A} - m\frac{d\vec{\omega}}{dt} \times \vec{r'} - 2m\vec{\omega} \times \vec{v'} - m\vec{\omega} \times (\vec{\omega} \times \vec{r'}) = m\vec{a'} \ . \tag{17.37}$$

For a person in the rotated system S' who wants to use Newton's second law of motion, we find that in addition to the sum of external forces, we must also include several fictive forces. The fictive forces have no reactions, and they are not real forces, but useful mathematical tools we use if we want to use Newton's laws in an accelerated coordinate system.

We recognize several of the fictive forces. The fictive force:

$$\vec{F}_A = -m\vec{A} \text{ (linear acceleration force)} \ , \tag{17.38}$$

is related to the linear acceleration of S' relative to S. For motions described on the surface of the Earth in a coordinate system that has the center of the Earth as the origin, and which rotates with the Earth, this component is zero.

The term:

$$\vec{F}_\alpha = -m\frac{\vec{\omega}}{dt} \times \vec{r'} \ , \tag{17.39}$$

is due to a change in angular velocity, either its magnitude or speed, and for many practical applications, such as for motions on the surface of the Earth, this term is zero, since the direction and magnitude of the angular velocity do not change (over the time periods we typically address).

We recognize **the centrifugal force**:

$$\vec{F}_S = -m\vec{\omega} \times (\vec{\omega} \times \vec{r'}) \ \text{ (centrifugal force)} \ . \tag{17.40}$$

This fictive force acts outward in a direction perpendicular to the axis of rotation.

The last term is **the Coriolis force**:

$$\vec{F}_C = -2m\vec{\omega} \times \vec{v'} \ \text{ (Coriolis force)} \ . \tag{17.41}$$

This term does not depend on the position – there is no $\vec{r'}$ term – but it depends on the velocity of the object.

In the following, we address applications and interpretations of these fictive forces.

17.4 Variation in g at the surface of the Earth

Because the Earth is rotating around an axis through its center, the surface of the Earth is an accelerated reference system. This is bad news if our laboratory (reference) system is placed on the surface of the Earth – as it typically is. In principle, we should therefore always include the effects of fictive forces in a rotating system. In practice, we only need to include these effects if we need very precise measurements, or if we are looking at processes that occur over a time frame comparable to the rotation time of the Earth (one day), or if we look at motions occuring over a significant distance.

Let us now study the effects of the fictive forces quantitatively, starting with the centrifugal force: How large are the effects of the centrifugal force compared with other relevant forces – such as the force of gravity – on the surface on the Earth?

How do we measure the force of gravity on an object? We use a device that measures forces, such as a spring weight. From the weight we can read off the normal force on the object from the weight (which from Newton's third law is the same as the force from the object on the weight).

What is the weight of an object of mass m placed on a weight at the equator? The object is affected by the gravitational force, $\vec{W}$, and the normal force, $\vec{N}$. Since the object is in an accelerated system, S', we must also include fictive forces.

Whenever we include fictive forces we must be very careful in describinng the inertial system and the accelerated system. In particular, it is important where we place the origin of the accelerated system, since the centrifugal force depends on the position vector, $\vec{r}'$, measured in the accelerated system. When we address motion on the surface of the Earth, we usually use a reference system that is rotating along with the Earth, and with its origin at the center of the Earth. In this case the reference system is not accelerated but only rotated relative to an inertial system placed at the center of the Earth[1].

We address each of the fictive forces separately:

- Since the reference system is not accelerated, $\vec{A} = 0$, and there is no fictive force due to a linear acceleration.

- Since the Earth is rotating with a constant angular velocity, $\vec{w}$, there is no fictive force due to changes in the angular velocity.

- Since the object is not moving relative to the surface of the Earth, it has zero velocity $\vec{v}'$ as measured in the rotating coordinate system on the surface of the Earth. Therefore, the Coriolis force on the object is zero:

$$\vec{F}_C = -2m\vec{\omega} \times \vec{v}' = 0 \ . \tag{17.42}$$

- The object is located at a position $\vec{r}'$ that does not change in time. Without loss of generality, we may assume that the object is located at $\vec{r}' = R\hat{\imath}'$, where R is the radius of the Earth. The angular velocity of the Earth is constant, $\vec{\omega} = \omega\hat{k} = \omega\hat{k}'$. Therefore, we find the centrifugal force to be:

$$\begin{aligned} \vec{F}_S &= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}') \\ &= -m\omega\hat{k} \times \left(\omega\hat{k} \times R\hat{\imath}'\right) \\ &= -m\omega\hat{k} \times \left(-\omega R\hat{\jmath}'\right) \\ &= m\omega^2 R\hat{\imath}' \ . \end{aligned} \tag{17.43}$$

We can now apply Newton's second law in the accelerated system:

$$\sum \vec{F} + \vec{F}_S = m\vec{a}' = 0 \ , \tag{17.44}$$

where the acceleration is zero since the object is not moving. We insert the external and fictive forces:

$$\vec{W} + \vec{N} + \vec{F}_S = 0 \ , \tag{17.45}$$

[1] Notice that we here do not include the effect of the accelerated motion of the Earth along its path around the Sun.

and the normal force is therefore

$$\begin{aligned}\vec{N} &= -\vec{W} - \vec{F}_S = -\left(-mg\hat{i}'\right) - m\omega^2 R\hat{i}' \\ &= m\left(g - \omega^2 R\right)\hat{i}' = mg^*\hat{i}' \ .\end{aligned} \tag{17.46}$$

Here we have introduced the effective acceleration of gravity, g^*:

$$g^* = g - \omega^2 R \ . \tag{17.47}$$

For the Earth the radius is $R = 6378\text{km}$, and

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \cdot 3600\text{s}} \simeq 7.27 \cdot 10^{-5}\text{rad/s} \ , \tag{17.48}$$

which gives:

$$\omega^2 R = 0.03\text{m/s}^2 \ . \tag{17.49}$$

Hence the correction to the acceleration of gravity is small, but not always negligible!

Summary – Chapter 17

Fictive forces

You are observing the motion of an object in a reference system S' which may be translated and rotating relative to an inertial reference system S. If you want to apply Newton's laws to describe the motion of the object in your reference system, you must introduce a set of fictive forces:

$$\sum \vec{F} + \vec{F}_A + \vec{F}_\omega + \vec{F}_C + \vec{F}_S = m\vec{a}' \ ,$$

where the various forces are:

- The force due to relative **translational** motion:

  $$\vec{F}_A = -m\vec{A} \ ,$$

  where $\vec{A}$ is the acceleration of the S' system measured in the S-system.

- The force due to a **change in rotational motion**:

  $$\vec{F}_\omega = -m\frac{d\vec{\omega}}{dt} \times \vec{r'} \ .$$

- The **Coriolis force**:

  $$\vec{F}_C = -2m\vec{\omega} \times \vec{v'} \ .$$

- The **centrifugal force**:

  $$\vec{F}_S = -m\vec{\omega} \times \left(\vec{\omega} \times \vec{r'}\right) \ .$$

Exercises – Chapter 17

17.1: Angular velocity of the Earth. Determine the angular velocity of the Earth for its rotation about its own axis (as seen from an inertial system).

17.2: Coriolis force on the Foucault pendulum. Make a sketch showing the velocity and the Coriolis force on a pendulum as its passes its lowest point:

(a) When the plane of oscillations is East-West.

(b) When the plane of oscillations is North-South.

It is useful to decompose velocities and forces in two ways: In a reference system where one axis is parallel to the Earth's rotational axis, and in a reference system where one axis is parallel to the surface normal on the Earth.

(c) For both cases above, find an expression for the part of the Coriolis force that is responsible for turning the plane of oscillation for the pendulum. Oslo is approximately at a latitude of 60°.

(d) Estimate the maximum of the Coriolis force for the pendulum in the main hall of the Physics Building. Compare with the gravitational force.

17.3: Rocket from the North Pole. A rocket is launched from the North Pole in a low orbit close to the surface of the Earth. The rocket travels a length of 4000km in 25 minutes. The radius of the Earth is $R = 6378$km. How far from the target does the rocket hit if the effects of the rotation of the Earth is neglected?

17.4: Effective gravity. Find the effective gravity in Oslo (at 60° North), and compare with the value at the equator.

17.5: Coriolis force on a car. Find the Coriolis force on a 1200kg car driving straight North from Oslo (at 60° North) with a speed of 90km/h.

17.6: Ball from CNN Tower. A metallic sphere is dropped from a window in the lower observatory deck in CNN Tower in Toronto. The height of the tower is 553m, while the height of the lower observational deck is only 342m above ground. By how much does the ball miss its target which is directely towards the center of the Earth? Toronto is at 44° North.

17.7: Children on a carousel. Two children are standing on two opposite sides of a carousel with a diameter of 6.0m. The carousel is rotating 12 times a minute. One of the children throws a ball of mass 0.50kg directly towards the other kid (as seen by the kid throwing the ball). The kid throws the ball with a velocity of 6.0m/s relative to the motion of the kid.

(a) Find the Coriolis force on the ball while in the air.

(b) Find where (on the carousel) the ball leaves the carousel.

17.8: Tilting of the tide. Due to the tidal forces, water is pushed northwards throught a channel of width d at a position λ degrees North. Show that the the height of the water on the eastern side of the channel is $2dv\omega \sin \lambda / g$ higher than on the western side of the channel. Here v is the flow velocity of the water, and ω is the angular velocity of the Earth.

Chapter 18

Theory of special relativity

Our intuition about physics is based on our experience with the physical world. In mechanics, we learn to fine-tune our observational capabilities and develop the appropriate conceptual framework to interpret experimental results. In practice, we learn to use Newton's laws both to predict behavior and to interpret the world. Through your studies of mechanics you therefore learn to adapt your intuition to a Newtontian view of the world. This is often a slow process, but eventually you interpret what you observe around you using these "Newtonian glasses", and you find that everything fits – it all makes sense. However, our intuition is limited to phenomena we have experience from. This is why we often find phenomena in the microscopic world – the atom and subatomic world – counterintuitive – simply because our intuition is from the macroscopic world. But our experience is not only limited in scale, we are also typically only observing a limited range of relative velocities. This is why we often find relativistic effects counter-intuitive – because we do not have any experience from objects that move at velocities near the speed of light.

In this chapter we address effects that are just as real as any other physical processes and effects as you have experience from – but they seem counter-intuitive because we do not have any experience from such effects. They only appear if we perform fine tuned experiments where objects move at a significant portion of the speed of light. Since we do not have intuition from these behaviors, we must be particularly stringent in our thoughts and careful in our assumptions and reasoning – because we cannot always fall back on our intuition to test whether the results are correct, although we can test them by careful experiments.

In this chapter you will learn that *simultaneity* is relative – it depends on the reference system. Two events may be simultaneous in one system and not simultaneous in another system. This is not a result of a measurement error or the way a measurement is done – it is a completely real effect.

You also learn about *length contraction* – an objects length is largest in a system where the object is at rest, and *time dilation* – the time interval between two events is the smallest in a system where the clock is at rest (where time is measured in the same point). Again, these effects are real – they are not illusions due to a particular way the effects are measured.

We introduce the Lorentz-transformations, which we use to transform from one frame of reference to another, and we find that the Galileo-transformations is a good approximation for low relative velocities.

18.1 Einstein's postulates

The theory of special relativity is the result of two postulates that seem innocent and intuitive, but their consequences are not.

18.1.1 Einstein's first postulate

Einstein's first postulate is that

> The laws of physics are identical in all inertial frames of reference.

We have already seen that Newton's laws are valid in all inertial reference systems – we also find that the motion predicted by Newton's laws is the same for any inertial system. But Einsteins postulate is more general: All laws of physics must be the same in all inertial reference frames. In particular, this should also be true for elecro-magnetism, which actually then also implies the second postulate.

18.1.2 Einstein's second postulate

Einstein's second postulate is that

> The speed of light in vacuum is the same in all inertial reference frames, and it does not depend on the velocity of the source.

Let us see how this corresponds to our experience and intuition, by first recalling the transformation between two coordinate systems.

18.1.3 The galilean transformation

You are sitting in a car driving at a velocity $\vec{u}$ along the ground, and throw a ball forward with an initial velocity $\vec{v}_0$ relative to the car. How do you find the motion of the ball relative to the car and to the ground? We introduce a coordinate system, S, placed on the ground, and coordinate system S' on the car. We may describe the position of the ball with $\vec{r}(t)$ in the system S, and with the position $\vec{r}'(t')$ in the S' system.

- Notice that two different observers in system S – for example two different persons standing at different positions on the ground – agree on the position $\vec{r}(t)$ as a function of time of the ball independently of their method of observation. We can assume that the observers are competent – they know the laws of physics and use them in their measurements of position and time.
- Notice that two observers in system S' – such as two persons standing at different positions in the car – also agree on the position $\vec{r}'(t')$ of the ball as a function of time.

We relate the two coordinate systems by the position $\vec{R}(t)$ of the origin of system S' measured in system S, so that:

$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t') \ , \tag{18.1}$$

in addition, we assume that the time is the same in both systems, so that $t = t'$. Taking the first and second time derivative allows us to relate the velocities and the accelerations of the ball in the two systems:

$$\vec{v}(t) = \frac{d\vec{R}}{dt} + \vec{v}'(t) = \vec{V} + \vec{v}'(t) \ , \tag{18.2}$$

$$\vec{a}(t) = \frac{d\vec{V}}{dt} + \vec{a}'(t) = \vec{A} + \vec{a}'(t) \ , \tag{18.3}$$

where $\vec{A} = 0$ if the system is an inertial system.

These transformations are called the **galilean transformations** between two inertial systems. We can use them to find the velocity of the ball relative to the ground:

$$\vec{v} = \vec{V} + \vec{v}_0' \ , \tag{18.4}$$

so if the car drives at a velocity of 50km/h and you throw the ball forward with an initial velocity of 50km/h relative to the car, the velocity of the ball relative to the ground is:

$$v = V + v_0' = 50\text{km/h} + 50\text{km/h} = 100\text{km/h} \ . \tag{18.5}$$

18.1.4 The light paradox

However, the results of the galilean transformations are not consistent with Einstein's second postulate: If a space ship flying by you with a speed of $V = 1000\mathrm{m/s}$ is firing a light beam with velocity c' forwards, then Einstein's second postulate states that the speed of light in vacuum in the space ship reference system, c', is the same as the speed of light in vacuum in your system, c. However, from the galilean transformations we find:

$$c = V + c' \neq c' \ . \tag{18.6}$$

This inconsistency is not due to an error in the measurements – it is a real inconsistency that needs to be addressed and resolved. And it becomes even more apparent in our next example.

18.2 Simultaneity of events

We have already started the definition of one of the most important definitions in our study of special relativity: What is an event? An event is an action – and event – that can be localized in space and time. That is, it can be described as occuring at a particular place, (x, y, z) and at a particular time, t. In many cases, we use a dramatic event for illustration purposes – such as a lightning strike – but an event may also simply be describing the position of an object at a particular space-time coordinate, such as the position of an object at the time t, (x, y, z, t).

Notice that we use 4 coordinates to specify an event: (x, y, z, t): It occurs at a specific place, (x, y, z), and a specific time, t. Similarly, we use four coordinates to describe an event in the S' system: (x', y', z', t').

We call two events **simultaneous** if they occur at the same time in a given reference system.

18.2.1 An event and its observations

Notice also that we must discern between an event, and the observation of an event. For example, if John sees a lightning strike at a time t_a at a distance a, as illustrated in figure 18.1, it means that the lightning strike occured at the time

$$t^* = t_a - \frac{a}{c} \ , \tag{18.7}$$

where c is the speed of light. However, Joan observes the same lightning strike at a time t_b, and she is standing a distance b from the lightning strike. Again, the lightning strike occured at the time

$$t^* = t_b - \frac{b}{c} \ . \tag{18.8}$$

They observe the lightning strike at different times – but they both agree on the time t^* when the strike occured. What is more fundamental – the time of observation or the time the event occurred? The most fundamental time is the time of the event, since the time of observation depends on the position of the observer and the means of observation.

In a given reference system, all observers agree on the position x, y, z and the time t of an event.

All the observers are intelligent and have a good grasp of the laws of physics, so that they can find out at what time and place an event occurred. It is therefore

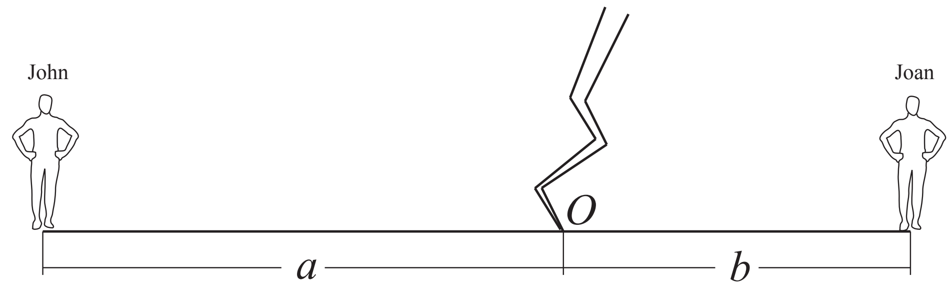

Figure 18.1: *A lightning strikes at the point O at the time t^*. John is standing at a distance a from O, and observes the lightning strike at a time t_a. Joan is standing at a distance b from O, and observes the lightning strike at the time t_b.*

only the space-time coordinate of the event, (x, y, z, t), that matters – not the position of the observer. Similarly, all observers in another reference frame, S', agree on the space-time position of an event in that reference frame, (x', y', z', t'), but these are generally not the same as in the first reference frame.

18.2.2 The train experiment

We are now ready to discuss Einstein's famous train experiment. This is only a thought experiment, but it clearly demonstrates the relativity of simultaneity.

A train is moving with a constant velocity u – which can be close to the speed of light – along a straight track along the x-axis. John is standing on the ground and Mary is standing on the train.

Two lightning strikes hits each end of the train, making a mark on the ground. John observes two flashes of light from the lightning striking the ground. He observes both flashes at the same time, and he carefully measures up the positions of the lightning strikes, and find that he is standing exactly in the middle between the positions of the two lightning strikes.

Since John was standing still on the ground at equal distance to each of the lightning strikes, and he observed the light from the two lightnings at the same time, he concludes that the two lightnings struk at the same time. That is, he concludes that event 2 – the lightning strike at position $x = x_1$ with space-time coordinates $(x_1, 0, 0, t_1)$, and the event 2 – the lightning strike at position $x = x_2$ with space-time coordinates $(x_2, 0, 0, t_2)$ occured at the same time: $t_1 = t_2$. The two events were therefore simultaneous.

In addition, John is able to infer that Mary must have observed event 2 before event 1 from the following argument. Mary was standing in the middle of the train, and is moving along with the train. At the time $t = t_1 = t_2$ in John's system, light is emitted from point 1 and point 2. But since Mary is moving towards point 2, the light wave from point 2 will reach her before the light wave from point 1. Therefore, Mary must observe event 2 before event 1. (See figure 18.2 for an illustration).

But what happens in Mary's system? If the events are simultaneous in her system, and because the speed of light (in vacuum) is constant, she observes the two signals at the same time. Hmmm. The two events cannot both occur at the same time and one before the other – only one thing will actually happen. We can simply ask Mary afterwards what happened – and either she observed one event before the other, or she observed them at the same time.

Something must therefore be wrong in our conclusions. We know with certainty that the events are simultaneous in John's system. But the event do not have to be simultaneous in Mary's system. We must therefore conclude that the two events are *not simultaneous* in Mary's system.

In Mary's system the two events occur at space-time coordinates $(x'_1, 0, 0, t'_1)$, and $(x'_2, 0, 0, t'_2)$, and we conclude that

$$t'_1 \neq t'_2 \ . \tag{18.9}$$

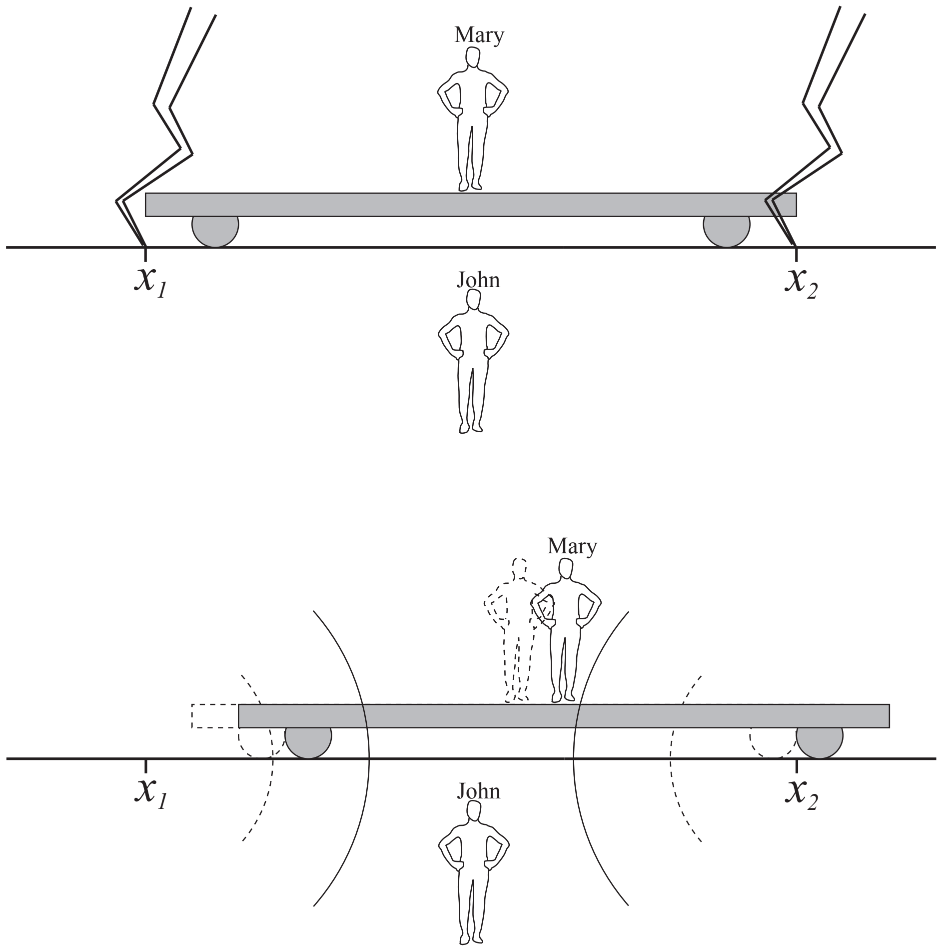

Figure 18.2: *John is standing on the ground, observing Joan passing on a high-speed train. A lightning strikes at each end of the train at the same time in John's system. The bottom figure shows the subsequent motion of the train and the light as seen by John. Since Joan is moving towards the right, she observes the light from the right event before the light from the left event.*

We can be even more precise. Since John has concluded that the light from event 2 reaches Mary before the light from event 1, we can conclude that for Mary

$$t'_2 < t'_1 \; . \tag{18.10}$$

Event 2 must have occured before event 1 in Mary's system in order for the light from event 2 to reach Mary before the light from event 1.

Our conclusion is therefore that

time, and simultaneity, is different in different inertial systems

This may seem counterintuitive. And it is – because we do not have any experience with such situations we do not have any well developed intuition. However, the effect is real. The world is actually in this way. It is not an illusion, and it is not an effect of errors in measurements. Time is relative.

18.3 Lorentz transformations

We are now ready to introduce a generalization of the galilean transformations that is also valid for large velocities. Let us address an event in two inertial systems S and S', where S' moves with a velocity u in the positive x-direction in the S-system. For simplicity, we assume that the axis are parallel in both systems, and at the time $t = t' = 0$ the origin in the S and S' systems coinside.

An event at (x, y, z, t) in the S system corresponds to an event at (x', y', z', t') in the S' system, and the two sets of coordinates are related by the **Lorentz transformations:**

$$\begin{aligned} x' &= \gamma\,(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{u}{c^2}x\right)\ , \end{aligned} \tag{18.11}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\ . \tag{18.12}$$

The reverse transformations are found be replacing x with x' and u with $-u$, giving:

$$\begin{aligned} x &= \gamma\,(x' + ut') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{u}{c^2}x'\right)\ . \end{aligned} \tag{18.13}$$

Advanced Material

Derivation of the Lorentz transformations

The coordinate system S' is moving with a constant velocity u relative to the inertial system S. The two systems are aligned, so that the x-axis is directed along the x'-axis, and similar for the other axes. At the time $t = t' = 0$ the origin of both coordinate systems are at the same position. We find the Lorentz transformation by addressing a light front sent out from the origin at the time $t = t' = 0$. At the time t, the light front has reached the point P: (x, y, z, t) in the S system, which corresponds to the point P': (x', y', z', t'), in the S'-system – that is, in the S' system the light front takes the time t' to reach this point.

According to Einstein's postulates, the speed of light is the same in both systems, therefore:

$$x^2 + y^2 + z^2 = (ct)^2\ , \tag{18.14}$$

and

$$\left(x'\right)^2 + \left(y'\right)^2 + \left(z'\right)^2 = \left(c't'\right)^2\ , \tag{18.15}$$

How do we get from x' to x? We realize that due to symmetry:

$$y' = y\ ,\ z' = z\ . \tag{18.16}$$

For the galilean transformations we found that:

$$x = X + x' = ut + x'\ \Rightarrow\ x' = x - ut\ . \tag{18.17}$$

Let us assume that the general solution has the same form, but we allow a prefactor A:

$$x' = A\,(x - ut)\ , \tag{18.18}$$

and we make a similar assumption for the time:

$$t' = B\,(t - Cx)\ . \tag{18.19}$$

We insert these expressions into equation 18.15, getting:

$$\begin{aligned} \left(x'\right)^2 + \left(y'\right)^2 + \left(z'\right)^2 &= \left(ct'\right)^2 \\ A^2\,(x - ut)^2 + y^2 + z^2 &= c^2B^2\,(t - Cx)^2 \\ A^2\left(x^2 - 2xut + u^2t^2\right) + y^2 + z^2 &= c^2B^2\left(t^2 - 2tCx + C^2x^2\right) \end{aligned} \tag{18.20}$$

$$\Bigg(\underbrace{A^2 - c^2B^2C^2}_{=1}\Bigg)x^2 + \Bigg(\underbrace{2CB^2c^2 - 2A^2u}_{=0}\Bigg)xt + y^2 + z^2 = \Bigg(\underbrace{c^2B^2 - A^2u^2}_{=1}\Bigg)t^2\ ,$$

since this must be true for any choise of x,y,z,t. We therefore get a set of equations:

$$A^2 - c^2B^2c^2 = 1 \ , \ 2CB^2c^2 - 2A^2u = 0 \ , \ c^2B^2 - A^2u^2 = 1 \ . \tag{18.21}$$

After some algebra we find the solutions:

$$A = B = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \ , \ C = \frac{u}{c^2} \ . \tag{18.22}$$

We have therefore found the Lorentz transformation.

18.3.1 Length contraction

Since we know the Lorentz transformations, they are our starting point to discuss physical effects. First, let us look at how the length of an object depends on the reference system.

How do we measure the length of an object? We find the end points x_1 and x_2 of the object at *the same time* $t_1 = t_2$, and define the length as the distance from point 1 to point 2:

$$L = x_2 - x_1 \ . \tag{18.23}$$

Now, we want to measure the length of a rod that moves with the velocity u along the x-axis. (We want to measure the length in the direction it is moving). First, we notice that if we introduce a reference system S' that moves with the velocity u along the x-axis, then the rod is at rest in the S' system.

It is easy to measure the length of the rod in the S' system. Here, the rod does not move, so we simple mark both ends, x'_1 and x'_2 and measure the distance between the two points:

$$L' = x'_2 - x'_1 \ . \tag{18.24}$$

Because the object is at rest in this system, this relation is always true. (The object does not move and its length does not change with time). This means that we may measure the positions x'_2 and x'_1 at any times t'_1 and t'_2 we like. The length in this system is fundamental, and we call the length the **resting length** of the object.

What is the length of the rod in the system S? In order to measure the length of the rod in the system S, we need to mark each end-point x_1 and x_2 at the same time – otherwise the object will have moved in between our measurements. That is, we record the end positions at the time $t = t_1 = t_2$ in the S system.

We use the Lorentz transformations to relate the two measurements. In the S' system, we have marked the points:

$$x'_2 = \gamma\left(x_2 - ut_2\right) \ , \tag{18.25}$$

and

$$x'_1 = \gamma\left(x_1 - ut_1\right) \ . \tag{18.26}$$

The length is therefore:

$$L' = x'_2 - x'_1 = \gamma\left(\underbrace{x_2 - x_1}_{=L}\right) + \gamma u\left(\underbrace{t_1 - t_2}_{=0}\right) \ , \tag{18.27}$$

since $t_2 = t_1$. We have therefore found a relation between the length L measured for the moving rod compared to the resting length L' of the rod:

$$L' = L_0 = \gamma L \ , \tag{18.28}$$

and therefore

$$L = \frac{1}{\gamma}L_0 = \underbrace{\sqrt{1-\frac{u^2}{c^2}}}_{<1}\,L_0 \ . \tag{18.29}$$

The rod is therefore shorter in the system where it is moving compared to the system where it is at rest. We call this effect **length contraction**.

In the classical limit where $u \ll c$, we find that $\gamma \simeq 1$, and therefore that $L \simeq L_0$, which is what our intuition tells us: Our intuition is based on observations in the $u \ll c$ limit.

18.3.2 Time dilation

Let us use the same method to address the time interval between two events. A watch is at rest at the position x_1', y_1', z_1' in the S' system, and the S' system is moving with a velocity u relative to the S system, with the axes alinged and motion only along the x-axes.

It is simple to record the time in the S' system: We record the time at the same position at the first event, t_1', and then the time of the second event, t_2'. The time interval is therefore

$$\Delta t' = t_2' - t_1' = \Delta t_0 \ , \tag{18.30}$$

which we call the **resting time**, since the watch is at rest in this system.

However, in the S system the watch is moving. The events that occured at the same position in the S' system, therefore occur at different positions in the S system, at the positions: (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2). We relate the coordinates by the Lorentz transformations:

$$t_1 = \gamma \left(t_1' + \frac{u}{c^2} x_1' \right) \ , \tag{18.31}$$

and

$$t_2 = \gamma \left(t_2' + \frac{u}{c^2} x_2' \right) \ , \tag{18.32}$$

The time interval between the two events in the system S where the watch is moving is therefore:

$$\Delta t = t_2 - t_1 = \gamma \left(t_2' - t_1' \right) - \frac{u}{c^2} \gamma \left(\underbrace{x_2' - x_1'}_{=0} \right) = \gamma \Delta t' = \gamma \Delta t_0 \ . \tag{18.33}$$

where $\gamma \geq 1$, and therefore:

$$\Delta t \geq \Delta t_0 \ . \tag{18.34}$$

The time between the two events is therefore longer in a system where the events do not take place in the same place, compared with the system where the events occur in the same place. This effect is called **time dilation**. The time period between two events is therefore *shortest* in the resting system.

Example 18.1: Muon decay

Problem: A muon created as a high energy particle from cosmic radiation enters the atmosphere. The Muon has a velocity $v = 0.990c$ relative to the Earth. The Muons decay time is $\tau = 2.2 \cdot 10^{-6}$s in a system where the Muon is at rest. How far does the Muon move before decaying?

Solution: We introduce the system S as the system of the Earth, and the system S' as the resting system for the Muon. The system S' therefore moves with a velocity $u = 0.990c$ relative to the Earth – relative to system S.

In the system S', the Muon decays after a time $\Delta t' = 2.2 \cdot 10^{-6}$s. How long time is this in the system S?

We apply time dilation. The time interval in a system S where the particle is moving with a velocity u is:

$$\begin{aligned} \Delta t = \gamma \Delta t' &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \Delta t' \\ &= \frac{1}{\sqrt{1 - (0.99)^2}} \Delta t' \simeq 7 \Delta t' = 16 \cdot 10^{-6}\text{s} \ . \end{aligned} \tag{18.35}$$

The Muon therefore decays after a time $\Delta t = 16 \cdot 10^{-6}$s in the Earth system, S.

How far has the Muon moved in this time?

$$\Delta x = u \Delta t \simeq 4.6\text{km} \ . \tag{18.36}$$

We may also find the answer by applying the Lorentz transformations directly: The Muon is moving along the x-axis of the Earth system. We introduce a system S' that moves with the Muon, that is, we introduce a system S' that moves along the x-axis with the velocity u, and we ensure that the two coordinate systems coinside at the time $t = t' = 0$ when the Muon was created. The Muon therefore starts at the point $x_1' = 0$ and $x_1 = 0$ at the time $t_1 = t_1' = 0$.

In the S' system the Muon does not move. Therefore it decays at the time $t_2' = \tau$ at the position $x_2' = 0$. Where and when does this occur in the Earth system? We apply the Lorentz transformations to find x_2 and t_2:

$$x_2 = \gamma \left(x_2' + u t_2' \right) = \gamma u \tau \ , \tag{18.37}$$

and

$$t_2 = \gamma \left(t_2' + \frac{u}{c^2} x_2' \right) = \gamma \tau \ , \tag{18.38}$$

Comment: Generally, we advice you to make a rule always to use the Lorentz transformations to map between events in two inertial systems. The main challenge is then usually to realize what *events* you need to introduce to convert the problem into a problem in transformation, as illustrated in this example.

Example 18.2: Train of thought

Let us revisit the train problem using the Lorentz transformations. In the system S the two lightning strikes are the two events, (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2), where we know that the events are simultaneous – that is $t_1 = t_2$.

When and where do these events occur in Mary's system – the S' system? Since the two reference systems have overlapping axis, we simply assume that they were coinsiding at the time $t = t' = 0$. We can therefore apply the Lorentz transformations to relate the events in the S and the S' systems. In Mary's system (the S' system) the events occur at:

$$x_1' = \gamma \left(x_1 - u t_1 \right) \ , \tag{18.39}$$

$$t_1' = \gamma \left(t_1 - \frac{u}{c^2} x_1 \right) \ , \tag{18.40}$$

and

$$x_2' = \gamma \left(x_2 - u t_2 \right) \ , \tag{18.41}$$

$$t_2' = \gamma \left(t_2 - \frac{u}{c^2} x_2 \right) \ . \tag{18.42}$$

We can therefore compare the time of occurence of the two events in Mary's system. The time interval between the two events in Mary's system is:

$$\begin{aligned} \Delta t' &= t_2' - t_1' \\ &= \gamma \left(\underbrace{t_2 - t_1}_{=0} \right) - \gamma \frac{u}{c^2} \left(x_2 - x_1 \right) \\ &= -\gamma \frac{u}{c^2} \left(x_2 - x_1 \right) \\ &\leq 0 \ , \end{aligned} \tag{18.43}$$

since $x_2 - x_1 \geq 0$.

From this argument we conclude that in Mary's system event 2 occured before event 1. Notice that this conclusion does not depend on the position of Mary and John in their respective coordinate systems – it only depends on the space-time positions of the two events.

In addition, the Lorentz transformations allows us to pinpoint the times and the positions of the events in Mary's system.

18.4 Velocity transformations

The instantaneous velocity of an object is the limit of the displacement of the time interval as the time interval goes to zero. We have now learned that when we transform between two inertial systems we need to transform both the spatial *and* the temporal coordinates using the Lorentz transformation. Therefore, if we compare the velocity measured in a system S with the velocity measured in a system S' that moves with a velocity u relative to the S-system, we must include the effect that the interval between two events also is different in the two reference systems.

Let us consider the standard situation for the Lorentz transformations. We address the motion of a point P in an inertial system S by its position $\vec{r}(t)$ as a function of time, where both the position and the time are measured in the system S. Another system S' is aligned so that the origin and all the axes overlap at the time $t = 0$. The system S' travels with the velocity u relative to the system S. For example, the system S may be a system created by an observer "at rest" in outer space, and the system S' corresponds to a spaceship travelling past the observer. The space ship has a velocity u relative to the observer's system. We can then always align the axes of the coordinate systems, and assume that the velocity u is directed along the x-axis without any loss of generality.

In this case, a consequence of the Lorentz transformations is that the velocity on an object, $\vec{v} = (v_x, v_y, v_z)$, in the S system can be related to the velocity, $\vec{v}' = (v_x', v_y', v_z')$ in the S' system through:

$$v_x' = \frac{dx'}{dt'} = \frac{v_x - u}{1 - \frac{u}{c^2} v_x} \ , \tag{18.44}$$

$$v_y' = \frac{dy'}{dt'} = \frac{v_y}{1 - \frac{u}{c^2} v_x} \ , \tag{18.45}$$

and

$$v'_z = \frac{dz'}{dt'} = \frac{v_z}{1 - \frac{u}{c^2} v_x} \ . \tag{18.46}$$

Notice that for the velocity transformations, the results are non-trivial also in the y' and z' directions because time also changes through the Lorentz transformations.

Advanced Material

Derivation of the velocity transform

How do we measure the velocity of an object in the S'-system? We find its position $\vec{r}'$ at a time t'_0 and at a time a small time interval $\Delta t'$ later, at $t'_1 = t'_0 + \Delta t'$, and then go to the limit when $\Delta t' \to 0$.

Let us first look at motion along the x-axis. The average velocity of the object from time t'_0 to t'_1 is :

$$\bar{v}'_x = \frac{x'(t'_1) - x'(t'_0)}{t'_1 - t'_0} \ . \tag{18.47}$$

We use the Lorentz transformation to relate these times and positions to the times and positions in the system S:

$$x'(t'_1) = x'_1 = \gamma \left(x_1 - u t_1 \right) \ , \tag{18.48}$$

and

$$t'_1 = \gamma \left(t_1 - \frac{u}{c^2} x_1 \right) \ . \tag{18.49}$$

We insert these results into equation 18.47, getting:

$$\frac{\Delta x'}{\Delta t'} = \frac{\gamma \left(x_1 - u t_1 \right) - \gamma \left(x_0 - u t_0 \right)}{\gamma \left(t_1 - \frac{u}{c^2} x_1 \right) - \gamma \left(t_0 - \frac{u}{c^2} x_0 \right)} \ , \tag{18.50}$$

$$\frac{\Delta x'}{\Delta t'} = \frac{\left(x_1 - x_0 \right) - u \left(t_1 - t_0 \right)}{\left(t_1 - t_0 \right) - \frac{u}{c^2} \left(x_1 - x_0 \right)} \ , \tag{18.51}$$

$$\frac{\Delta x'}{\Delta t'} = \frac{\frac{x_1 - x_0}{t_1 - t_0} - u}{1 - \frac{u}{c^2} \frac{x_1 - x_0}{t_1 - t_0}} \ , \tag{18.52}$$

$$\frac{\Delta x'}{\Delta t'} = \frac{\frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} - u}{1 - \frac{u}{c^2} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}} \ , \tag{18.53}$$

$$\frac{\Delta x'}{\Delta t'} = \frac{v_x - u}{1 - \frac{u}{c^2} v_x} \ . \tag{18.54}$$

Which proves the velocity transformation in the x-direction. You can find similar results in the y and the z directions, but in this case, it is only the time that is transformed.

Summary – Chapter 18

Einstein's postulates

Einstein introduced two general postulates that must be true for all inertial systems:

- The laws of physics are identical in all inertial frames of reference.
- The speed of light in vacuum is the same in all inertial reference frames, and it does not depend on the velocity of the source.

Event

- An **event** is an occurence that can be localized in space and time to a space-time coordinate (x, y, z, t). The event may simply be that an object is located at the position (x, y, z) at the time t.
- All observers in the same inertial system agree on when and where an event occur – the event is independent of the observation of the event.
- Simultaneity is relative. Two events that are simultaneous in one inertial system does not have to be simultaneous in another inertial system.

The Lorentz transformations

For two coordinate system S and S' where the axis are directed in the same directions in the two systems, where the origin in both systems overlap at the time $t = t' = 0$, and where the S' has a velocity u along the x-axis relative to the S system, the space-time coordinates of an event in system S is related to the space-time coordinates of the same event in system S' through the Lorentz trasformations:

$$x' = \gamma\,(x - ut)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{u}{c^2}x\right) \ ,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \ .$$

Length contraction

- The length of an object in a system where the object is at rest is called the rest length, L_0 of the object.
- The length of an object that moves with a velocity u along the x-axis in the system S is
$$L = \frac{1}{\gamma}L_0 \ .$$
- Since $\gamma > 1$ the length of a moving object will always be smaller than its rest length. Objects are contracted when moving. We call this effect **length contraction**.

Time dilation

- The time between two events in a system where the event occurs at the same place is called the rest time, Δt_0, of a process.
- The time between two events in a system where the time is measured in different positions is:
$$\Delta t = \gamma \Delta t_0 \ ,$$
where $\gamma \geq 1$, and therefore $\Delta t \geq \Delta t_0$.
- The time between two events in a system where the watch is moving is larger than the time between the same two events in a system where the events occur in the same position is therefore longer – we call this effect **time dilation**.

Velocity transformations

For two inertial systems moving relative to each other as required for the Lorentz transformation, the velocity of an object measured in system S' is related to the velocity of the object in the system S through:

$$v'_x = \frac{dx'}{dt'} = \frac{v_x - u}{1 - \frac{u}{c^2}v_x} \ ,$$

$$v'_y = \frac{dy'}{dt'} = \frac{v_y}{1 - \frac{u}{c^2}v_x} \ ,$$

and

$$v'_z = \frac{dz'}{dt'} = \frac{v_z}{1 - \frac{u}{c^2}v_x} \ .$$

Exercises – Chapter 18

18.1: Testing length contraction. Ole and Mary wants to test the concept of length contraction. Mary enters her spaceship and flies past Ole, who is standing still on the ground, at a velocity close to the speed of light. Exactly at the moment she passes Ole, she fires two lasers, and each laser makes a mark on the ground. One laser is fastened to the front for the spaceship, and marks the position of the the front of the spaceship, and the other laser is fastened at the back of the spaceship, and marks the end of the spaceship. You can assume that the mark on the ground from a laser is generated at the same time as the laser is fired.

Afterward, Ole measures the distance between the two marks, and compares with the length of the spaceship when it has stopped and is at rest. To his surprise, he finds that the distance between the marks on the ground is longer than the length of the spaceship at rest. He concludes that the spaceship is longer when it moves, and not shorter as expected from length contraction! How can you explain Ole's measurements?

18.2: A passing spacecraft. Ole is standing on the ground and observes two lightning strokes simultaneously. Lightning A at $x = 0$km and lightning B at $x = 30$km. Mary passes in a spaceship with the velocity $u = 0.8c$ in positive x-direction. In Ole's system she is at the point $x = 60$km when the lightnings occur. Do the lightnings occur at the same time in Mary's system? If not, which lightning occurs first?